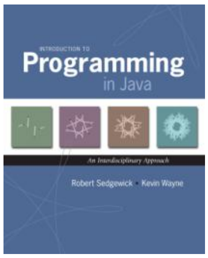
2.3 Recursion



Overview

What is recursion? When one function calls itself directly or indirectly.

Why learn recursion?
- New mode of thinking.
- Powerful programming paradigm.

Many computations are naturally self-referential.
- Mergesort, FFT, gcd, depth-first search.
- Linked data structures.
- A folder contains files and other folders.

Closely related to mathematical induction.

Reproductive Parts
M. C. Escher, 1948

Greatest Common Divisor

Gcd. Find largest integer that evenly divides into p and q.

Ex. gcd(4032, 1272) = 24.

$4032 = 2^6 \times 3^2 \times 7^1$
$1272 = 2^3 \times 3^1 \times 53^1$
$\text{gcd} = 2^3 \times 3^1 = 24$

Applications.
- Simplify fractions: 1272/4032 = 53/168.
- RSA cryptosystem.

Greatest Common Divisor

Gcd. Find largest integer d that evenly divides into p and q.

Euclid's algorithm. [Euclid 300 BCE]

$$\gcd(p,q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \,\%\, q) & \text{otherwise} \end{cases}$$

← base case
← reduction step, converges to base case

```
gcd(4032, 1272) = gcd(1272, 216)
                = gcd(216, 192)
                = gcd(192, 24)
                = gcd(24, 0)
                = 24.
```

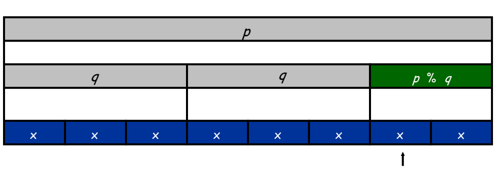
4032 = 3 × 1272 + 216

Greatest Common Divisor

Gcd. Find largest integer d that evenly divides into p and q.

$$\gcd(p,q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \,\%\, q) & \text{otherwise} \end{cases}$$

← base case
← reduction step, converges to base case



p = 8x
q = 3x
gcd(p, q) = x

Greatest Common Divisor

Gcd. Find largest integer d that evenly divides into p and q.

$$\gcd(p,q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \,\%\, q) & \text{otherwise} \end{cases}$$

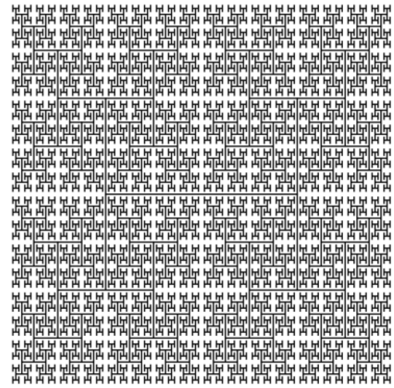
← base case
← reduction step, converges to base case

Java implementation.

```
public static int gcd(int p, int q) {
   if (q == 0) return p;
   else return gcd(q, p % q);
}
```

← base case
← reduction step

Recursive Graphics

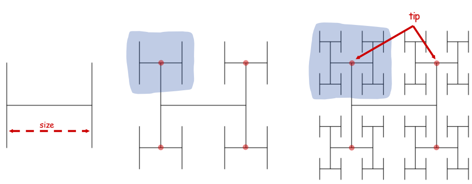


Htree

H-tree of order n.

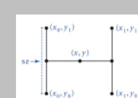
- Draw an H.
- Recursively draw 4 H-trees of order n-1, one connected to each tip.

and half the size



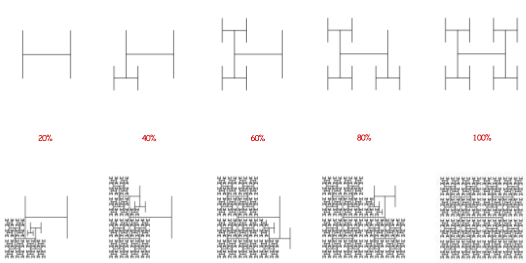
Htree in Java

```java
public class Htree {
   public static void draw(int n, double sz, double x, double y) {
      if (n == 0) return;
      double x0 = x - sz/2, x1 = x + sz/2;
      double y0 = y - sz/2, y1 = y + sz/2;

      StdDraw.line(x0,  y, x1,  y);   // draw the H, centered on(x, y)
      StdDraw.line(x0, y0, x0, y1);
      StdDraw.line(x1, y0, x1, y1);

      draw(n-1, sz/2, x0, y0);        // recursively draw 4 half-size Hs
      draw(n-1, sz/2, x0, y1);
      draw(n-1, sz/2, x1, y0);
      draw(n-1, sz/2, x1, y1);
   }

   public static void main(String[] args) {
      int n = Integer.parseInt(args[0]);
      draw(n, .5, .5, .5);
   }
}
```



Animated H-tree

Animated H-tree. Pause for 1 second after drawing each H.



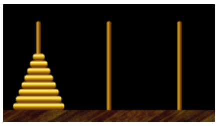
Towers of Hanoi

http://en.wikipedia.org/wiki/Image:Hanoiklein.jpg

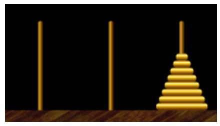
Towers of Hanoi

Move all the discs from the leftmost peg to the rightmost one.

- Only one disc may be moved at a time.
- A disc can be placed either on empty peg or on top of a larger disc.



start



finish

Towers of Hanoi demo

Edouard Lucas (1883)

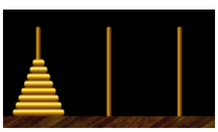
Towers of Hanoi Legend

Q. Is world going to end (according to legend)?

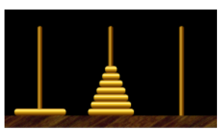
- 64 golden discs on 3 diamond pegs.
- World ends when certain group of monks accomplish task.

Q. Will computer algorithms help?

Towers of Hanoi: Recursive Solution

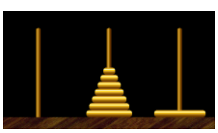


Move n-1 smallest discs right.



Move largest disc left.

cyclic wrap-around



Move n-1 smallest discs right.

Towers of Hanoi: Recursive Solution

```
public class TowersOfHanoi {

   public static void moves(int n, boolean left) {
      if (n == 0) return;
      moves(n-1, !left);
      if (left) System.out.println(n + " left");
      else      System.out.println(n + " right");
      moves(n-1, !left);
   }

   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      moves(N, true);
   }

}
```

`moves(n, true)` : move discs 1 to n one pole to the left
`moves(n, false)`: move discs 1 to n one pole to the right

smallest disc

Towers of Hanoi: Recursive Solution

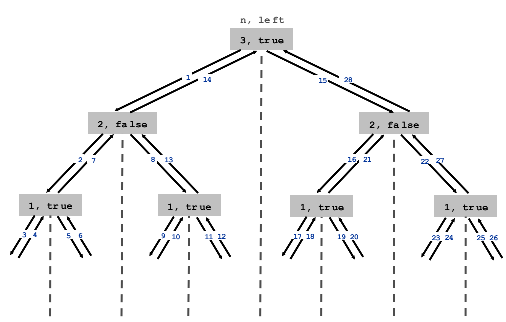
```
% java TowersOfHanoi 3
1 left
2 right
1 left
3 left
1 left
2 right
1 left
```

```
% java TowersOfHanoi 4
1 right
2 left
1 right
3 right
1 right
2 left
1 right
4 left
1 right
2 left
1 right
3 right
1 right
2 left
1 right
```

every other move is smallest disc

subdivisions of ruler

Towers of Hanoi: Recursion Tree



Towers of Hanoi: Properties of Solution

Remarkable properties of recursive solution.
- Takes $2^n - 1$ moves to solve n disc problem.
- Sequence of discs is same as subdivisions of ruler.
- Every other move involves smallest disc.

Recursive algorithm yields non-recursive solution!
- Alternate between two moves:
  - move smallest disc to right if n is even (to left if n is odd)
  - make only legal move not involving smallest disc

Recursive algorithm may reveal fate of world.
- Takes 585 billion years for n = 64 (at rate of 1 disc per second).
- Reassuring fact: any solution takes at least this long!

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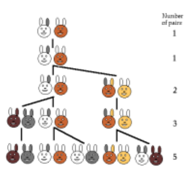
Fibonacci Numbers

Fibonacci Numbers

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{otherwise} \end{cases}$$



Fibonacci rabbits


L. P. Fibonacci
(1170 - 1250)

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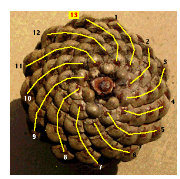
Fibonacci Numbers and Nature

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{otherwise} \end{cases}$$


pinecone


cauliflower

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A Possible Pitfall With Recursion

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{otherwise} \end{cases}$$

A natural for recursion?

```
public static long F(int n) {
   if (n == 0) return 0;
   if (n == 1) return 1;
   return F(n-1) + F(n-2);
}
```

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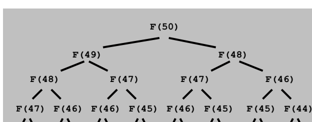
Recursion Challenge 1 (difficult but important)

Q. Is this an efficient way to compute F(50)?

```
public static long F(int n) {
   if (n == 0) return 0;
   if (n == 1) return 1;
   return F(n-1) + F(n-2);
}
```

A. No, no, no! This code is spectacularly inefficient.



recursion tree for naive Fibonacci function

F(50) is called once.
F(49) is called once.
F(48) is called 2 times.
F(47) is called 3 times.
F(46) is called 5 times.
F(45) is called 8 times.
...
F(1) is called 12,586,269,025 times. (F(50))

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Recursion Challenge 2 (easy and also important)

Q. Is this a more efficient way to compute F(50)?

```java
public static long F(int n) {
   if (n == 0) return 0;
   long[] F = new long[n+1];
   F[0] = 0;
   F[1] = 1;
   for (int i = 2; i <= n; i++)
      F[i] = F[i-1] + F[i-2];
   return F[n];
}
```

FYI: classic math

$$F(n) = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} = \left\lfloor \phi^n / \sqrt{5} \right\rceil$$

ϕ = golden ratio ≈ 1.618

A. Yes. This code does it with 50 additions.
Lesson. Don't use recursion to engage in exponential waste.

Context. This is a special case of an important programming technique known as dynamic programming (stay tuned).

Summary

How to write simple recursive programs?
- Base case, reduction step.
- Trace the execution of a recursive program.
- Use pictures.

Why learn recursion?
- New mode of thinking.
- Powerful programming tool.

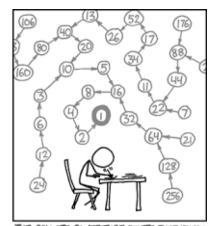
Extra Slides

Collatz Sequence

Collatz sequence.
- If n is 1, stop.
- If n is even, divide by 2.
- If n is odd, multiply by 3 and add 1.

Ex. 35 106 53 160 80 40 20 10 5 16 8 4 2 1.

```java
public static void collatz(int n) {
   System.out.print(n + " ");
   if (n == 1) return;
   if (n % 2 == 0) collatz(n / 2);
   collatz(3*n + 1);
}
```

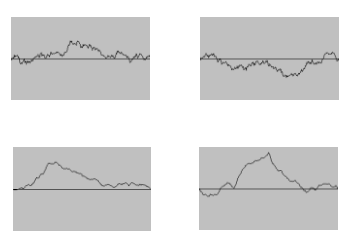


THE COLLATZ CONJECTURE STATES THAT IF YOU PICK A NUMBER, AND IF IT'S EVEN DIVIDE IT BY TWO AND IF IT'S ODD MULTIPLY IT BY THREE AND ADD ONE, AND YOU REPEAT THIS PROCEDURE LONG ENOUGH, EVENTUALLY YOUR FRIENDS WILL STOP CALLING TO SEE IF YOU WANT TO HANG OUT.

Fractional Brownian Motion

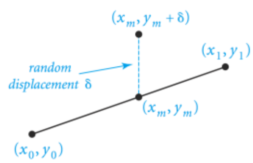
Fractional Brownian Motion

Physical process which models many natural and artificial phenomenon.
- Price of stocks.
- Dispersion of ink flowing in water.
- Rugged shapes of mountains and clouds.
- Fractal landscapes and textures for computer graphics.



Simulating Brownian Motion

Midpoint displacement method.
- Maintain an interval with endpoints (x_0, y_0) and (x_1, y_1).
- Divide the interval in half.
- Choose δ at random from Gaussian distribution.
- Set $x_m = (x_0 + x_1)/2$ and $y_m = (y_0 + y_1)/2 + \delta$.
- Recur on the left and right intervals.



Simulating Brownian Motion: Java Implementation

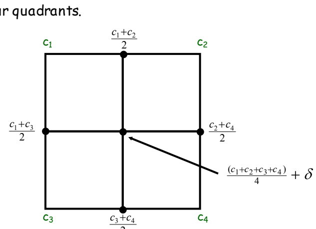
Midpoint displacement method.
- Maintain an interval with endpoints (x_0, y_0) and (x_1, y_1).
- Divide the interval in half.
- Choose δ at random from Gaussian distribution.
- Set $x_m = (x_0 + x_1)/2$ and $y_m = (y_0 + y_1)/2 + \delta$.
- Recur on the left and right intervals.

```
public static void curve(double x0, double y0,
                         double x1, double y1, double var) {
   if (x1 - x0 < 0.01) {
      StdDraw.line(x0, y0, x1, y1);
      return;
   }
   double xm = (x0 + x1) / 2;
   double ym = (y0 + y1) / 2;
   ym += StdRandom.gaussian(0, Math.sqrt(var));
   curve(x0, y0, xm, ym, var/2);
   curve(xm, ym, x1, y1, var/2);
}
```

variance halves at each level;
change factor to get different shapes

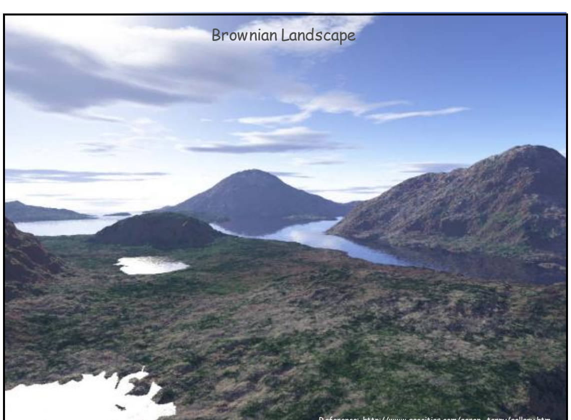
Plasma Cloud

Plasma cloud centered at (x, y) of size s.
- Each corner labeled with some grayscale value.
- Divide square into four quadrants.
- The grayscale of each new corner is the average of others.
  - center: average of the four corners + random displacement
  - others: average of two original corners
- Recur on the four quadrants.



Plasma Cloud

Brownian Landscape



Brown

Robert Brown (1773-1858)