A Note On Logical PERs and Reducibility. Logical Relations Strike Again!

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Abstract

We prove a general theorem for establishing properties expressed by binary relations on typed (first-order) λ-terms, using a variant of the reducibility method and logical PERs. As an application, we prove simultaneously that β-reduction in the simply-typed λ-calculus is strongly normalizing, and that the Church-Rosser property holds (and similarly for $\beta\eta$-reduction).

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1 Introduction

Logical relations are an important tool used in proving some deep results about various typed λ-calculi and their models. A special form of the concept of a logical relation first appeared in Harvey Friedman's seminal paper [4]. General logical relations were defined and used extensively in the pioneering work of Plotkin [18] and Statman [19, 21, 20], and later on in a more general setting by Breazu-Tannen and Coquand [2], Mitchell [15], Mitchell and Moggi [16], and Abramsky [1], among others. As the name indicates, logical relations are certain kinds of relations, and they are used to prove relational properties of terms. On the other hand, reducibility is a tool used in proving properties of terms in various typed λ-calculi. Typically, it is used to prove strong normalization or normalization, but it can be used to prove other properties as well. The method was pioneered by Tait [22] for the simply-typed λ-calculus, and brilliantly extended to various higher-order typed λ-calculi by Girard [9, 10] (see also Tait [23]). Various expositions and analyses of such proofs are given in Mitchell [15], Krivine [14], Huet [11], and Gallier [5, 6, 7, 8], among others. Another crucial concept is that of a partial equivalence relation, or *PER*. PER's were introduced by Hyland [12] and Mulry [17]. PERs are a major tool in defining categories of domains in an effective setting (see Freyd, Mulry, Rosolini, and Scott [3]). PERs also often show up as logical relations, and are called *logical PERs* (see Breazu-Tannen and Coquand [2]).

In this note, we prove a general theorem for establishing properties expressed by binary relations on typed (first-order) λ-terms, using a variant of the reducibility method and of logical PERs. This note is written much in the spirit of our earlier papers [6, 8]. Our goal is to elucidate the conditions under which the technology of reducibility and of logical relations works. We do this by finding sufficient conditions that a binary relation $\mathcal{R}$ on typed λ-terms need to satisfy for establishing that $\mathcal{R}$ holds, using reducibility. The conditions presented in this paper were inspired by a paper by Koletsos [13].

In this short note, we restrict out attention to the simply-typed λ-calculus, but there is little doubt that our method can be generalized to all the first-order types (as in [6]), or to type intersection disciplines (as in [8]). As an illustration, it is easy to show simultaneously that β-reduction is strongly normalizing and that the Church-Rosser property holds (and similarly for $\beta\eta$-reduction).

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The generalization to second-order types (or more general types) is much more problematic (for a discussion of some of the problems, see Breazu-Tannen and Coquand [2]), and is left as an open problem.

2 $\mathcal{R}$-Logical Candidates for the Arrow Type Constructor $\rightarrow$

Let $\mathcal{T}$ denote the set of (simple) types. Recall that the set of simple types is defined inductively from a set of base types and using the type constructor $\rightarrow$, i.e. a base type b is a type, and $(\sigma \rightarrow \tau)$ is a type whenever σ and τ are types.

The presentation will be simplified if we adopt the definition of simply-typed λ-terms where all the variables are explicitly assigned types once and for all. More precisely, we have a family $\mathcal{X} = (X_\sigma)_{\sigma \in \mathcal{T}}$ of variables, where each X_σ is a countably infinite set of variables of type σ, and $X_\sigma \cap X_\tau = \emptyset$ whenever $\sigma \neq \tau$. Using this definition, there is no need to drag contexts along, and the most important feature of the proof, namely the reducibility method, is easier to grasp. Recall that an untyped λ-term is either a variable x, an application (MN), or a λ-abstraction $\lambda x\colon \sigma.\, M$. The terms of the typed λ-calculus $\lambda^{\rightarrow}$ (also called simply-typed λ-terms) are the λ-terms that respect certain type-checking rules reviewed below.

▶ **Definition 1.** *Given a λ-term M and a type σ, we define the binary relation $M\colon \sigma$ (read, M has type σ) using the following type-checking rules:*

$$x\colon \sigma, \quad \textit{when } x \in X_\sigma,$$

(we can also have $c\colon \sigma$, where c is a constant of type σ, if there is a set of constants that have been preassigned types).

$$\frac{x\colon \sigma \quad M\colon \tau}{\lambda x\colon \sigma.\, M\colon (\sigma \rightarrow \tau)} \quad \textit{(abstraction)}$$

$$\frac{M\colon (\sigma \rightarrow \tau) \quad N\colon \sigma}{(MN)\colon \tau} \quad \textit{(application)}$$

From now on, when we refer to a λ-term, we mean a λ-term that type-checks. We let Λ_σ denote the set of λ-terms of type σ, and $\Lambda^{\rightarrow} = (\Lambda_\sigma)_{\sigma \in \mathcal{T}}$, also called the set of *simply-typed λ-terms*. In this section, the only reduction rule considered is β-reduction:

$$(\lambda x\colon \sigma.\, M)N \longrightarrow_\beta M[N/x].$$

Equations between λ-terms of the same type σ are denoted as $M \doteq N\colon \sigma$, and equational provability is defined as follows.

▶ **Definition 2.** *The axioms and inference rules of the equational β-theory of the typed λ-calculus $\lambda^{\rightarrow}$ are defined below.*

$$x \doteq x\colon \sigma \quad \textit{(reflexivity)},$$

where x is any variable of type σ. We also have axioms $c \doteq c\colon \sigma$, where c is a constant of type σ, when typed constants are present.

$$(\lambda x\colon \sigma.\, M)N \doteq M[N/x]\colon \tau \quad (\beta)$$

$$\frac{M_1 \doteq M_2\colon \sigma}{M_2 \doteq M_1\colon \sigma} \quad \textit{(symmetry)}$$

$$\frac{M_1 \doteq M_2\colon \sigma \quad M_2 \doteq M_3\colon \sigma}{M_1 \doteq M_3\colon \sigma} \quad \textit{(transitivity)}$$

$$\frac{M_1 \doteq M_2\colon (\sigma \to \tau) \quad N_1 \doteq N_2\colon \sigma}{(M_1 N_1) \doteq (M_2 N_2)\colon \tau} \quad \textit{(congruence)}$$

$$\frac{M_1 \doteq M_2\colon \tau}{\lambda x\colon \sigma.\, M_1 \doteq \lambda x\colon \sigma.\, M_2\colon (\sigma \to \tau)} \quad (\xi)$$

The notation $\vdash_\beta M \doteq N\colon \sigma$ means that the equation $M \doteq N\colon \sigma$ is provable from the above axioms and inference rules.

The equational $\beta\eta$-theory of the typed λ-calculus $\lambda^{\to}$ is obtained by adding the following axiom to the above axioms and inference rules.

$$\lambda x\colon \sigma.\, (Mx) \doteq M\colon (\sigma \to \tau) \quad (\eta)$$

where $x \notin FV(M)$.

The notation $\vdash_{\beta\eta} M \doteq N\colon \sigma$ means that the equation $M \doteq N\colon \sigma$ is provable from all the axioms, including (η), and the inference rules.

Given any term M, we can easily show by induction on the structure of M that the equation $M \doteq M\colon \sigma$ is provable using the (*reflexivity*) axioms and the rules (*congruence*) and (ξ). Thus, reflexivity holds for all terms, not just variables and constants. The reason for using a restricted form of the reflexivity axioms is that this makes the proof of Lemma 10 simpler.

It turns out that the behavior of a term depends heavily on the nature of the last typing inference rule used in typing this term. A term created by an introduction rule, or I-term, plays a crucial role, because when combined with another term, a new redex is created. On the other hand, for a term created by an elimination rule, or simple term, no new redex is created when this term is combined with another term. This motivates the following definition.

▶ **Definition 3.** *An I-term is a term of the form $\lambda x\colon \sigma.\, M$. A simple term (or neutral term) is a term that is not an I-term. Thus, a simple term is either a variable x, a constant c, or an application MN. A term M is stubborn iff it is simple and, either M is irreducible, or M' is a simple term whenever $M \xrightarrow{+}_\beta M'$ (equivalently, M' is* **not** *an I-term).*

Let $\mathcal{R} = (R_\sigma)_{\sigma \in \mathcal{T}}$ be a family of nonempty binary relations, where $R_\sigma \subseteq \Lambda_\sigma \times \Lambda_\sigma$.

▶ **Definition 4.** *Properties (P0)-(P3) are defined as follows:*

(P0) Every relation R_σ is a per, i.e., R_σ is symmetric and transitive.

(P1) $\langle x,\, x\rangle \in R_\sigma$, $\langle c,\, c\rangle \in R_\sigma$, for every variable x and constant c of type σ.

(P2) If $\langle M_1,\, M_2\rangle \in R_\sigma$ and $M_1 \longrightarrow_\beta M_1'$, then $\langle M_1',\, M_2\rangle \in R_\sigma$.

(P3) If M_1 and M_2 are simple, $\langle M_1,\, M_2\rangle \in R_{\sigma\to\tau}$, $\langle N_1,\, N_2\rangle \in R_\sigma$, and either $\langle(\lambda x\colon \sigma. M_1')N_1,\, M_2N_2\rangle \in R_\tau$ whenever $M_1 \xrightarrow{+}_\beta \lambda x\colon \sigma.\, M_1'$ and M_2 is stubborn, or $\langle(\lambda x\colon \sigma.\, M_1')N_1,\, (\lambda x\colon \sigma.\, M_2')N_2\rangle \in R_\tau$ whenever $M_1 \xrightarrow{+}_\beta \lambda x\colon \sigma.\, M_1'$ and $M_2 \xrightarrow{+}_\beta \lambda x\colon \sigma.\, M_2'$, then $\langle M_1N_1,\, M_2N_2\rangle \in R_\tau$.

From now on, we only consider families of relations $\mathcal{R}$ satisfying conditions (P0)-(P3) of Definition 4.

▶ **Definition 5.** *For any type σ, a nonempty relation $C \subseteq \Lambda_\sigma \times \Lambda_\sigma$ is a $\mathcal{R}$-logical candidate iff it satisfies the following conditions:*

(R0) C is a per.

(R1) $C \subseteq R_\sigma$.

(R2) If $\langle M_1,\ M_2\rangle \in C$ and $M_1 \longrightarrow_\beta M_1'$, then $\langle M_1',\ M_2\rangle \in C$.

(R3) If M_1 and M_2 are simple, $\langle M_1,\ M_2\rangle \in R_\sigma$, and either $\langle \lambda x\colon \gamma.\, M_1',\ M_2\rangle \in C$ whenever $M_1 \xrightarrow{+}_\beta \lambda x\colon \gamma.\, M_1'$ and M_2 is stubborn, or $\langle \lambda x\colon \gamma.\, M_1',\ \lambda x\colon \gamma.\, M_2'\rangle \in C$ whenever $M_1 \xrightarrow{+}_\beta \lambda x\colon \gamma.\, M_1'$ and $M_2 \xrightarrow{+}_\beta \lambda x\colon \gamma.\, M_2'$, then $\langle M_1,\ M_2\rangle \in C$.

Note that (R3) and (P1) imply that for every type σ, any $\mathcal{R}$-logical candidate C of type σ contains all pairs $\langle x,\ x\rangle$ and $\langle c,\ c\rangle$ for all variables and all constants of type σ. More generally, (R3) implies that C contains all pairs $\langle M_1,\ M_2\rangle$ of stubborn terms in R_σ, and (P1) guarantees that pairs $\langle x,\ x\rangle$ and $\langle c,\ c\rangle$ are in R_σ (for every type σ).

By (P3), if $\langle M_1,\ M_2\rangle \in R_{\sigma\to\tau}$ is a pair of stubborn terms and $\langle N_1,\ N_2\rangle \in R_\sigma$ is any pair of terms, then $\langle M_1N_1,\ M_2N_2\rangle \in R_\tau$. Furthermore, M_1N_1 and M_2N_2 are also stubborn since they are simple terms and since they can only reduce to an I-term (a λ-abstraction) if M_1 or M_2 reduce to a λ-abstraction, i.e. an I-term. Thus, if $\langle M_1,\ M_2\rangle \in R_{\sigma\to\tau}$ is a pair of stubborn terms and $\langle N_1,\ N_2\rangle \in R_\sigma$ is any pair of terms, then $\langle M_1N_1,\ M_2N_2\rangle \in R_\tau$ is a pair of stubborn terms. Also, observe that if $M_1 \xrightarrow{+}_\beta M_1'$, $M_2 \xrightarrow{+}_\beta M_2'$, and $\langle M_1,\ M_2\rangle \in C$, then $\langle M_1',\ M_2'\rangle \in C$. This follows from (R2) and (R0), since (R0) implies symmetry and transitivity.

Given a family of relations $\mathcal{R}$, for every type σ, we define the relation $[\![\sigma]\!]$ as follows.

▶ **Definition 6.** *The logical relations $[\![\sigma]\!]$ are defined as follows:*

$$
\begin{aligned}
[\![\sigma]\!] &= R_\sigma, \qquad \sigma \text{ a base type},\\
[\![\sigma \to \tau]\!] &= \{\langle M_1,\ M_2\rangle \mid \langle M_1,\ M_2\rangle \in R_{\sigma\to\tau},\ \text{and for all } N_1,\ N_2,\\
&\qquad \text{if } \langle N_1,\ N_2\rangle \in [\![\sigma]\!] \text{ then } \langle M_1N_1,\ M_2N_2\rangle \in [\![\tau]\!]\}.
\end{aligned}
$$

▶ **Lemma 7.** *If $\mathcal{R}$ is a family of relations satisfying conditions (P0)-(P3), then each $[\![\sigma]\!]$ is a $\mathcal{R}$-logical candidate that contains all pairs of stubborn terms in R_σ.*

Proof. We proceed by induction on types. If σ is a base type, $[\![\sigma]\!] = R_\sigma$, and obviously, every pair of stubborn terms in R_σ is in $[\![\sigma]\!]$. Since $[\![\sigma]\!] = R_\sigma$, (R0) and (R1) are trivial, (R2) follows from (P2), and (R3) is also trivial.[1]

We now consider the induction step.

(R0). By the definition of $[\![\sigma \to \tau]\!]$, symmetry and transitivity are straightforward.

(R1). By the definition of $[\![\sigma \to \tau]\!]$, (R1) is trivial.

(R2). Let $\langle M_1,\ M_2\rangle \in [\![\sigma \to \tau]\!]$, and assume that $M_1 \longrightarrow_\beta M_1'$. Since $\langle M_1,\ M_2\rangle \in R_{\sigma\to\tau}$ by (R1), we have $\langle M_1',\ M_2\rangle \in R_{\sigma\to\tau}$ by (P2). For any $\langle N_1,\ N_2\rangle \in [\![\sigma]\!]$, since

[1] In fact, if $[\![\sigma]\!] = R_\sigma$, (R3) holds trivially even at nonbase types. This remark is useful is we allow type variables.

$\langle M_1,\ M_2\rangle \in [\![\sigma \to \tau]\!]$, we have $\langle M_1N_1,\ M_2N_2\rangle \in [\![\tau]\!]$, and since $M_1 \longrightarrow_\beta M_1'$ we have $M_1N_1 \longrightarrow_\beta M_1'N_1$. Then, applying the induction hypothesis at type τ, (R2) holds for $[\![\tau]\!]$, and thus $\langle M_1'N_1,\ M_2N_2\rangle \in [\![\tau]\!]$. Thus, we have shown that $\langle M_1',\ M_2\rangle \in R_{\sigma\to\tau}$ and that if $\langle N_1,\ N_2\rangle \in [\![\sigma]\!]$, then $\langle M_1'N_1,\ M_2N_2\rangle \in [\![\tau]\!]$. By the definition of $[\![\sigma \to \tau]\!]$, this shows that $\langle M_1',\ M_2\rangle \in [\![\sigma \to \tau]\!]$, and (R2) holds at type $\sigma \to \tau$.

(R3). Let $\langle M_1,\ M_2\rangle \in R_{\sigma\to\tau}$, and assume that $\langle \lambda x\colon \sigma.\ M_1',\ \lambda x\colon \sigma.\ M_2'\rangle \in [\![\sigma \to \tau]\!]$ whenever $M_1 \xrightarrow{+}_\beta \lambda x\colon \sigma.\ M_1'$ and $M_2 \xrightarrow{+}_\beta \lambda x\colon \sigma.\ M_2'$, or that $\langle \lambda x\colon \sigma.\ M_1',\ M_2\rangle \in [\![\sigma \to \tau]\!]$ whenever $M_1 \xrightarrow{+}_\beta \lambda x\colon \sigma.\ M_1'$ and M_2 is stubborn, where M_1 and M_2 are simple terms. We prove that for every $\langle N_1,\ N_2\rangle$, if $\langle N_1,\ N_2\rangle \in [\![\sigma]\!]$, then $\langle M_1N_1,\ M_2N_2\rangle \in [\![\tau]\!]$. First, we prove that $\langle M_1N_1,\ M_2N_2\rangle \in R_\tau$, and for this we use (P3). First, assume that M_1 and M_2 are stubborn, and let $\langle N_1,\ N_2\rangle$ be in $[\![\sigma]\!]$. By (R1), $\langle N_1,\ N_2\rangle \in R_\sigma$. By the induction hypothesis, all pairs of stubborn terms in R_τ are in $[\![\tau]\!]$. Since we have shown that $\langle M_1N_1,\ M_2N_2\rangle$ is a pair of stubborn terms in R_τ whenever $\langle M_1,\ M_2\rangle \in R_{\sigma\to\tau}$ is pair of stubborn terms and $\langle N_1,\ N_2\rangle \in R_\tau$, we have $\langle M_1,\ M_2\rangle \in [\![\sigma \to \tau]\!]$.

Now, assume that M_1 or M_2 is not stubborn. Since by (R0), each $[\![\sigma]\!]$ is symmetric, we only need to consider the case where M_1 is not stubborn and M_2 is stubborn. This case is similar to the next case, because M_2N_2 is stubborn for any N_2, and we leave it as an exercise.

Consider $\langle M_1, M_2\rangle \in R_{\sigma\to\tau}$ where M_1 and M_2 are non stubborn. If $M_1 \xrightarrow{+}_\beta \lambda x\colon \sigma.\ M_1'$ and $M_2 \xrightarrow{+}_\beta \lambda x\colon \sigma. M_2'$, then by assumption, $\langle \lambda x\colon \sigma. M_1',\ \lambda x\colon \sigma. M_2'\rangle \in [\![\sigma \to \tau]\!]$, and for any $\langle N_1,\ N_2\rangle \in [\![\sigma]\!]$, we have $\langle (\lambda x\colon \sigma.\ M_1')N_1,\ (\lambda x\colon \sigma.\ M_2')N_2\rangle \in [\![\tau]\!]$. Since by (R1), $\langle N_1,\ N_2\rangle \in R_\sigma$ and $\langle (\lambda x\colon \sigma.\ M_1')N_1,\ (\lambda x\colon \sigma.\ M_2')N_2\rangle \in R_\tau$, by (P3), we have $\langle M_1N_1,\ M_2N_2\rangle \in R_\tau$. Now, there are two cases.

If τ is a base type, then $[\![\tau]\!] = R_\tau$ and $\langle M_1N_1,\ M_2N_2\rangle \in [\![\tau]\!]$.

If τ is not a base type, then the terms M_1N_1 and M_2N_2 are simple. We prove that $\langle M_1N_1, M_2N_2\rangle \in [\![\tau]\!]$ using (R3) (which by induction, holds at type τ). The case where M_1N_1 and M_2N_2 are stubborn follows from the induction hypothesis. The case where M_1N_1 is not stubborn and M_2N_2 is stubborn is similar to the next case, but simpler (and the symmetric case follows by (R0)).

If both M_1N_1 and M_2N_2 are not stubborn terms, observe that if $M_1N_1 \xrightarrow{+}_\beta Q_1$ and $M_2N_2 \xrightarrow{+}_\beta Q_2$, where $Q_1 = \lambda y\colon \gamma.\ P_1$ and $Q_2 = \lambda y\colon \gamma.\ P_2$ are I-terms, then the reductions are necessarily of the form

$$M_1N_1 \xrightarrow{+}_\beta (\lambda x\colon \sigma.\ M_1')N_1' \longrightarrow_\beta M_1'[N_1'/x] \xrightarrow{*}_\beta Q_1,$$

and

$$M_2N_2 \xrightarrow{+}_\beta (\lambda x\colon \sigma.\ M_2')N_2' \longrightarrow_\beta M_2'[N_2'/x] \xrightarrow{*}_\beta Q_2,$$

where $M_1 \xrightarrow{+}_\beta \lambda x\colon \sigma.\ M_1'$, $M_2 \xrightarrow{+}_\beta \lambda x\colon \sigma.\ M_2'$, $N_1 \xrightarrow{*}_\beta N_1'$, and $N_2 \xrightarrow{*}_\beta N_2'$. Since by assumption, $\langle \lambda x\colon \sigma.\ M_1',\ \lambda x\colon \sigma.\ M_2'\rangle \in [\![\sigma \to \tau]\!]$ whenever $M_1 \xrightarrow{+}_\beta \lambda x\colon \sigma.\ M_1'$ and $M_2 \xrightarrow{+}_\beta \lambda x\colon \sigma.\ M_2'$, and by the induction hypothesis applied at type σ, by (R2) and (R0),[2] $\langle N_1',\ N_2'\rangle \in [\![\sigma]\!]$, we conclude that $\langle (\lambda x\colon \sigma.\ M_1')N_1',\ (\lambda x\colon \sigma.\ M_2')N_2'\rangle \in [\![\tau]\!]$. By the induction hypothesis applied at type τ, by (R2) and (R0), we have $\langle Q_1,\ Q_2\rangle \in [\![\tau]\!]$, and by (R3), we have $\langle M_1N_1,\ M_2N_2\rangle \in [\![\tau]\!]$.

[2] Symmetry and transitivity are needed, but they follow from (R0).

Since $\langle M_1,\ M_2\rangle \in R_{\sigma\to\tau}$ and $\langle M_1N_1,\ M_2N_2\rangle \in [\![\tau]\!]$ whenever $\langle N_1,\ N_2\rangle \in [\![\sigma]\!]$, we conclude that $\langle M_1,\ M_2\rangle \in [\![\sigma \to \tau]\!]$. ◀

For the proof of the next lemma, we need to add two new conditions (P4) and (P5) to (P0)-(P3).

▶ **Definition 8.** *Properties (P4) and (P5) are defined as follows:*

(P4) *If* $\langle M_1,\ M_2\rangle \in R_\tau$, *then* $\langle \lambda x\colon \sigma.\, M_1,\ \lambda x\colon \sigma.\, M_2\rangle \in R_{\sigma\to\tau}$.
(P5) *If* $\langle N_1, N_2\rangle \in R_\sigma$ *and* $\langle M_1[N_1/x], M_2[N_2/x]\rangle \in R_\tau$, *then* $\langle (\lambda x\colon \sigma.\, M_1)N_1, (\lambda x\colon \sigma.\, M_2)N_2\rangle \in R_\tau$.

▶ **Lemma 9.** *If* $\mathcal{R}$ *is a family of relations satisfying conditions (P0)-(P5) and for every* $\langle N_1, N_2\rangle$, $(\langle N_1, N_2\rangle \in [\![\sigma]\!]$ *implies* $\langle M_1[N_1/x],\ M_2[N_2/x]\rangle \in [\![\tau]\!])$, *then* $\langle \lambda x\colon \sigma.M_1, \lambda x\colon \sigma.M_2\rangle \in [\![\sigma \to \tau]\!]$.

Proof. We prove that $\langle \lambda x\colon \sigma.\, M_1,\ \lambda x\colon \sigma.\, M_2\rangle \in R_{\sigma\to\tau}$ and that for every every $\langle N_1,\ N_2\rangle$, if $\langle N_1,\ N_2\rangle \in [\![\sigma]\!]$, then $\langle (\lambda x\colon \sigma.\, M_1)N_1,\ (\lambda x\colon \sigma.\, M_2)N_2\rangle \in [\![\tau]\!]$. We will need the fact that the sets of the form $[\![\sigma]\!]$ have the properties (R0)-(R3), but this follows from Lemma 7, since (P0)-(P3) hold. First, we prove that $\langle (\lambda x\colon \sigma.\, M_1)N_1,\ (\lambda x\colon \sigma.\, M_2)N_2\rangle \in R_{\sigma\to\tau}$.

Since by Lemma 7, $\langle x,\ x\rangle \in [\![\sigma]\!]$ for every variable of type σ, by the assumption of Lemma 9, $\langle M_1[x/x],\ M_2[x/x]\rangle = \langle M_1,\ M_2\rangle \in [\![\tau]\!]$. Then, by (R1), $\langle M_1,\ M_2\rangle \in R_\tau$, and by (P4), we have $\langle \lambda x\colon \sigma.\, M_1,\ \lambda x\colon \sigma.\, M_2\rangle \in R_{\sigma\to\tau}$.

Next, we prove that for every every $\langle N_1,\ N_2\rangle \in [\![\sigma]\!]$, then $\langle (\lambda x\colon \sigma.M_1)N_1,\ (\lambda x\colon \sigma.M_2)N_2\rangle \in [\![\tau]\!]$. Assume that $\langle N_1,\ N_2\rangle \in [\![\sigma]\!]$. Then, by the assumption of Lemma 9, we deduce that $\langle M_1[N_1/x],\ M_2[N_2/x]\rangle \in [\![\tau]\!]$. Thus, by (R1), we have $\langle N_1,\ N_2\rangle \in R_\sigma$ and $\langle M_1[N_1/x], M_2[N_2/x]\rangle \in R_\tau$. By (P5), we have $\langle (\lambda x\colon \sigma.\, M_1)N_1,\ (\lambda x\colon \sigma.\, M_2)N_2\rangle \in R_\tau$. Now, there are two cases.

If τ is a base type, then $[\![\tau]\!] = R_\tau$. But we just showed that $\langle (\lambda x\colon \sigma.M_1)N_1, (\lambda x\colon \sigma.M_2)N_2\rangle \in R_\tau$, so we have $\langle (\lambda x\colon \sigma.\, M_1)N_1,\ (\lambda x\colon \sigma.\, M_2)N_2\rangle \in [\![\tau]\!]$.

If τ is not a base type, then $(\lambda x\colon \sigma.\, M_1)N_1$ and $(\lambda x\colon \sigma.\, M_2)N_2$ are simple. Thus, we prove that $\langle (\lambda x\colon \sigma.\, M_1)N_1,\ (\lambda x\colon \sigma.\, M_2)N_2\rangle \in [\![\tau]\!]$ using (R3). The case where $(\lambda x\colon \sigma.\, M_1)N_1$ and $(\lambda x\colon \sigma.\, M_2)N_2$ are stubborn is trivial. The case where $(\lambda x\colon \sigma.\, M_1)N_1$ is not stubborn and $(\lambda x\colon \sigma.\, M_2)N_2$ is stubborn is similar to the next case and simpler (and the symmetric case follows by (R0)).

If $(\lambda x\colon \sigma.\, M_1)N_1$ and $(\lambda x\colon \sigma.\, M_2)N_2$ are not stubborn and if $(\lambda x\colon \sigma.\, M_1)N_1 \xrightarrow{+}_\beta Q_1$ and $(\lambda x\colon \sigma.\, M_2)N_2 \xrightarrow{+}_\beta Q_2$, where $Q_1 = \lambda y\colon \gamma.\, P_1$ and $Q_2 = \lambda y\colon \gamma.\, P_2$ are I-terms, then the reductions are necessarily of the form

$$(\lambda x\colon \sigma.\, M_1)N_1 \xrightarrow{*}_\beta (\lambda x\colon \sigma.\, M_1')N_1' \longrightarrow_\beta M_1'[N_1'/x] \xrightarrow{*}_\beta Q_1,$$

$$(\lambda x\colon \sigma.\, M_2)N_2 \xrightarrow{*}_\beta (\lambda x\colon \sigma.\, M_2')N_2' \longrightarrow_\beta M_2'[N_2'/x] \xrightarrow{*}_\beta Q_2,$$

where $M_1 \xrightarrow{*}_\beta M_1'$, $M_2 \xrightarrow{*}_\beta M_2'$, $N_1 \xrightarrow{*}_\beta N_1'$, and $N_2 \xrightarrow{*}_\beta N_2'$. But $\langle M_1[N_1/x], M_2[N_2/x]\rangle \in [\![\tau]\!]$, and since

$$M_1[N_1/x] \xrightarrow{*}_\beta M_1'[N_1'/x] \xrightarrow{*}_\beta Q_1,$$

and

$$M_2[N_2/x] \xrightarrow{*}_\beta M_2'[N_2'/x] \xrightarrow{*}_\beta Q_2,$$

by (R2) and (R0), we have $\langle Q_1,\ Q_2\rangle \in [\![\tau]\!]$. Since $\langle(\lambda x\colon \sigma.\, M_1)N_1,\ (\lambda x\colon \sigma.\, M_2)N_2\rangle \in R_\tau$ and $\langle Q_1,\ Q_2\rangle \in [\![\tau]\!]$ whenever $(\lambda x\colon \sigma.\, M_1)N_1 \xrightarrow{+}_\beta Q_1$ and $(\lambda x\colon \sigma.\, M_2)N_2 \xrightarrow{+}_\beta Q_2$, by (R3), we have $\langle(\lambda x\colon \sigma.\, M_1)N_1,\ (\lambda x\colon \sigma.\, M_2)N_2\rangle \in [\![\tau]\!]$. ◀

▶ **Lemma 10.** *Given a family of relations $\mathcal{R}$ satisfying conditions (P0)-(P5), for every pair $\langle M_1,\ M_2\rangle$ of type σ, for every pair of substitutions φ_1 and φ_2 such that $\langle\varphi_1(y),\ \varphi_2(y)\rangle \in [\![\gamma]\!]$ for every $y\colon \gamma \in FV(M_1) \cup FV(M_2)$, if $\vdash_\beta M_1 \doteq M_2\colon \sigma$, then $\langle M_1[\varphi_1],\ M_2[\varphi_2]\rangle \in [\![\sigma]\!]$.*

Proof. First, we prove the lemma, but in the case where $M_1 \doteq M_2\colon \sigma$ is provable in the proof system of Definition 2 **without using** the axioms (β) or (η). We proceed by induction on the proof of $M_1 = M_2$.

$$x \doteq x\colon \sigma \quad (\textit{reflexivity})$$

Obvious, since by assumption, $\langle\varphi_1(x),\ \varphi_2(x)\rangle \in [\![\sigma]\!]$.

$$\frac{M_1 \doteq M_2\colon \sigma}{M_2 \doteq M_1\colon \sigma} \quad (\textit{symmetry})$$

By the induction hypothesis, $\langle M_1[\varphi_1],\ M_2[\varphi_2]\rangle \in [\![\sigma]\!]$. Since by Lemma 7 (R0), every $[\![\gamma]\!]$ is symmetric, we also have $\langle M_2[\varphi_2],\ M_1[\varphi_1]\rangle \in [\![\sigma]\!]$.

$$\frac{M_1 \doteq M_2\colon \sigma \quad M_2 \doteq M_3\colon \sigma}{M_1 \doteq M_3\colon \sigma} \quad (\textit{transitivity})$$

By the induction hypothesis, $\langle M_1[\varphi_1],\ M_2[\varphi_2]\rangle \in [\![\sigma]\!]$ and $\langle M_2[\varphi_2],\ M_3[\varphi_3]\rangle \in [\![\sigma]\!]$. Since by Lemma 7 (R0), every $[\![\gamma]\!]$ is transitive, we also have $\langle M_1[\varphi_1],\ M_3[\varphi_3]\rangle \in [\![\sigma]\!]$.

$$\frac{M_1 \doteq M_2\colon (\sigma \to \tau) \quad N_1 \doteq N_2\colon \sigma}{(M_1N_1) \doteq (M_2N_2)\colon \tau} \quad (\textit{congruence})$$

By the induction hypothesis, $\langle M_1[\varphi_1],\ M_2[\varphi_2]\rangle \in [\![\sigma \to \tau]\!]$ and $\langle N_1[\varphi_1],\ N_2[\varphi_2]\rangle \in [\![\sigma]\!]$. By the definition of $[\![\sigma \to \tau]\!]$, we get $\langle M_1[\varphi_1]N_1[\varphi_1],\ M_2[\varphi_2]N_2[\varphi_2]\rangle \in [\![\tau]\!]$, which shows that

$$\langle(M_1N_1)[\varphi_1],\ (M_2N_2)[\varphi_2]\rangle \in [\![\tau]\!],$$

since $M_1[\varphi_1]N_1[\varphi_1] = (M_1N_1)[\varphi_1]$ and $M_2[\varphi_2]N_2[\varphi_2] = (M_2N_2)[\varphi_2]$.

$$\frac{M_1 \doteq M_2\colon \tau}{\lambda x\colon \sigma.\, M_1 \doteq \lambda x\colon \sigma.\, M_2\colon (\sigma \to \tau)} \quad (\xi)$$

Consider any $\langle N_1,\ N_2\rangle \in [\![\sigma]\!]$, and any substitutions φ_1 and φ_2 such that $\langle\varphi_1(y), \varphi_2(y)\rangle \in [\![\gamma]\!]$ for every $y\colon \gamma \in (FV(M_1) \cup FV(M_2) - \{x\})$. Thus, the substitutions $\varphi_1[x := N_1]$ and $\varphi_2[x := N_2]$ have the property that $\langle\varphi_1(y),\ \varphi_2(y)\rangle \in [\![\gamma]\!]$ for every $y\colon \gamma \in FV(M_1) \cup FV(M_2)$. By suitable α-conversion, we can assume that x does not occur in any $\varphi_1(y)$ or $\varphi_2(y)$ for every $y \in \mathit{dom}(\varphi_1) \cup \mathit{dom}(\varphi_2)$, that N_1 is substitutable for x in M_1, and that N_2 is substitutable for x in M_2. Then, $M_1[\varphi_1[x := N_1]] = M_1[\varphi_1][N_1/x]$ and $M_2[\varphi_2[x := N_2]] = M_2[\varphi_2][N_2/x]$. By the induction hypothesis applied to $\langle M_1,\ M_2\rangle$, $\varphi_1[x := N_1]$, and $\varphi_2[x := N_2]$, we have

$$\langle M_1[\varphi_1[x := N_1]],\ M_2[\varphi_2[x := N_2]]\rangle \in [\![\tau]\!],$$

that is, $\langle M_1[\varphi_1][N_1/x],\ M_2[\varphi_2][N_2/x]\rangle \in [\![\tau]\!]$. Consequently, by Lemma 9,

$$\langle(\lambda x\colon \sigma.\ M_1[\varphi_1]),\ (\lambda x\colon \sigma.\ M_2[\varphi_2])\rangle \in [\![\sigma \to \tau]\!],$$

that is,

$$\langle(\lambda x\colon \sigma.\ M_1)[\varphi_1],\ (\lambda x\colon \sigma.\ M_2)[\varphi_2]\rangle \in [\![\sigma \to \tau]\!],$$

since $(\lambda x\colon \sigma.\ M_1[\varphi_1]) = (\lambda x\colon \sigma.\ M_1)[\varphi_1]$ and $(\lambda x\colon \sigma.\ M_2[\varphi_2]) = (\lambda x\colon \sigma.\ M_2)[\varphi_2]$.

This concludes the proof in the case where $M_1 \doteq M_2\colon \sigma$ is provable in the proof system of Definition 2 **without using** the axioms (β) or (η). We now show that the lemma holds when the axioms (β) are also used.

We noted (just after Definition 2) that the equation $M \doteq M\colon \sigma$ is provable using the (*reflexivity*) axioms and the rules (*congruence*) and (ξ), for every term M. Thus, by the previous proof, we have that $\langle M[\varphi_1],\ M[\varphi_2]\rangle \in [\![\sigma]\!]$ for every term M of type σ. In particular, this holds for the term $(\lambda x\colon \sigma.\ M)N$, and by (R2), we have

$$\langle((\lambda x\colon \sigma.\ M)N)[\varphi_1],\ M[N/x][\varphi_2]\rangle \in [\![\tau]\!].$$

But this shows that the lemma also holds for every axiom (β), concluding the proof. ◀

▶ **Theorem 11.** *If $\mathcal{R}$ is a binary relation on λ-terms satisfying conditions (P0)-(P5) listed below*

(P0) Every relation R_σ is a per, i.e., R_σ is symmetric and transitive;
(P1) $\langle x,\ x\rangle \in R_\sigma$, $\langle c,\ c\rangle \in R_\sigma$, for every variable x and constant c of type σ;
(P2) If $\langle M_1,\ M_2\rangle \in R_\sigma$ and $M_1 \longrightarrow_\beta M_1'$, then $\langle M_1',\ M_2\rangle \in R_\sigma$;
(P3) If M_1 and M_2 are simple, $\langle M_1, M_2\rangle \in R_{\sigma\to\tau}$, $\langle N_1, N_2\rangle \in R_\sigma$, and either $\langle(\lambda x\colon \sigma.\ M_1')N_1,\ M_2N_2\rangle \in R_\tau$ whenever $M_1 \xrightarrow{+}_\beta \lambda x\colon \sigma.\ M_1'$ and M_2 is stubborn, or $\langle(\lambda x\colon \sigma.\ M_1')N_1,\ (\lambda x\colon \sigma.\ M_2')N_2\rangle \in R_\tau$ whenever $M_1 \xrightarrow{+}_\beta \lambda x\colon \sigma.\ M_1'$ and $M_2 \xrightarrow{+}_\beta \lambda x\colon \sigma.\ M_2'$, then $\langle M_1N_1,\ M_2N_2\rangle \in R_\tau$;
(P4) If $\langle M_1,\ M_2\rangle \in R_\tau$, then $\langle \lambda x\colon \sigma.\ M_1,\ \lambda x\colon \sigma.\ M_2\rangle \in R_{\sigma\to\tau}$;
(P5) If $\langle N_1, N_2\rangle \in R_\sigma$ and $\langle M_1[N_1/x], M_2[N_2/x]\rangle \in R_\tau$, then $\langle(\lambda x\colon \sigma.\, M_1)N_1, (\lambda x\colon \sigma.\, M_2)N_2\rangle \in R_\tau$;

then for every provable equation $\vdash_\beta M_1 \doteq M_2\colon \sigma$, we have $\langle M_1,\ M_2\rangle \in \mathcal{R}_\sigma$ (in other words, every equation provable in the equational β-theory of $\lambda^\to$ satisfies the binary predicate defined by $\mathcal{R}$).

Proof. Apply Lemma 10 to every β-provable equation $M_1 \doteq M_2\colon \sigma$ and to the pair of identity substitutions, which is legitimate since $\langle x,\ x\rangle \in [\![\gamma]\!]$ for every variable of type γ (by Lemma 7). Thus, $\langle M_1,\ M_2\rangle \in [\![\sigma]\!]$ for every β-provable equation $M_1 \doteq M_2\colon \sigma$, and thus $\langle M_1,\ M_2\rangle \in \mathcal{R}_\sigma$. ◀

Remark: The proof of Lemma 10 actually shows that each $\mathcal{R}_\sigma$ is reflexive.

As an application of Theorem 19, it is easy to prove strong normalization and the Church-Rosser property for $\longrightarrow_\beta$. To do this consider the relation $\mathcal{R}$ defined as $\langle M_1,\ M_2\rangle \in \mathcal{R}$ iff $M_1 \xleftrightarrow{*}_\beta M_2$, and both M_1 and M_2 reduce to the same unique normal form. Properties (P0)-(P5) are easily verified, using the same techniques as in Gallier [6]. Of course, this is a bit of an overkill for the simply-typed λ-calculus.

We now show how to extend the previous results to the $\beta\eta$-equational theory of $\lambda^\to$.

3 Adding η-Reduction

The rule of η-reduction is an oriented version of axiom (η):

$$\lambda x\colon \sigma.\, (Mx) \longrightarrow_{\eta} M,$$

where $x \notin FV(M)$. We will denote the reduction relation defined by β-reduction and η-reduction as $\longrightarrow_{\beta\eta}$.

The definition of an I-term remains identical to that given in Definition 3, and similarly for stubborn terms. Properties (P0)-(P3) also remain the same, but they are stated with respect to the new reduction relation $\xrightarrow{+}_{\beta\eta}$.

▶ **Definition 12.** *Properties (P0)-(P3) are defined as follows:*

(P0) Every relation R_σ is a per, i.e., R_σ is symmetric and transitive.
(P1) $\langle x,\ x\rangle \in R_\sigma$, $\langle c,\ c\rangle \in R_\sigma$, for every variable x and constant c of type σ.
(P2) If $\langle M_1,\ M_2\rangle \in R_\sigma$ and $M_1 \longrightarrow_{\beta\eta} M_1'$, then $\langle M_1',\ M_2\rangle \in R_\sigma$.
(P3) If M_1 and M_2 are simple, $\langle M_1, M_2\rangle \in R_{\sigma\to\tau}$, $\langle N_1, N_2\rangle \in R_\sigma$, and either $\langle(\lambda x\colon \sigma.\, M_1')N_1, M_2N_2\rangle \in R_\tau$ whenever $M_1 \xrightarrow{+}_{\beta\eta} \lambda x\colon \sigma.\, M_1'$ and M_2 is stubborn, or $\langle(\lambda x\colon \sigma.\, M_1')N_1, (\lambda x\colon \sigma.\, M_2')N_2\rangle \in R_\tau$ whenever $M_1 \xrightarrow{+}_{\beta\eta} \lambda x\colon \sigma.\, M_1'$ and $M_2 \xrightarrow{+}_{\beta\eta} \lambda x\colon \sigma.\, M_2'$, then $\langle M_1N_1, M_2N_2\rangle \in R_\tau$.

From now on, we only consider families of relations $\mathcal{R}$ satisfying conditions (P0)-(P3) of Definition 12. Definition 5 remains the same, except that it uses the new reduction relation $\longrightarrow_{\beta\eta}$.

▶ **Definition 13.** *For any type σ, a nonempty relation $C \subseteq \Lambda_\sigma \times \Lambda_\sigma$ is a $\mathcal{R}$-logical candidate iff it satisfies the following conditions:*

(R0) C is a per.
(R1) $C \subseteq R_\sigma$.
(R2) If $\langle M_1,\ M_2\rangle \in C$ and $M_1 \longrightarrow_{\beta\eta} M_1'$, then $\langle M_1',\ M_2\rangle \in C$.
(R3) If M_1 and M_2 are simple, $\langle M_1,\ M_2\rangle \in R_\sigma$, and either $\langle \lambda x\colon \gamma.\, M_1',\ M_2\rangle \in C$ whenever $M_1 \xrightarrow{+}_{\beta\eta} \lambda x\colon \gamma.\, M_1'$ and M_2 is stubborn, or $\langle \lambda x\colon \gamma.\, M_1',\ \lambda x\colon \gamma.\, M_2'\rangle \in C$ whenever $M_1 \xrightarrow{+}_{\beta\eta} \lambda x\colon \gamma.\, M_1'$ and $M_2 \xrightarrow{+}_{\beta\eta} \lambda x\colon \gamma.\, M_2'$, then $\langle M_1,\ M_2\rangle \in C$.

Definition 6 remains unchanged, but we repeat it for convenience.

▶ **Definition 14.** *The logical relations $[\![\sigma]\!]$ are defined as follows:*

$$\begin{aligned} [\![\sigma]\!] &= R_\sigma, \qquad \sigma \text{ a base type},\\ [\![\sigma \to \tau]\!] &= \{\langle M_1,\ M_2\rangle \mid \langle M_1,\ M_2\rangle \in R_{\sigma\to\tau},\ \text{and for all } N_1,\ N_2,\\ &\qquad \text{if } \langle N_1,\ N_2\rangle \in [\![\sigma]\!] \text{ then } \langle M_1N_1,\ M_2N_2\rangle \in [\![\tau]\!]\}. \end{aligned}$$

Lemma 7 also holds.

▶ **Lemma 15.** *If $\mathcal{R}$ is a family of relations satisfying conditions (P0)-(P3), then each $[\![\sigma]\!]$ is a $\mathcal{R}$-logical candidate that contains all pairs of stubborn terms in R_σ.*

Proof. Careful inspection reveals that the proof of Lemma 7 remains unchanged. This is because, for a simple term M:

If $M \in \Lambda_{\sigma\to\tau}$ and there is a reduction $MN \xrightarrow{+}_{\beta\eta} Q$ where Q is an I-term, we must have $M \xrightarrow{+}_{\beta\eta} \lambda x\colon \sigma.\, M_1$, even w.r.t. the reduction relation $\xrightarrow{+}_{\beta\eta}$. ◀

Properties (P4) and (P5) are unchanged, but we repeat them for convenience.

▶ **Definition 16.** *Properties (P4) and (P5) are defined as follows:*

(P4) *If* $\langle M_1,\ M_2\rangle \in R_\tau$*, then* $\langle \lambda x\colon \sigma.\, M_1,\ \lambda x\colon \sigma.\, M_2\rangle \in R_{\sigma\to\tau}$*.*

(P5) *If* $\langle N_1, N_2\rangle \in R_\sigma$ *and* $\langle M_1[N_1/x], M_2[N_2/x]\rangle \in R_\tau$*, then* $\langle (\lambda x\colon \sigma.\, M_1)N_1, (\lambda x\colon \sigma.\, M_2)N_2\rangle \in R_\tau$*.*

Lemma 9 also extends to $\beta\eta$-reduction.

▶ **Lemma 17.** *If* $\mathcal{R}$ *is a family of relations satisfying conditions (P0)-(P5) and for every* $\langle N_1, N_2\rangle$*,* $(\langle N_1, N_2\rangle \in [\![\sigma]\!]$ *implies* $\langle M_1[N_1/x], M_2[N_2/x]\rangle \in [\![\tau]\!])$*, then* $\langle \lambda x\colon \sigma.\, M_1, \lambda x\colon \sigma.\, M_2\rangle \in [\![\sigma \to \tau]\!]$*.*

Proof. This time, a few changes to the proof of Lemma 9 have to be made to take η-reduction rules into account.

We need to reexamine the case where

$$(\lambda x\colon \sigma.\, M_1)N_1 \xrightarrow{+}_{\beta\eta} Q_1$$

and Q_1 is an I-term (and similarly for $(\lambda x\colon \sigma.\, M_2)N_2 \xrightarrow{+}_{\beta\eta} Q_2$). The reduction is necessarily of the form either

$$(\lambda x\colon \sigma.\, M_1)N_1 \xrightarrow{*}_{\beta\eta} (\lambda x\colon \sigma.\, M_1')N_1' \longrightarrow_{\beta\eta} M_1'[N_1'/x] \xrightarrow{*}_{\beta\eta} Q_1,$$

where $M_1 \xrightarrow{*}_{\beta\eta} M_1'$ and $N_1 \xrightarrow{*}_{\beta\eta} N_1'$, or

$$(\lambda x\colon \sigma.\, M_1)N_1 \xrightarrow{*}_{\beta\eta} (\lambda x\colon \sigma.\, (M_1'x))N_1' \longrightarrow_{\beta\eta} M_1'N_1' \xrightarrow{*}_{\beta\eta} Q_1,$$

where $M_1 \xrightarrow{*}_{\beta\eta} M_1'x$, with $x \notin FV(M_1')$, and $N_1 \xrightarrow{*}_{\beta\eta} N_1'$.

The first case is as in Lemma 9, we have

$$M_1[N_1/x] \xrightarrow{*}_{\beta\eta} M_1'[N_1'/x] \xrightarrow{*}_{\beta\eta} Q_1.$$

In the second case, as $x \notin FV(M_1')$, we have $M_1'N_1' = (M_1'x)[N_1'/x]$. Since $M_1 \xrightarrow{*}_{\beta\eta} M_1'x$ and $N_1 \xrightarrow{*}_{\beta\eta} N_1'$, we have

$$M_1[N_1/x] \xrightarrow{*}_{\beta\eta} (M_1'x)[N_1'/x] = M_1'N_1' \xrightarrow{*}_{\beta\eta} Q_1.$$

Thus, in all cases,

$$M_1[N_1/x] \xrightarrow{*}_{\beta\eta} Q_1 \quad \text{and} \quad M_2[N_2/x] \xrightarrow{*}_{\beta\eta} Q_2,$$

and since $\langle M_1[N_1/x],\ M_2[N_2/x]\rangle \in [\![\tau]\!]$, by (R2) and (R0), we have $\langle Q_1,\ Q_2\rangle \in [\![\tau]\!]$. ◀

Since Lemma 15 and Lemma 17 hold, so does the extension of Lemma 10 to $\beta\eta$-provability.

▶ **Lemma 18.** *Given a family of relations* $\mathcal{R}$ *satisfying conditions (P0)-(P5), for every pair* $\langle M_1,\ M_2\rangle$ *of type* σ*, for every pair of substitutions* φ_1 *and* φ_2 *such that* $\langle \varphi_1(y),\ \varphi_2(y)\rangle \in [\![\gamma]\!]$ *for every* $y\colon \gamma \in FV(M_1) \cup FV(M_2)$*, if* $\vdash_{\beta\eta} M_1 \doteq M_2\colon \sigma$*, then* $\langle M_1[\varphi_1],\ M_2[\varphi_2]\rangle \in [\![\sigma]\!]$*.*

Proof. The proof is similar to that of Lemma 10, but we also need to treat the case of the (η)-axioms. Recall that the proof shows that $\langle M[\varphi_1],\ M[\varphi_2]\rangle \in [\![\sigma]\!]$ for every term M of type σ. In particular, this holds for the term $\lambda x\colon \sigma.\ (Mx)$ where $x \notin FV(M)$. By (R2), we have

$$\langle((\lambda x\colon \sigma.\ (Mx))[\varphi_1],\ M[\varphi_2]\rangle \in [\![\tau]\!].$$

This concludes the proof. ◀

▶ **Theorem 19.** *If $\mathcal{R}$ is a binary relation on λ-terms satisfying conditions (P0)-(P5), then for every provable equation $\vdash_{\beta\eta} M_1 \doteq M_2\colon \sigma$, we have $\langle M_1,\ M_2\rangle \in \mathcal{R}_\sigma$ (in other words, every equation provable in the equational $\beta\eta$-theory of $\lambda^{\rightarrow}$ satisfies the binary predicate defined by $\mathcal{R}$).*

Proof. Apply Lemma 18 to every $\beta\eta$-provable equation $M_1 \doteq M_2\colon \sigma$ and to the pair of identity substitutions, which is legitimate since $\langle x,\ x\rangle \in [\![\gamma]\!]$ for every variable of type γ (by Lemma 15). Thus, $\langle M_1,\ M_2\rangle \in [\![\sigma]\!]$ for every $\beta\eta$-provable equation $M_1 \doteq M_2\colon \sigma$, and thus $\langle M_1,\ M_2\rangle \in \mathcal{R}_\sigma$, ◀

Several variations of Lemma 18 and Theorem 19 are possible. We can use $\beta\eta$-convertibility instead of $\beta\eta$-reduction in Definition 12, Definition 13, and Definition 16. We can drop symmetry from (R0) and (P0), or drop (R0) and (P0) altogether. In these last two cases, we obtain a version of Lemma 18 by suitably restricting provability. Further investigations are required.

As in the case of β-conversion, it is possible to prove strong normalization and the Church-Rosser property for $\longrightarrow_{\beta\eta}$, using Theorem 19. To do this consider the relation $\mathcal{R}$ defined as $\langle M_1,\ M_2\rangle \in \mathcal{R}$ iff $M_1 \overset{*}{\longleftrightarrow}_{\beta\eta} M_2$, and both M_1 and M_2 reduce to the same unique normal form. Properties (P0)-(P5) are easily verified.

Obviously, it would be interesting to find more general conditions than properties (P0)-(P5) for which our theorems still hold. We leave this an an open problem.

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