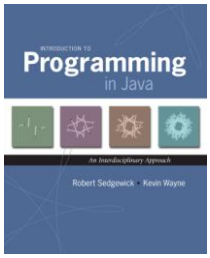
# 2.3 Recursion






## Overview

What is recursion? When one function calls itself directly or indirectly.

Why learn recursion?
- New mode of thinking.
- Powerful programming paradigm.

Many computations are naturally self-referential.
- Mergesort, FFT, gcd, depth-first search.
- Linked data structures.
- A folder contains files and other folders.

Closely related to mathematical induction.

Reproductive Parts
M. C. Escher, 1948



## Greatest Common Divisor

Gcd. Find largest integer that evenly divides into p and q.

Ex. gcd(4032, 1272) = 24.

$4032 = 2^6 \times 3^2 \times 7^1$
$1272 = 2^3 \times 3^1 \times 53^1$
$\gcd = 2^3 \times 3^1 = 24$

Applications.
- Simplify fractions: 1272/4032 = 53/168.
- RSA cryptosystem.



## Greatest Common Divisor

Gcd. Find largest integer d that evenly divides into p and q.

Euclid's algorithm. [Euclid 300 BCE]

$$\gcd(p, q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \% q) & \text{otherwise} \end{cases}$$

← base case
← reduction step, converges to base case

```
gcd(4032, 1272)  = gcd(1272, 216)
                 = gcd(216, 192)
                 = gcd(192, 24)
                 = gcd(24, 0)
                 = 24.
```

4032 = 3 × 1272 + 216



## Greatest Common Divisor

Gcd. Find largest integer d that evenly divides into p and q.

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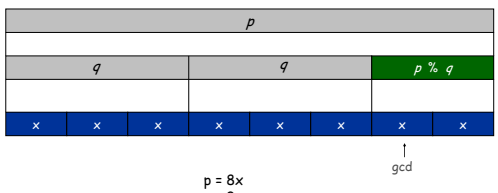



p = 8x
q = 3x
gcd(p, q) = x



## Greatest Common Divisor

Gcd. Find largest integer d that evenly divides into p and q.

$$\gcd(p, q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \% q) & \text{otherwise} \end{cases}$$

← base case
← reduction step, converges to base case

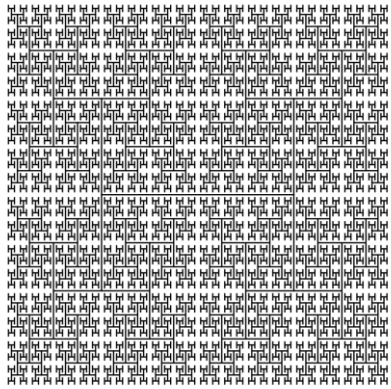
Java implementation.

```
public static int gcd(int p, int q) {
   if (q == 0) return p;
   else return gcd(q, p % q);
}
```

← base case
← reduction step

## Recursive Graphics



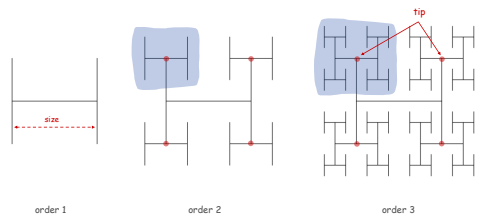


## Htree

H-tree of order n.

- Draw an H.
- Recursively draw 4 H-trees of order n-1, one connected to each tip.

and half the size

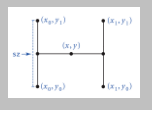



## Htree in Java

```
public class Htree {
   public static void draw(int n, double sz, double x, double y) {
      if (n == 0) return;
      double x0 = x - sz/2, x1 = x + sz/2;
      double y0 = y - sz/2, y1 = y + sz/2;

      StdDraw.line(x0,  y, x1,  y);
      StdDraw.line(x0, y0, x0, y1);
      StdDraw.line(x1, y0, x1, y1);

      draw(n-1, sz/2, x0, y0);
      draw(n-1, sz/2, x0, y1);
      draw(n-1, sz/2, x1, y0);
      draw(n-1, sz/2, x1, y1);
   }

   public static void main(String[] args) {
      int n = Integer.parseInt(args[0]);
      draw(n, .5, .5, .5);
   }
}
```

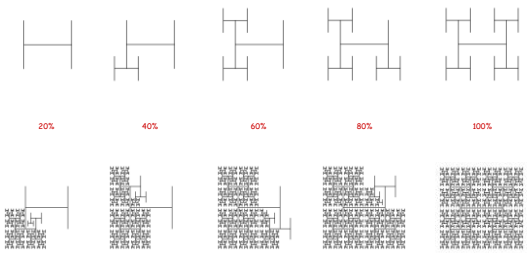
← draw the H, centered on (x, y)

← recursively draw 4 half-size Hs

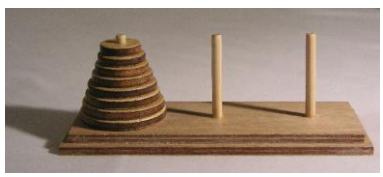



## Animated H-tree

Animated H-tree. Pause for 1 second after drawing each H.




## Towers of Hanoi



http://en.wikipedia.org/wiki/Image:Hanoiklein.jpg

## Towers of Hanoi

Move all the discs from the leftmost peg to the rightmost one.

- Only one disc may be moved at a time.
- A disc can be placed either on empty peg or on top of a larger disc.

start

finish

Towers of Hanoi demo

Edouard Lucas (1883)

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## Towers of Hanoi Legend

Q. Is world going to end (according to legend)?

- 64 golden discs on 3 diamond pegs.
- World ends when certain group of monks accomplish task.

Q. Will computer algorithms help?

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## Towers of Hanoi: Recursive Solution

Move n-1 smallest discs right.

Move largest disc left.

cyclic wrap-around

Move n-1 smallest discs right.

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## Towers of Hanoi: Recursive Solution

```
public class TowersOfHanoi {

   public static void moves(int n, boolean left) {
      if (n == 0) return;
      moves(n-1, !left);
      if (left) System.out.println(n + " left");
      else      System.out.println(n + " right");
      moves(n-1, !left);
   }

   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      moves(N, true);
   }

}
```

moves(n, true) : move discs 1 to n one pole to the left
moves(n, false): move discs 1 to n one pole to the right

smallest disc

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## Towers of Hanoi: Recursive Solution

```
% java TowersOfHanoi 3
1 left
2 right
1 left
3 left
1 left
2 right
1 left
```

```
% java TowersOfHanoi 4
1 right
2 left
1 right
3 right
1 right
2 left
1 right
4 left
1 right
2 left
1 right
3 right
1 right
2 left
1 right
```

every other move is smallest disc

subdivisions of ruler

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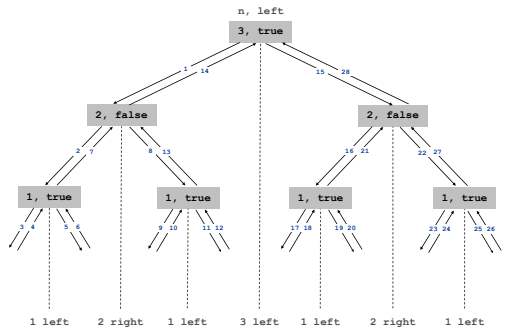
## Towers of Hanoi: Recursion Tree

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## Towers of Hanoi: Properties of Solution

Remarkable properties of recursive solution.

- Takes $2^n - 1$ moves to solve n disc problem.
- Sequence of discs is same as subdivisions of ruler.
- Every other move involves smallest disc.

Recursive algorithm yields non-recursive solution!

- Alternate between two moves:
  - move smallest disc to right if n is even
  - make only legal move not involving smallest disc

to left if n is odd

Recursive algorithm may reveal fate of world.

- Takes 585 billion years for n = 64 (at rate of 1 disc per second).
- Reassuring fact: any solution takes at least this long!

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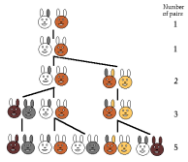
# Fibonacci Numbers

## Fibonacci Numbers

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{otherwise} \end{cases}$$



Fibonacci rabbits


L. P. Fibonacci
(1170 - 1250)

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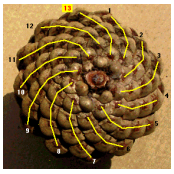
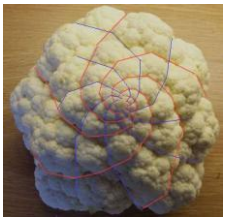
## Fibonacci Numbers and Nature

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{otherwise} \end{cases}$$


pinecone


cauliflower

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## A Possible Pitfall With Recursion

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{otherwise} \end{cases}$$

A natural for recursion?

```
public static long F(int n) {
   if (n == 0) return 0;
   if (n == 1) return 1;
   return F(n-1) + F(n-2);
}
```

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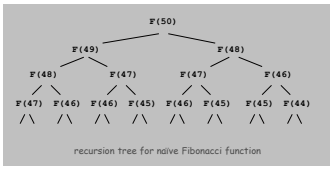
## Recursion Challenge 1 (difficult but important)

Q. Is this an efficient way to compute F(50)?

```
public static long F(int n) {
   if (n == 0) return 0;
   if (n == 1) return 1;
   return F(n-1) + F(n-2);
}
```

A. No, no, no! This code is spectacularly inefficient.



recursion tree for naive Fibonacci function

F(50) is called once.
F(49) is called once.
F(48) is called 2 times.
F(47) is called 3 times.
F(46) is called 5 times.
F(45) is called 8 times.
…
F(1) is called 12,586,269,025 times.

F(50)

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## Recursion Challenge 2 (easy and also important)

Q. Is this a more efficient way to compute F(50)?

```
public static long F(int n) {
   if (n == 0) return 0;
   long[] F = new long[n+1];
   F[0] = 0;
   F[1] = 1;
   for (int i = 2; i <= n; i++)
       F[i] = F[i-1] + F[i-2];
   return F[n];
}
```

FYI: classic math

$$F(n) = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} = \left\lfloor \phi^n / \sqrt{5} \right\rfloor$$

$\phi$ = golden ratio ≈ 1.618

A. Yes. This code does it with 50 additions.
Lesson. Don't use recursion to engage in exponential waste.

Context. This is a special case of an important programming technique known as dynamic programming (stay tuned).

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## Summary

How to write simple recursive programs?
- Base case, reduction step.
- Trace the execution of a recursive program.
- Use pictures.

Why learn recursion?
- New mode of thinking.
- Powerful programming tool.

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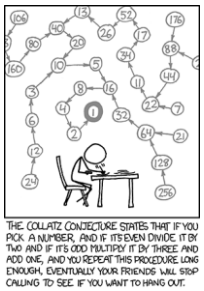
## Extra Slides

## Collatz Sequence

Collatz sequence.
- If n is 1, stop.
- If n is even, divide by 2.
- If n is odd, multiply by 3 and add 1.

Ex. 35 106 53 160 80 40 20 10 5 16 8 4 2 1.

```
public static void collatz(int n) {
  System.out.print(n + " ");
  if (n == 1) return;
  if (n % 2 == 0) collatz(n / 2);
  collatz(3*n + 1);
}
```

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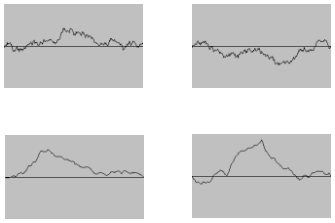
## Fractional Brownian Motion

## Fractional Brownian Motion

Physical process which models many natural and artificial phenomenon.
- Price of stocks.
- Dispersion of ink flowing in water.
- Rugged shapes of mountains and clouds.
- Fractal landscapes and textures for computer graphics.



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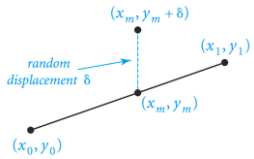
## Simulating Brownian Motion

Midpoint displacement method.

- Maintain an interval with endpoints $(x_0, y_0)$ and $(x_1, y_1)$.
- Divide the interval in half.
- Choose $\delta$ at random from Gaussian distribution.
- Set $x_m = (x_0 + x_1)/2$ and $y_m = (y_0 + y_1)/2 + \delta$.
- Recur on the left and right intervals.






## Simulating Brownian Motion: Java Implementation

Midpoint displacement method.

- Maintain an interval with endpoints $(x_0, y_0)$ and $(x_1, y_1)$.
- Divide the interval in half.
- Choose $\delta$ at random from Gaussian distribution.
- Set $x_m = (x_0 + x_1)/2$ and $y_m = (y_0 + y_1)/2 + \delta$.
- Recur on the left and right intervals.

```
public static void curve(double x0, double y0,
                         double x1, double y1, double var) {
   if (x1 - x0 < 0.01) {
      StdDraw.line(x0, y0, x1, y1);
      return;
   }
   double xm = (x0 + x1) / 2;
   double ym = (y0 + y1) / 2;
   ym += StdRandom.gaussian(0, Math.sqrt(var));
   curve(x0, y0, xm, ym, var/2);
   curve(xm, ym, x1, y1, var/2);
}
```

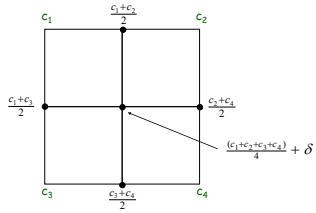
variance halves at each level; change factor to get different shapes



## Plasma Cloud

Plasma cloud centered at (x, y) of size s.

- Each corner labeled with some grayscale value.
- Divide square into four quadrants.
- The grayscale of each new corner is the average of others.
  - center: average of the four corners + random displacement
  - others: average of two original corners
- Recur on the four quadrants.

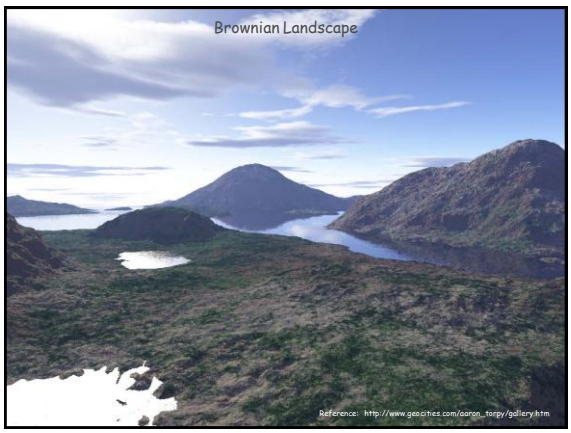





## Plasma Cloud



## Brownian Landscape




## Brown

Robert Brown (1773-1858)