Dynamic Programming

CIS 110, Spring 2012

University of Pennsylvania

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Dynamic programming records saves computation for reuse later.

- Programming: in the optimization sense ("Linear Programming")
- Dynamic: ". . . it's impossible to use [it] in a pejorative way." (Richard Bellman)
- The name was designed to sound cool to RAND management and the US Department of Defense
- A more descriptive term is look-up table

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When is dynamic programming useful?

- Exponential number of solutions
- Cost of soluion is recursively computed
- Different solutions recursively compute same values

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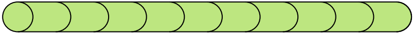
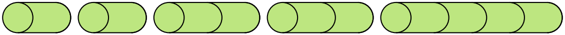
The trick is exposing the common subproblems

Rod Cutting

Start with a rod of integer length n . . .



. . . and cut it into several smaller pieces (of integer length).



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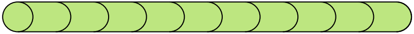

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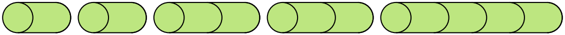
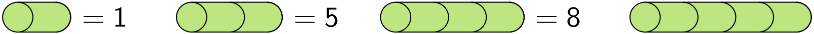
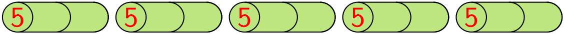


Now suppose each length has a different value:



$= 1$ $= 5$ $= 8$ $= 9$

$= 1 + 1 + 5 = 5 + 9 =$ 21



$= 5 + 5 + 5 = 5 + 5 =$ 25

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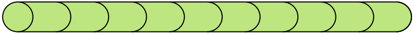

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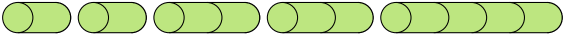

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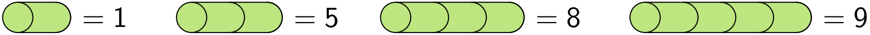

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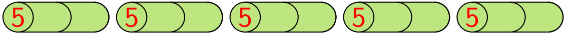

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How should we cut the rod into pieces?

- 2^{n-1} possibilities for a rod of length n

Rod Cutting

First rod-cutting strategy (brute-force):

- For every possible cut, compute the value of the left part plus the value of optimally cutting the right part. Take the best cut.

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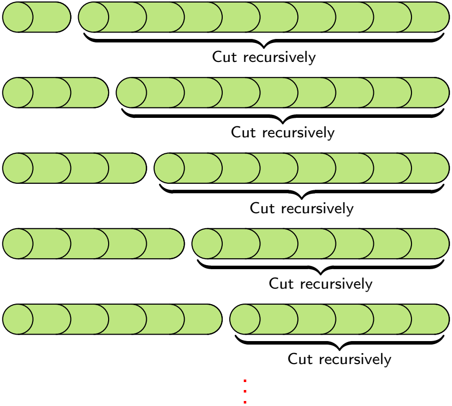

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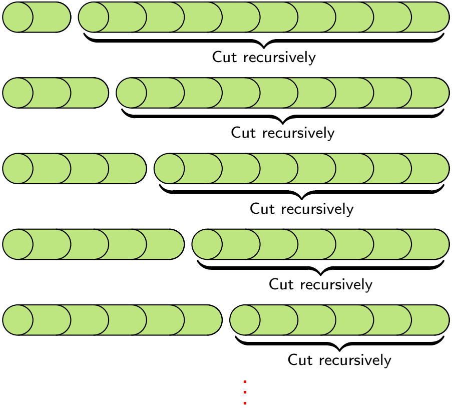

- Exponential number of recursive calls!

Rod Cutting

Second rod-cutting strategy (top-down):

- For every cut, compute value of left part and store it in a table
- Find value of optimal cut for right part in table
  - Compute it recursively if it doesn't exist yet

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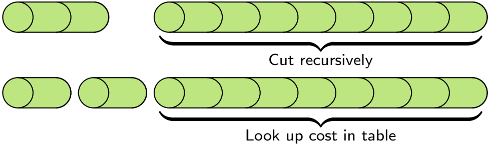

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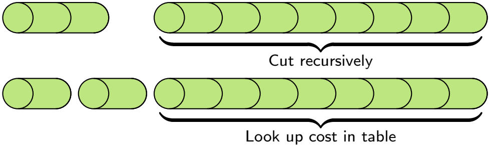


- Reduces computation from $O(2^n)$ to $O(n^2)$ (Why?)
- Requires an array of length n to store intermediate computations

Rod Cutting

Thrid rod-cutting strategy (bottom-up):

- Compute the value of a rod of length 1. Store it.
- Compute the value of a rod of length 2. You can only cut it into rods of length 1. The value of a rod of length 1 is already computed, so there is no recursion.
- Compute the value of successively longer rods up to length n. The optimal values of shorter rods are always computed first so there is no recursion.

Sequence Matching

- Human genes are coded by four bases: Adenine (A), Thymine (T), Guanine (G), Cytosine (C)
- DNA undergoes mutations with each copy:
  - Substitutions: replace one base with another
  - Deletions: some bases are dropped
- Suppose we isolate a gene in a new organism:

  `A A C A G T T A C C`

  Predict function by comparing to genes in know, organism:

  e.g. `T A A G G T C A`
- How similar are `A A C A G T T A C C` and `T A A G G T C A`?

Sequence Matching

How similar are `A A C A G T T A C C` and `T A A G G T C A`?

- How many mutations to change first sequence into second?
- How (un)likely is each mutation

Edit Distance: minimum cost to convert one string into another.

- Each change (mutation) has an associated cost:

Gap	2
Mismatch	1
Match	0

- Example matchings:

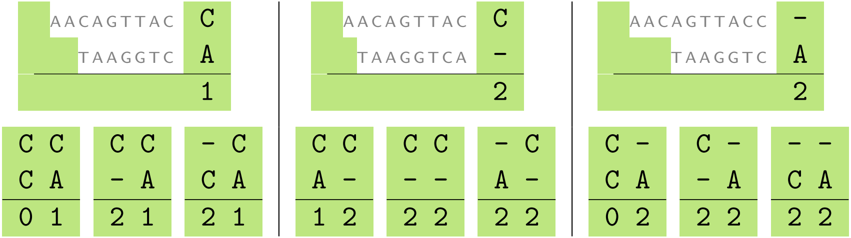
	A	A	C	A	G	T	T	A	C	C
	T	A	A	G	G	T	C	A	-	-
8	1	0	1	1	0	0	1	0	2	2

A	A	C	A	G	T	T	A	C	C	
T	A	-	A	G	G	T	-	C	A	
1	0	2	0	0	1	0	2	0	1	**7**

Edit Distance

Brute-force recursive solution:

- Start at end of sequence and work backwards



- Recurse until we have all possible matches, then find minimum
- Three recursive calls per node $\Rightarrow$ $O(3^n)$ matching cost
- We need to do better!

Edit Distance

Dynamic Programming recursive solution

- Consider a pair of characters in the middle:

```
A A C A G T T A C C
T A A G G T C A
```

- What is the cost of matching from this pair of Gs to the end?
  - Cost of matching `Gs` (0) + *lowest* cost of matching `T T A C C` to `T C A`.
  - Brute force solution computes *all possible* costs
- Idea: For each pair of characters, keep track of best match up to end

Dynamic Programming

Idea: For each pair of characters, keep track of best match to end

	A	A	C	A	G	T	T	A	C	C	-
T											16↓
A											14↓
A											12↓
G											10↓
G											8↓
T											6↓
C											4↓
A											2↓
-	20→	18→	16→	14→	12→	10→	8→	6→	4→	2→	0

Initialization:

- Cost of zero-length match (lower right) is zero
- Inserting a gap (move right or down in table) costs two

Dynamic Programming

Idea: For each pair of characters, keep track of best match to end

	A	A	C	A	G	T	T	A	C	C	-
T											16↓
A											14↓
A											12↓
G											10↓
G											8↓
T											6↓
C											4↓
A										1↘	2↓
-	20→	18→	16→	14→	12→	10→	8→	6→	4→	2→	0

Iteration:

- Work back from lower right
- Cost of cell (i, j) is

$$C(i,j) = \min(C(i+i,j)+2, C(i,j+1)+2, C(i+1,j+1)+\delta$$

where $\delta = 1$ if the i'th character of string A and j'th character of string B are identical.

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	A	A	C	A	G	T	T	A	C	C	-
T											16↓
A											14↓
A											12↓
G											10↓
G											8↓
T											6↓
C											4↓
A									3↘	1↘	2↓
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T											16↓
A											14↓
A											12↓
G											10↓
G											8↓
T											6↓
C											4↓
A								4↘	3↘	1↘	2↓
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T											16↓
A											14↓
A											12↓
G											10↓
G											8↓
T											6↓
C											4↓
A							6→	4↘	3↘	1↘	2↓
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A											14↓
A											12↓
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	A	A	C	A	G	T	T	A	C	C	-
T											$16\downarrow$
A											$14\downarrow$
A											$12\downarrow$
G											$10\downarrow$
G											$8\downarrow$
T											$6\downarrow$
C											$4\downarrow$
A				$12\searrow$	$10\rightarrow$	$8\rightarrow$	$6\rightarrow$	$4\searrow$	$3\searrow$	$1\searrow$	$2\downarrow$
-	$20\rightarrow$	$18\rightarrow$	$16\rightarrow$	$14\rightarrow$	$12\rightarrow$	$10\rightarrow$	$8\rightarrow$	$6\rightarrow$	$4\rightarrow$	$2\rightarrow$	0

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T											16↓
A											14↓
A											12↓
G											10↓
G											8↓
T											6↓
C											4↓
A			14→	12↘	10→	8→	6→	4↘	3↘	1↘	2↓
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	A	A	C	A	G	T	T	A	C	C	-
T	7↘	6↘	6↘	7↓	9↘	8↘	9↘	11↓	13↘	14↓	16↓
A	8↘	6↘	5↘	5↘	7↘	8↘	8↘	9↘	11↘	12↓	14↓
A	10↘	8↘	6→	4↘	5↘	6↘	7↘	7↘	9↘	10↓	12↓
G	12↘	10↘	8↘	6↘	4↘	4↘	5↘	6↘	7↘	8↓	10↓
G	13→	11→	9→	7→	5↘	4↘	3↘	4↘	5↘	6↓	8↓
T	15↘	13↘	11→	9→	7→	5↘	3↘	2↘	3↘	4↓	6↓
C	16→	14→	12↘	11↘	9↘	7↘	5↘	3→	1↘	2↘	4↓
A	18↘	16↘	14→	12↘	10→	8→	6→	4↘	3↘	1↘	2↓
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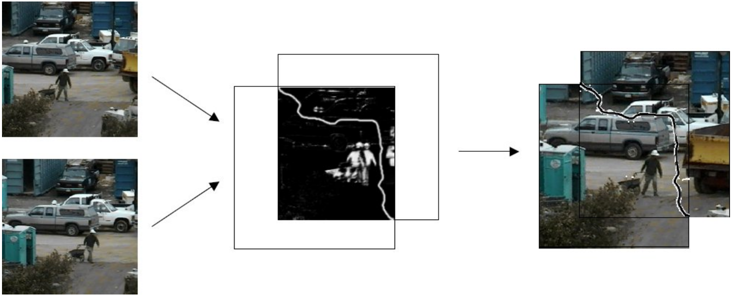
	A	A	C	A	G	T	T	A	C	C	-
T	7↘	6↘	6↘	7↓	9↘	8↘	9↘	11↓	13↘	14↓	16↓
A	8↘	6↘	5↘	5↘	7↘	8↘	8↘	9↘	11↘	12↓	14↓
A	10↘	8↘	6→	4↘	5↘	6↘	7↘	7↘	9↘	10↓	12↓
G	12↘	10↘	8↘	6↘	4↘	4↘	5↘	6↘	7↘	8↓	10↓
G	13→	11→	9→	7→	5↘	4↘	3↘	4↘	5↘	6↓	8↓
T	15↘	13↘	11→	9→	7→	5↘	3↘	2↘	3↘	4↓	6↓
C	16→	14→	12↘	11↘	9↘	7↘	5↘	3→	1↘	2↘	4↓
A	18↘	16↘	14→	12↘	10→	8→	6→	4↘	3↘	1↘	2↓
-	20→	18→	16→	14→	12→	10→	8→	6→	4→	2→	0

Recovering the best alignment:

- Final cost is in cell $(0, 0)$
- Follow arrows to reconstruct string
- $\rightarrow$ aligns letter in the current *column* with a gap
- $\downarrow$ aligns letter in the current *row* with a gap
- $\searrow$ matches letters in current row and column with each other
- Total running time: $O(mn)$!

Some More Examples

- Image Compositing
  - Given a set of overlapping images, what is the best way to stitch them?
  - Cut the images along an "invisible" seam, and splice them together.
  - The optimal seam can be found through dynamic programming.
    - Even better: shortest path



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Some More Examples

- Matrix parenthesization
  - Need to multiply a sequence of rectangular matrices
  - Which matrices should be multiplied first to minimize the number of operations

$$\left(\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right) \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 b_1 + a2 b_2 + a_3 b_3 \end{pmatrix} \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}$$

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- Seam carving (see demo)
  - Shrink an image by finding one row or column of pixels to remove
  - The seam doesn't have to be straight — it can wiggle
  - Use dynamic programming to find the best set of pixels to remove