Programming Languages and Techniques (CIS120)

Lecture 7

Binary Search Trees

(Chapters 7 & 8)

Announcements

- Read Chapters 7 & 8
  - Binary Search Trees
- Check out the background survey on Piazza
  - Help us improve CIS120!
- HW2 due *Tuesday* at midnight
- Midterm 1: Friday, September 27th
  - During lecture time (but different rooms)
  - Announcements about review session, etc., soon

Recap: Ordered Trees

Big idea: find things faster by searching less

Key Insight:

Ordered data can be searched more quickly

- This is why telephone books are arranged alphabetically
- Requires the ability to focus on (roughly) *half* of the current data

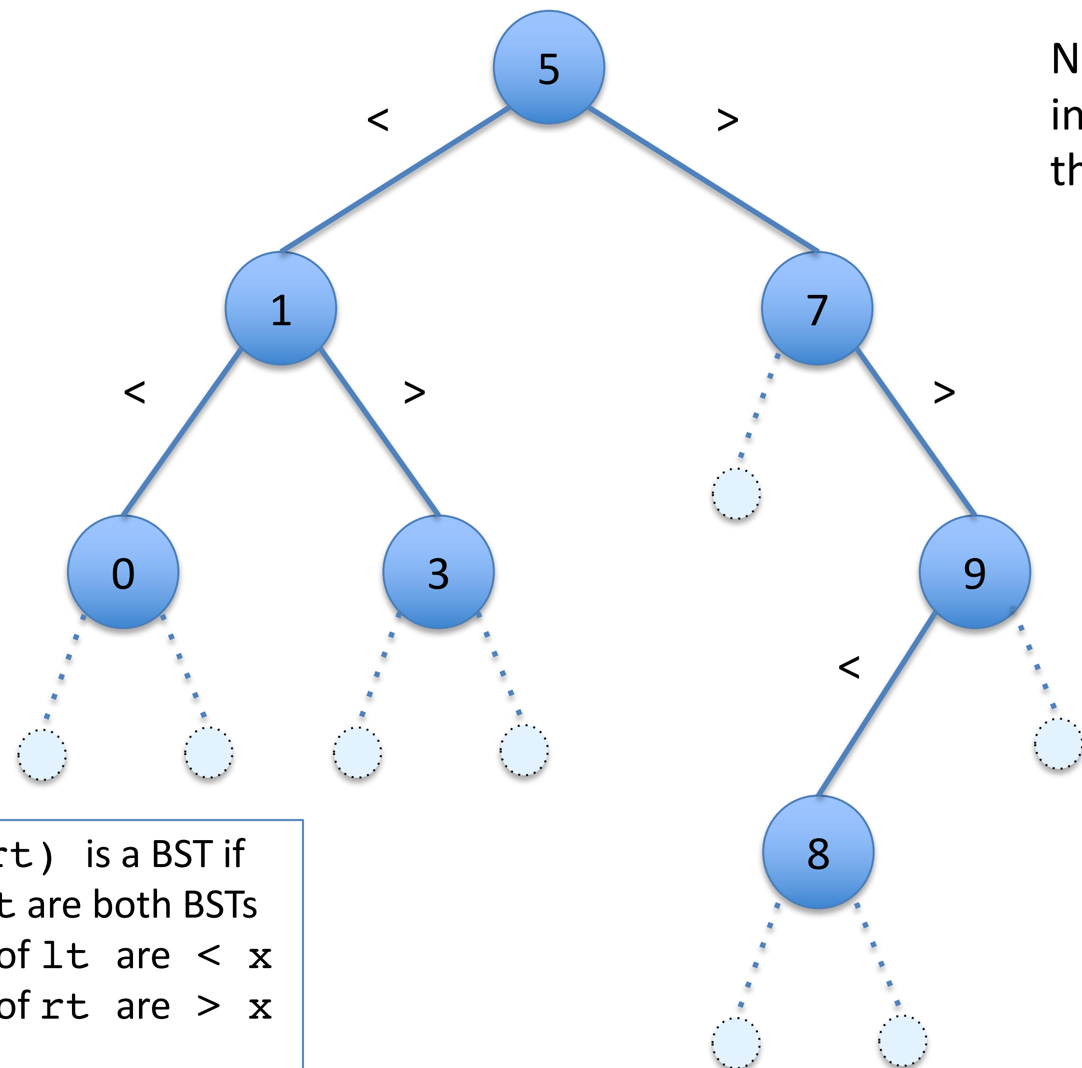
Binary Search Trees

- A *binary search tree* (BST) is a binary tree with some additional *invariants:*

- `Node(lt,x,rt)` is a BST if
  - `lt` and `rt` are both BSTs
  - all nodes of `lt` are `<` `x`
  - all nodes of `rt` are `>` `x`
- `Empty` is a BST

- *The BST invariant means that container functions can take time proportional to the **height** instead of the **size** of the tree.*

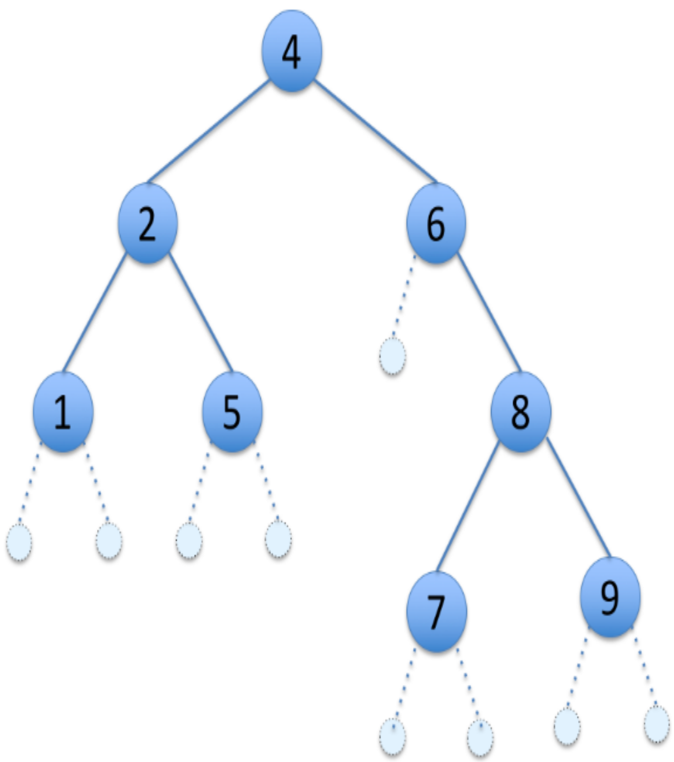
An Example Binary Search Tree



Note that the BST invariants hold for this tree.

- `Node(lt,x,rt)` is a BST if
  - `lt` and `rt` are both BSTs
  - all nodes of `lt` are `<` `x`
  - all nodes of `rt` are `>` `x`
- `Empty` is a BST

Is this a BST?

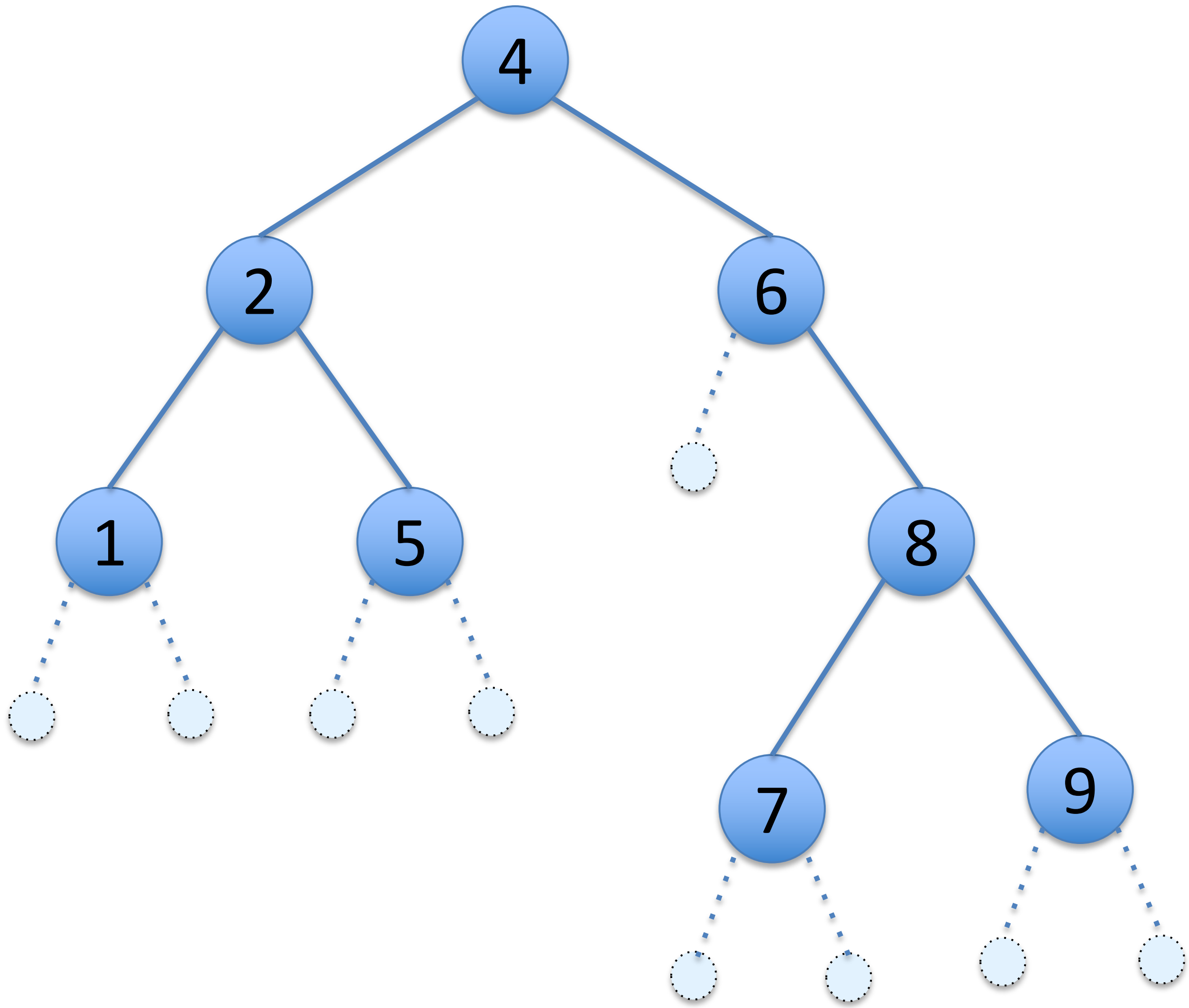


1. No

2. Yes

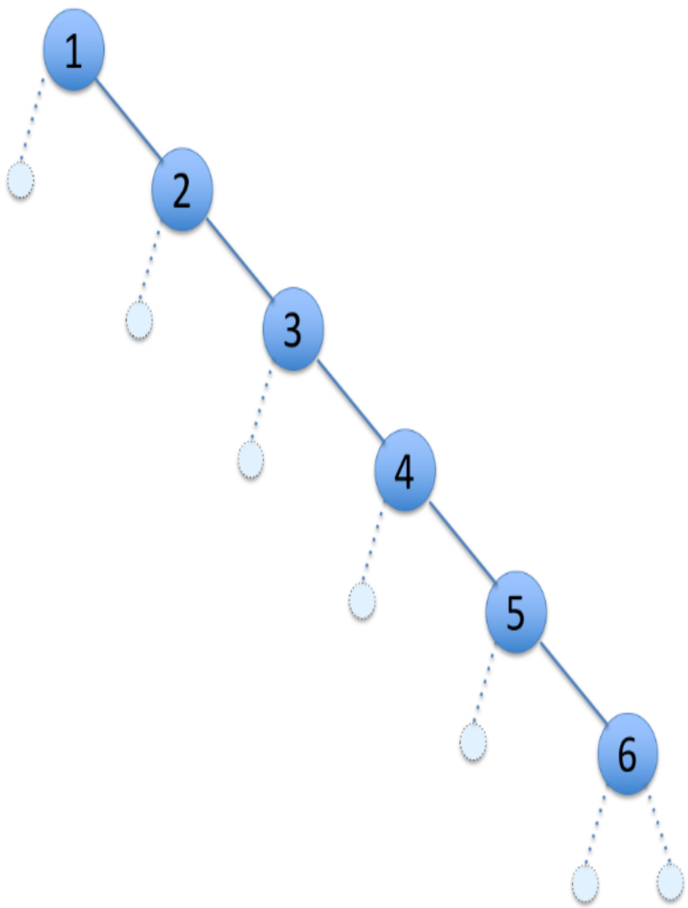
Is this a BST??

1. yes
2. no



Answer: no, 5 to the left of 4

Is this a BST?

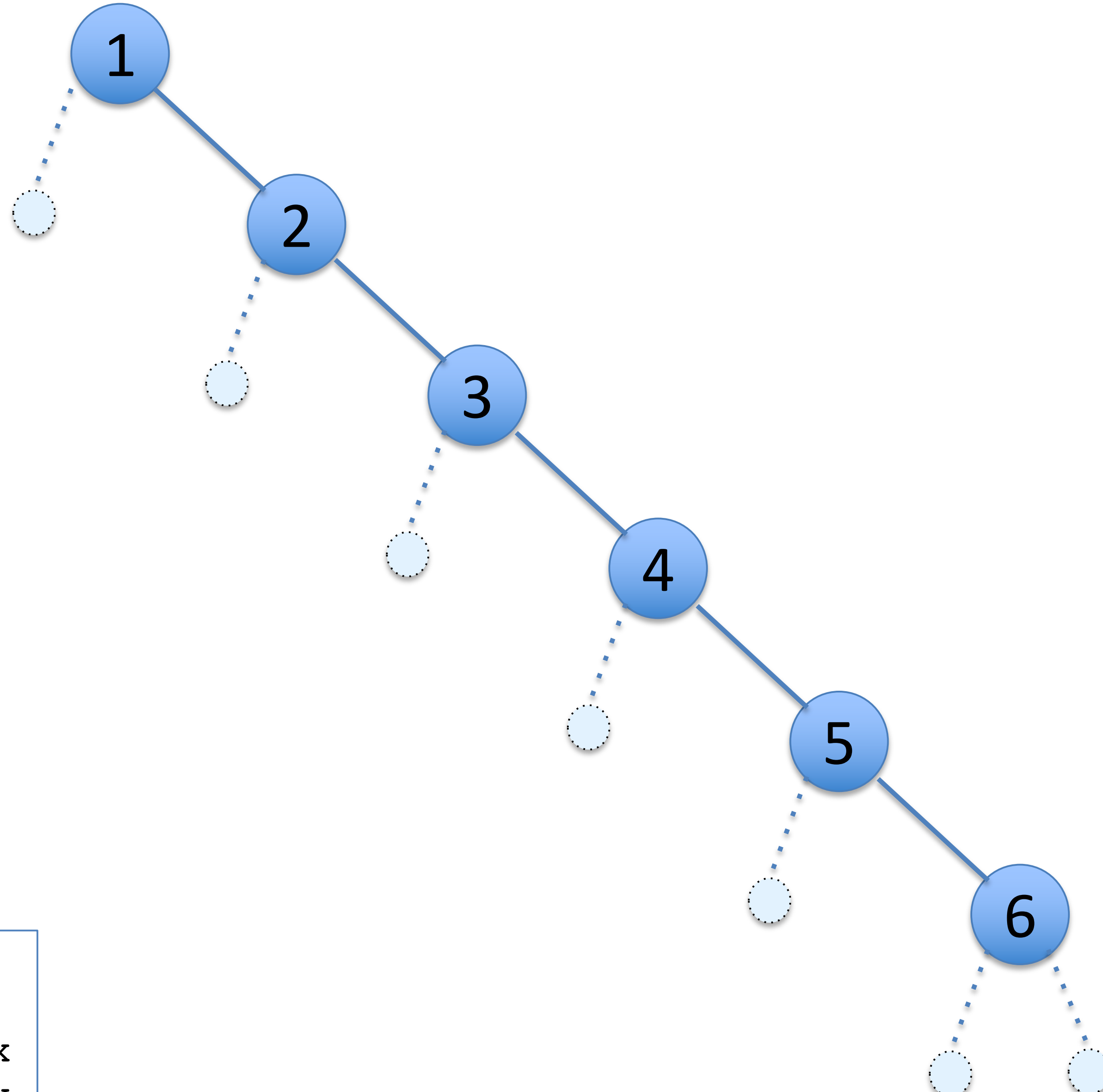


No

Yes

Is this a BST??

1. yes
2. no

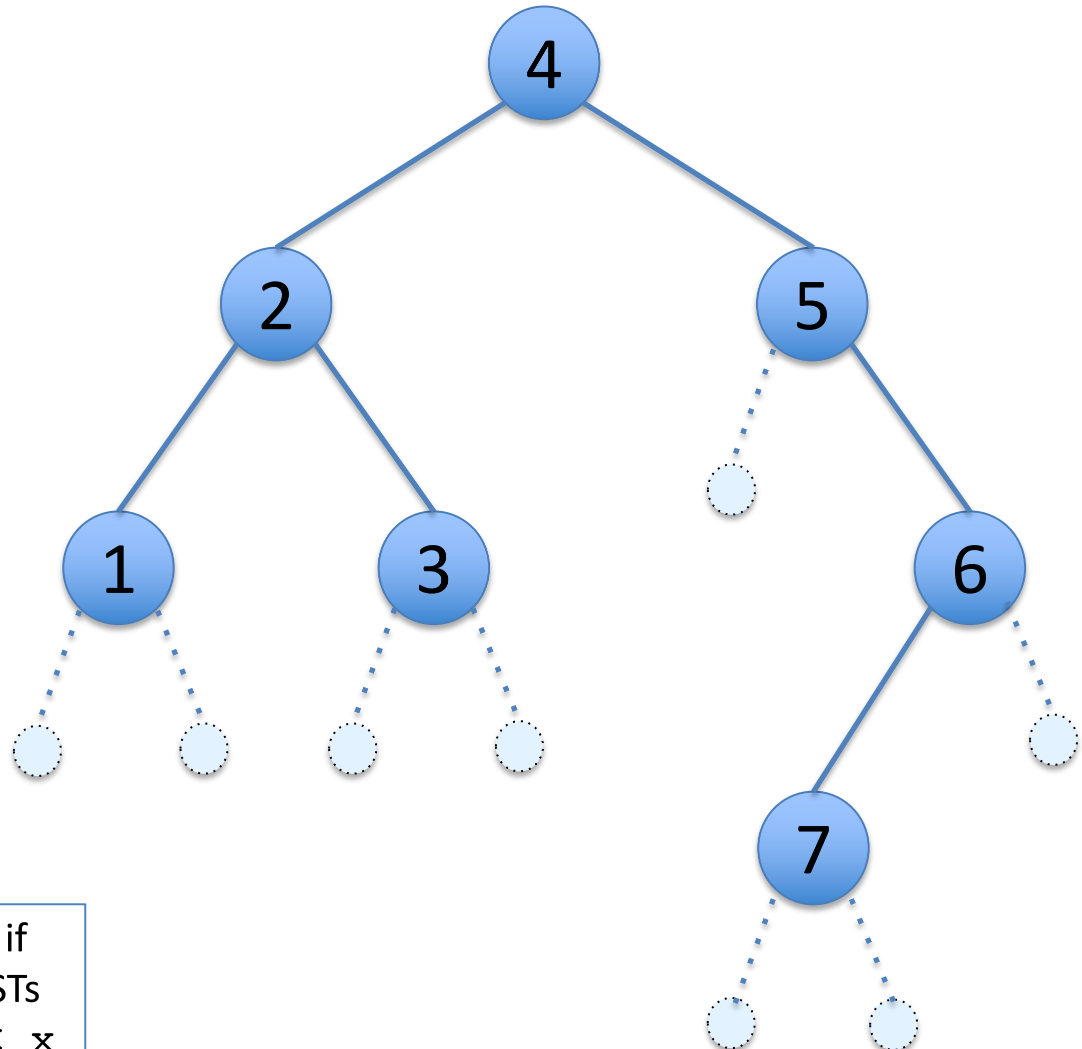


- `Node(lt,x,rt)` is a BST if
  - `lt` and `rt` are both BSTs
  - all nodes of `lt` are `<` `x`
  - all nodes of `rt` are `>` `x`
- `Empty` is a BST

Answer: Yes

Is this a BST??

1. yes
2. no

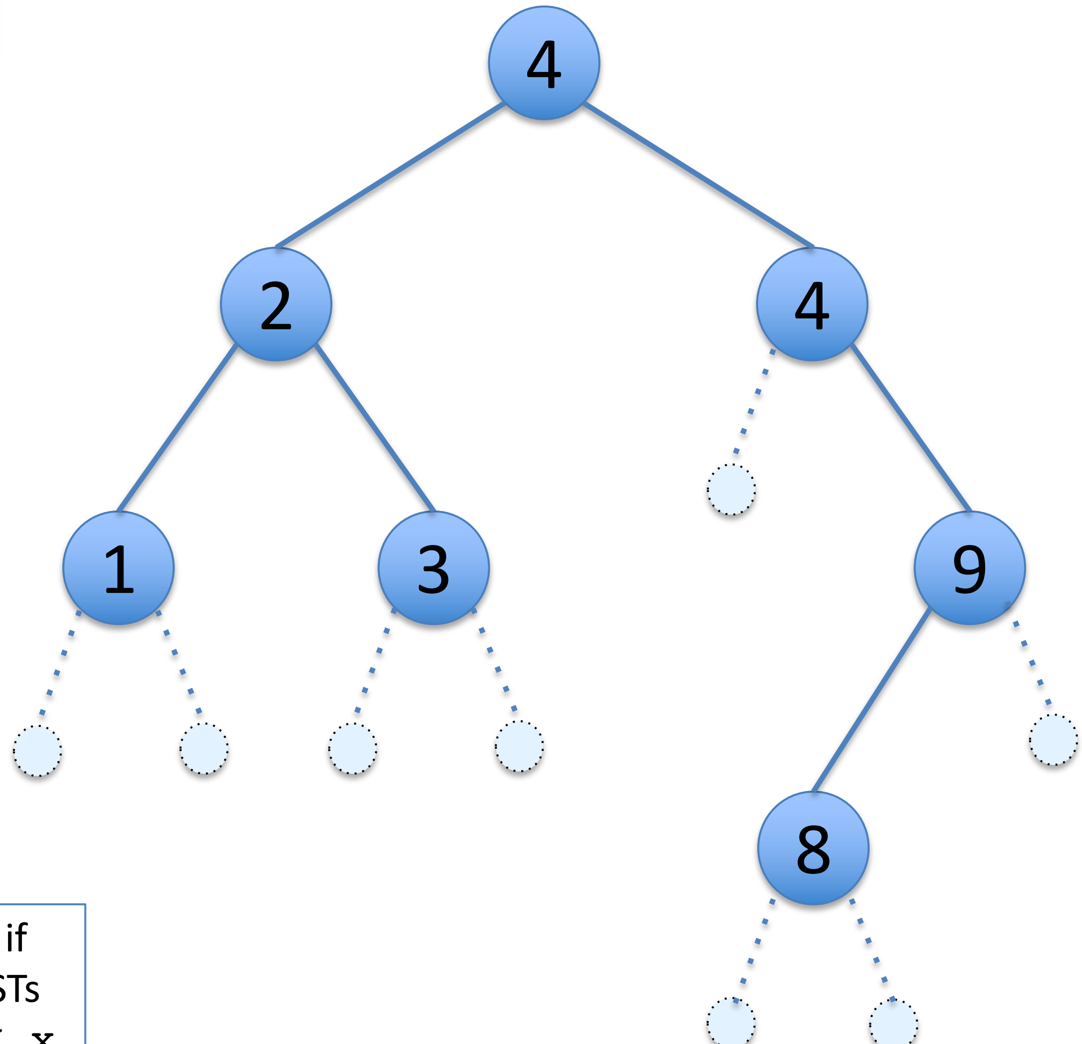


- `Node(lt,x,rt)` is a BST if
  - `lt` and `rt` are both BSTs
  - all nodes of `lt` are `<` `x`
  - all nodes of `rt` are `>` `x`
- `Empty` is a BST

Answer: no, 7 to the left of 6

Is this a BST??

1. yes
2. no

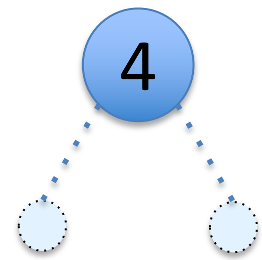


- `Node(lt,x,rt)` is a BST if
  - `lt` and `rt` are both BSTs
  - all nodes of `lt` are `<` `x`
  - all nodes of `rt` are `>` `x`
- `Empty` is a BST

Answer: no, 4 to the right of 4

Is this a BST??

1. yes
2. no

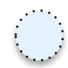


- `Node(lt,x,rt)` is a BST if
  - `lt` and `rt` are both BSTs
  - all nodes of `lt` are `<` `x`
  - all nodes of `rt` are `>` `x`
- `Empty` is a BST

Answer: yes

Is this a BST??

1. yes
2. no



- `Node(lt,x,rt)` is a BST if
  - `lt` and `rt` are both BSTs
  - all nodes of `lt` are `<` `x`
  - all nodes of `rt` are `>` `x`
- `Empty` is a BST

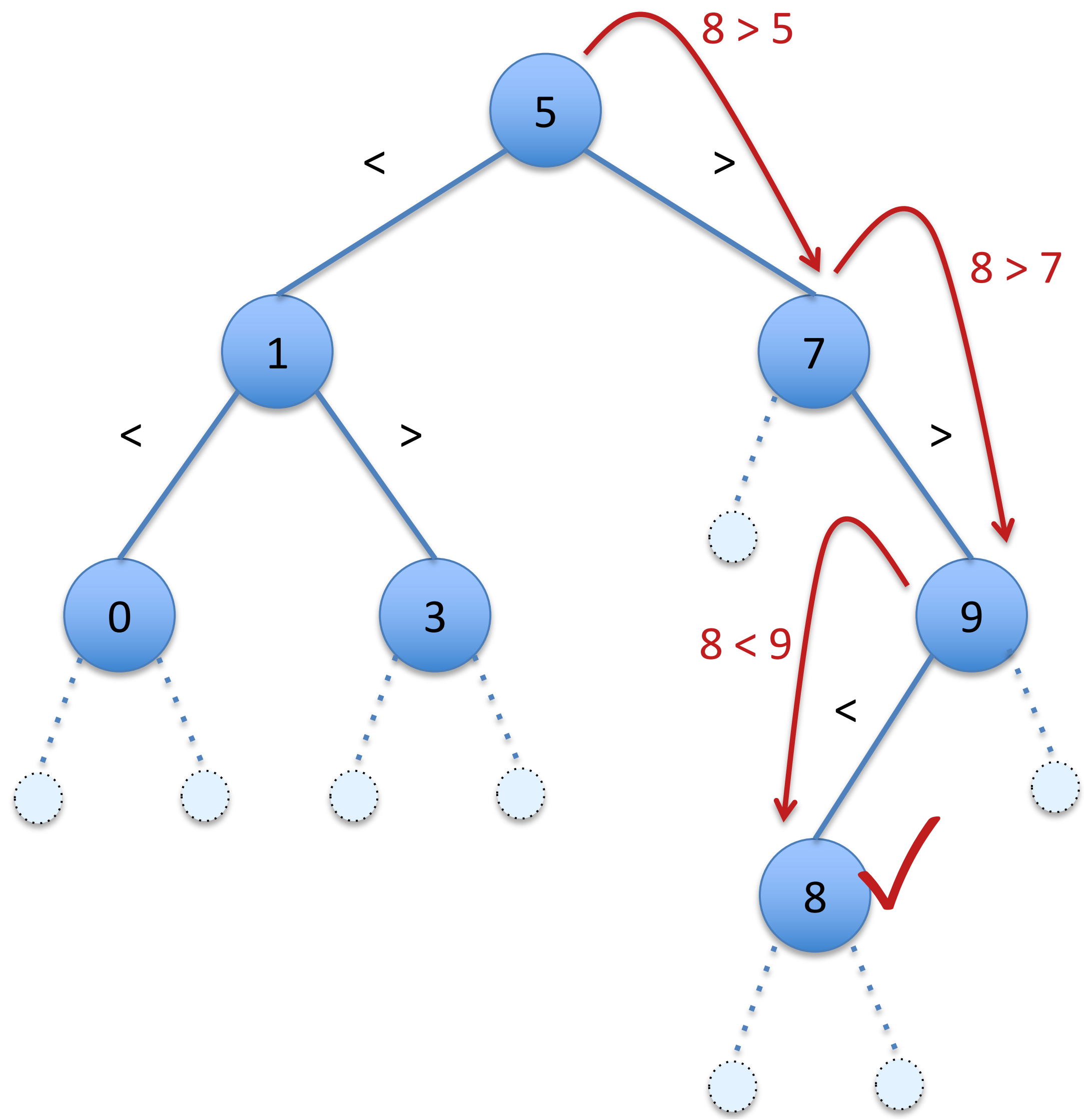
Answer: yes

Searching a BST

```
(* Assumes that t is a BST *)
let rec lookup (t:tree) (n:int) : bool =
  begin match t with
  | Empty -> false
  | Node(lt,x,rt) ->
      if x = n then true
      else if n < x then lookup lt n
      else lookup rt n
  end
```

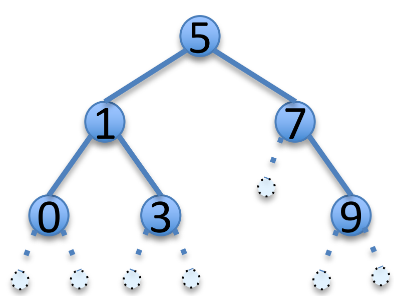
- The BST invariants guide the search.
- Note that lookup may return an incorrect answer if the input is *not* a BST!
  - This function *assumes* that the BST invariants hold of t.

Search in a BST: `(lookup t 8)`

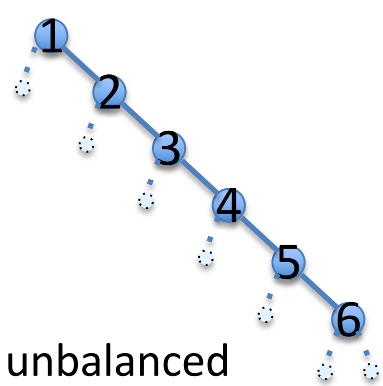


BST Performance

- `lookup` takes time proportional to the *height* of the tree.
  - not the *size* of the tree (as it did with `contains` for unordered trees)
- In a *balanced tree*, the lengths of the paths from the root to each leaf are (almost) *the same*.
  - no leaf is too far from the root
  - the height of the BST is minimized
  - the height of a balanced binary tree is roughly $\log_2(N)$ where N is the number of nodes in the tree



balanced



unbalanced

see bst.ml

UTOP DEMO

Manipulating BSTs

Inserting an element

insert : tree -> int -> tree

"insert t x" returns a new tree containing x
and all of the elements of t

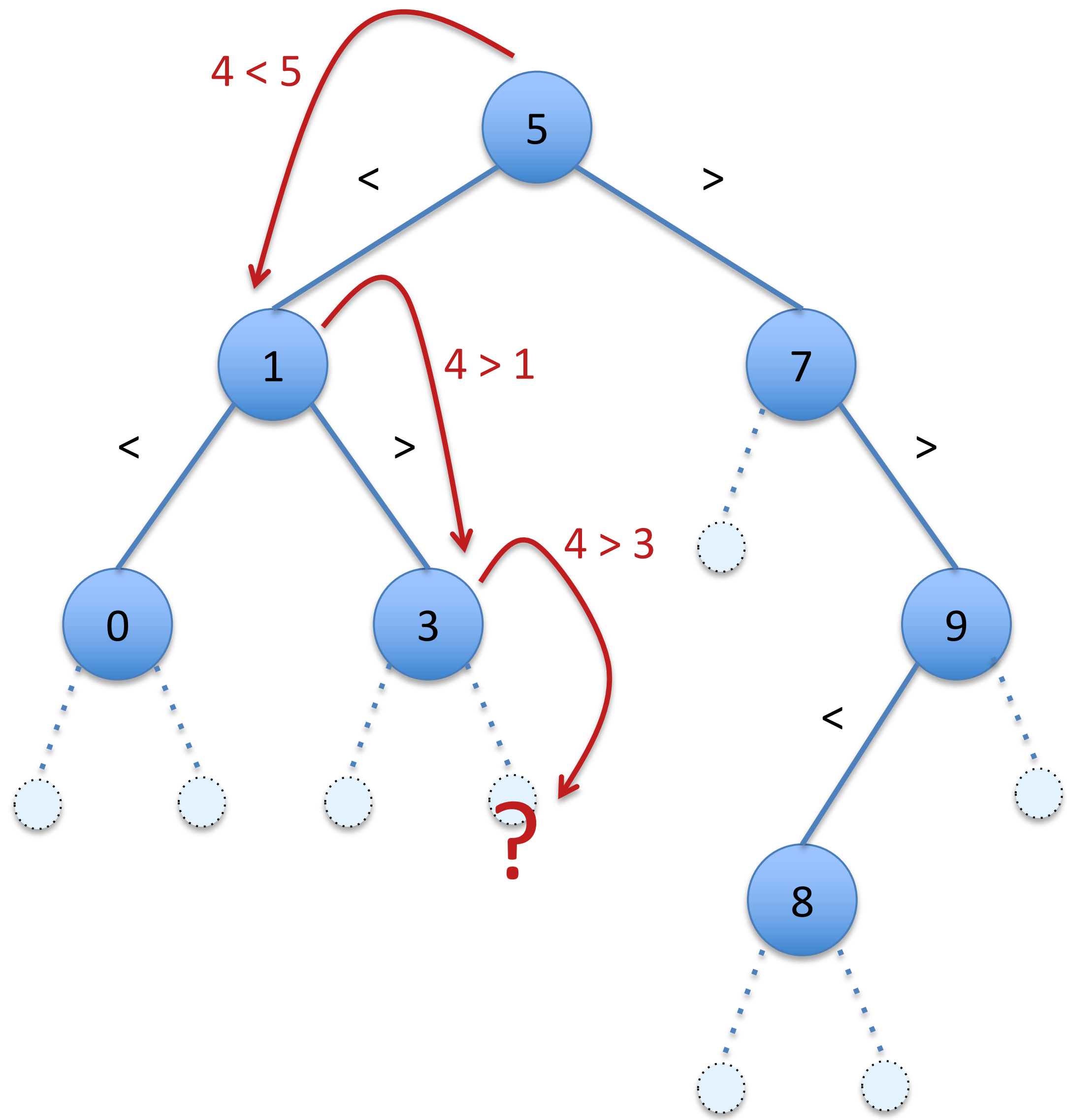
Inserting into a BST

- Challenge: can we make sure that the result of insert really is a BST?
  - i.e., the new element needs to be in the right place!

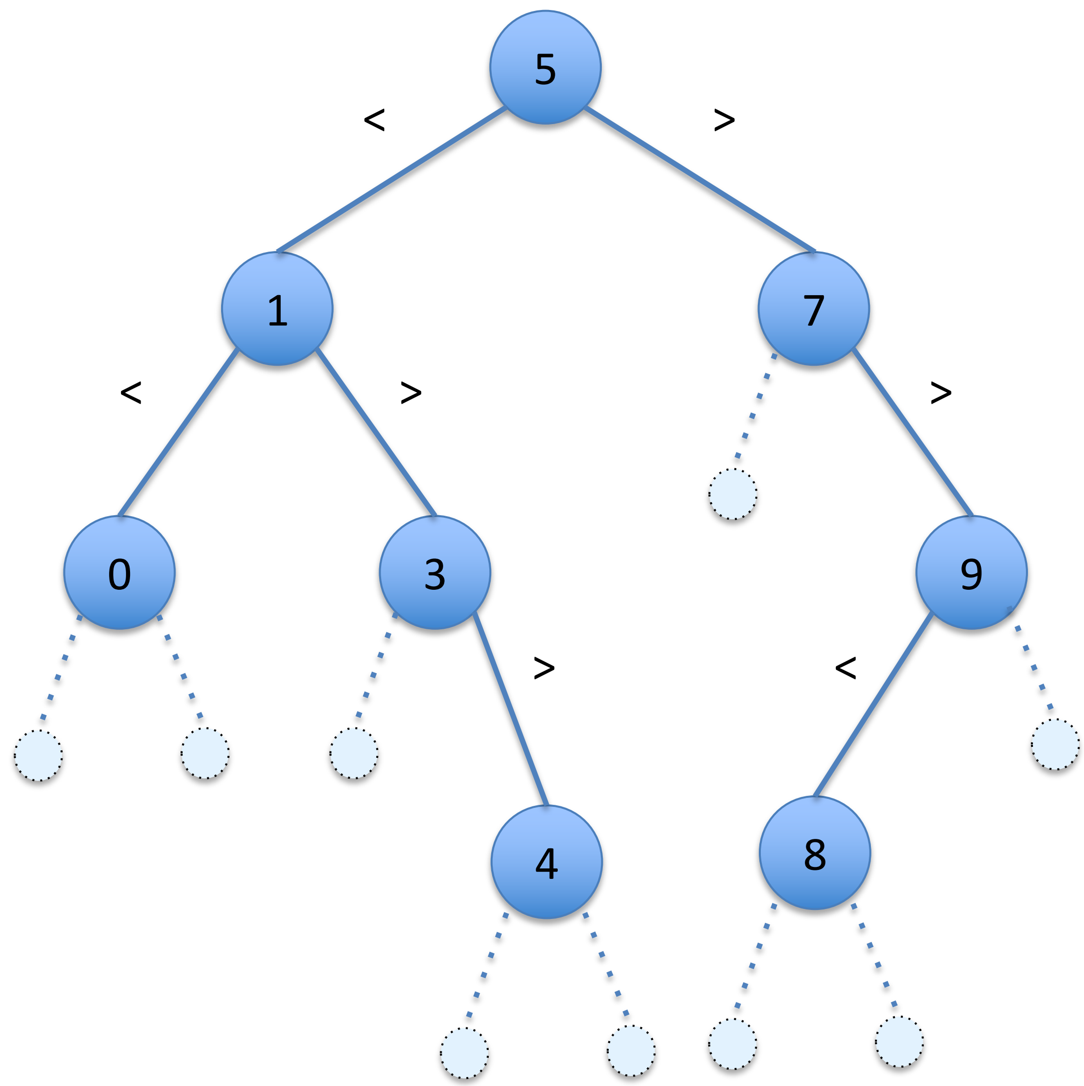
- Payoff: we can build a BST containing any set of elements
  - Starting with `Empty`, apply insert repeatedly
  - If insert *preserves* the BST invariants, then any tree we get from it will be a BST *by construction*
    - No need to check!
  - Later: we can also "rebalance" the tree to make lookup efficient (NOT in CIS 120; see CIS 121)

First step: find the right place…

Inserting a new node: `(insert t 4)`



Inserting a new node: `(insert t 4)`



Inserting Into a BST

```
(* Insert n into the BST t *)
let rec insert (t:tree) (n:int) : tree =
  begin match t with
  | Empty -> Node(Empty,n,Empty)
  | Node(lt,x,rt) ->
     if x = n then t
     else if n < x then Node(insert lt n, x, rt)
     else Node(lt, x, insert rt n)
  end
```

- Note the similarity to searching the tree.
- Assuming that t is a BST, the result is also a BST. (Why?)
- Note that the result is a *new* tree with (possibly) one more Node; the original tree is unchanged

Critical point!

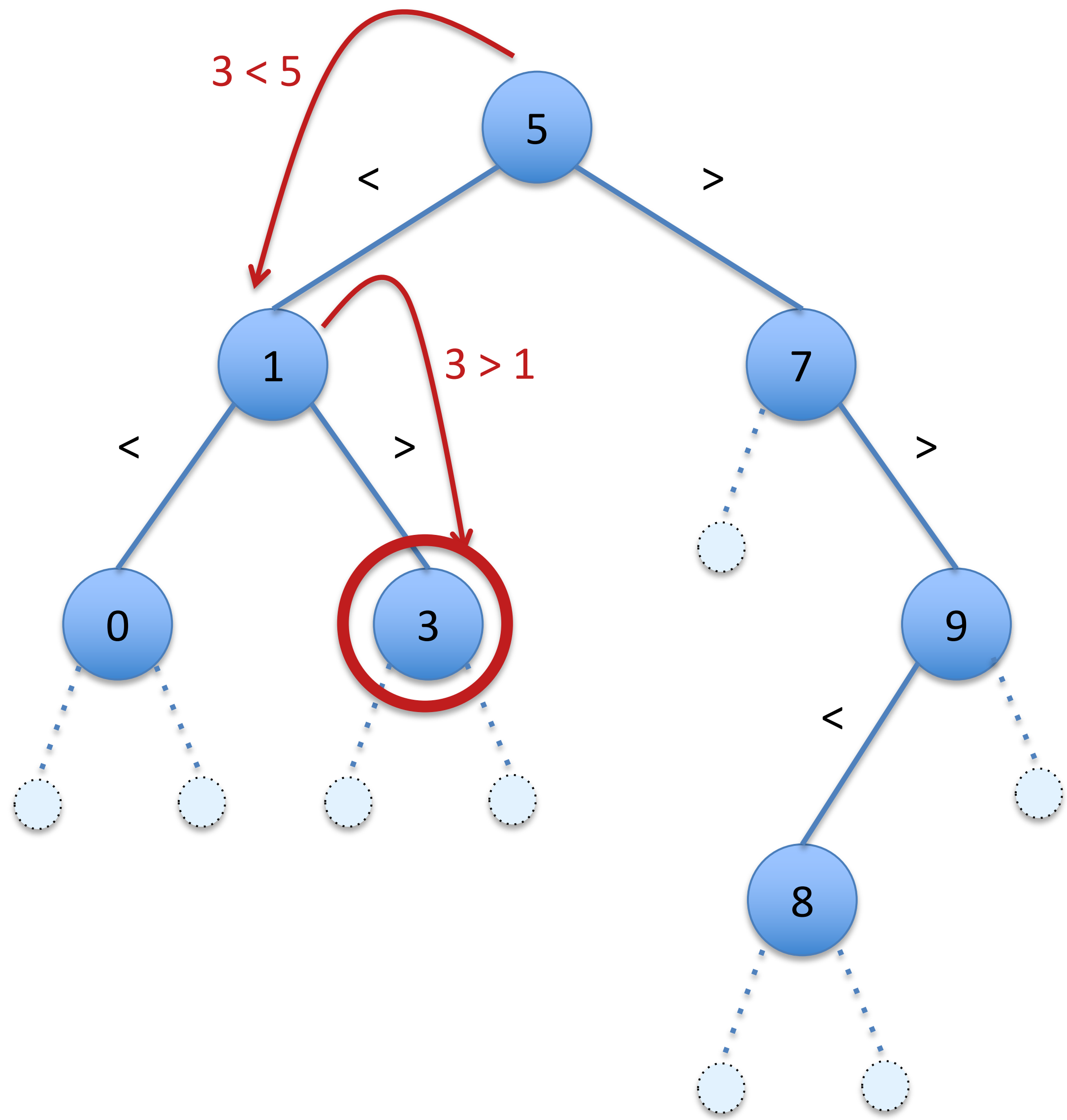
Manipulating BSTs

Deleting an element

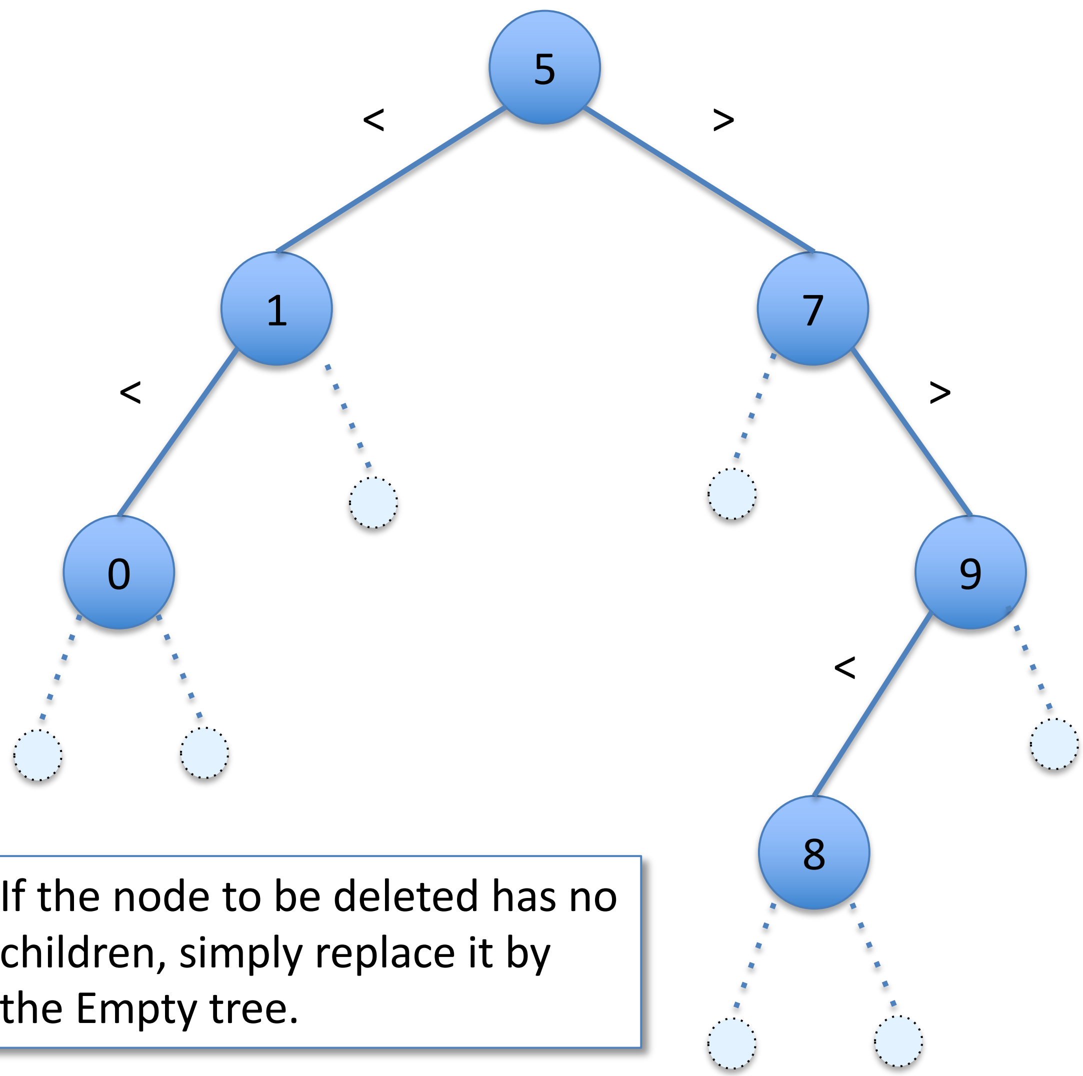
delete : tree -> int -> tree

"delete t x" returns a tree containing
all of the elements of t except for x

Deletion – No Children: `(delete t 3)`

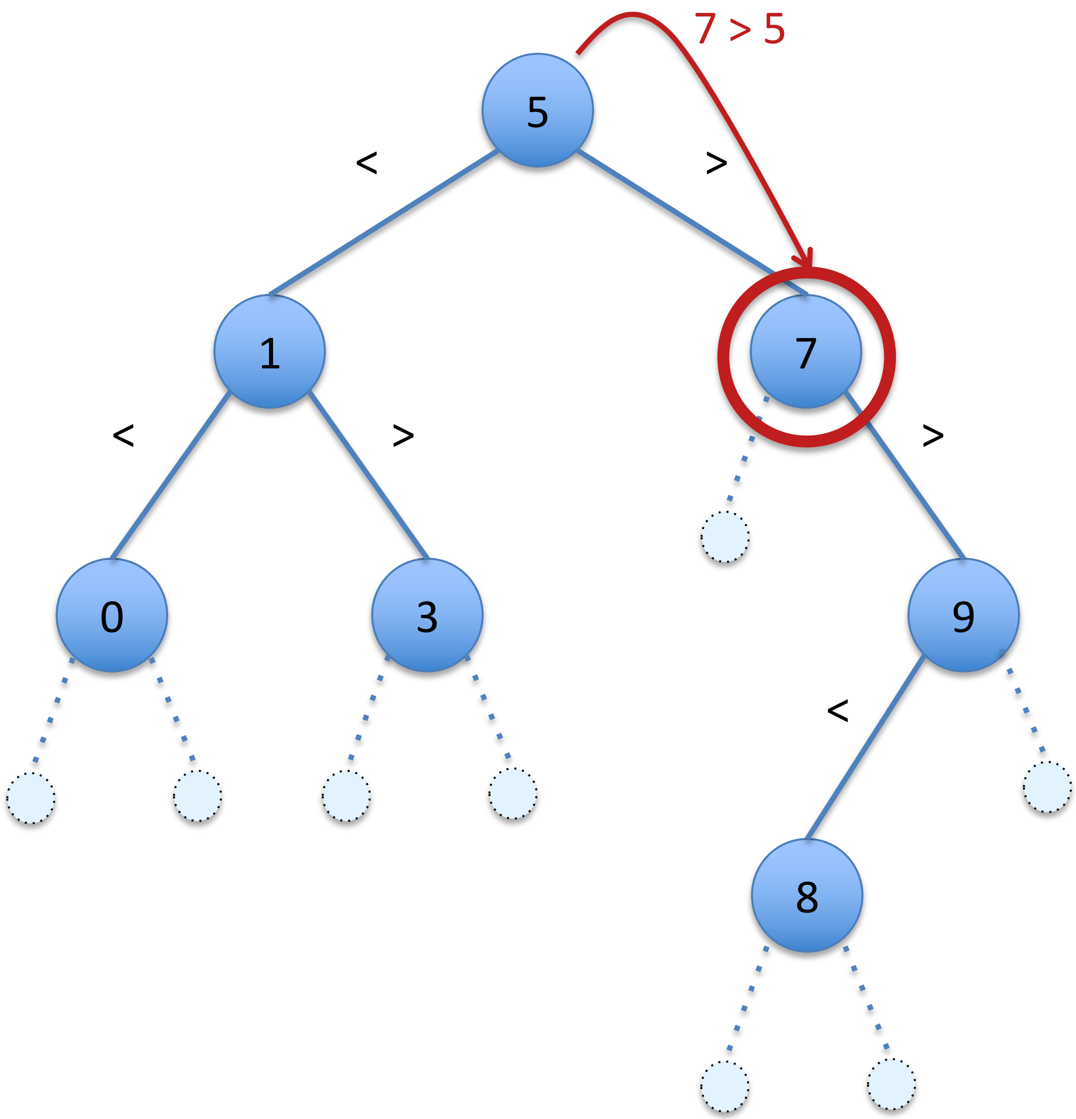


Deletion – No Children: `(delete t 3)`



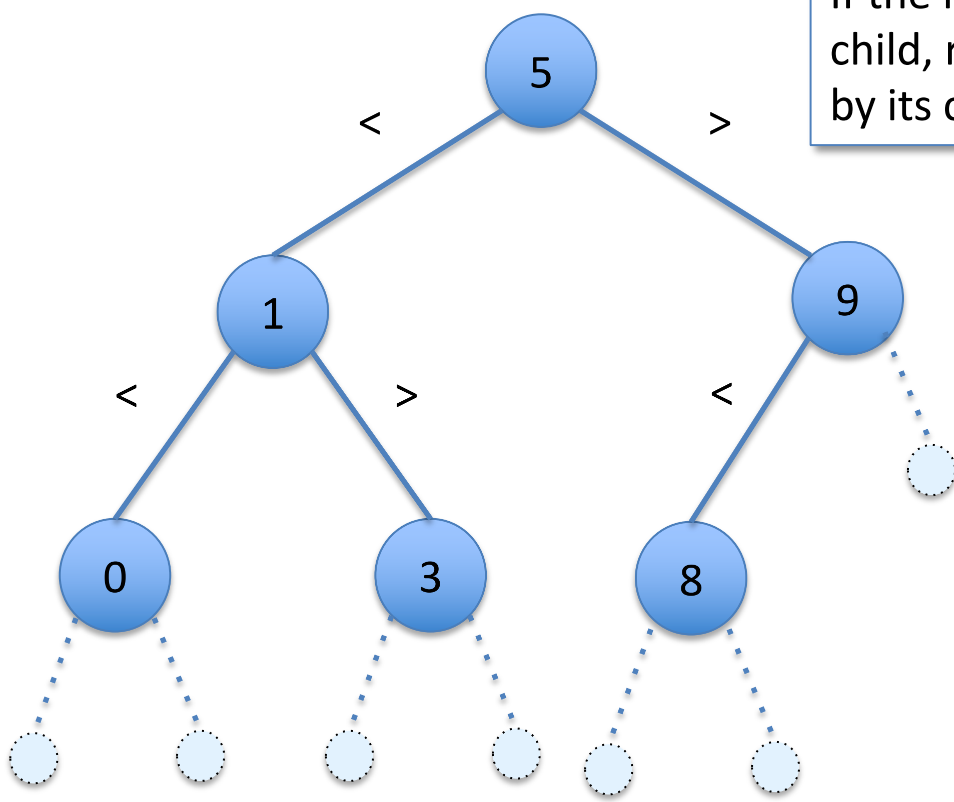
If the node to be deleted has no children, simply replace it by the Empty tree.

Deletion – One Child: `(delete t 7)`

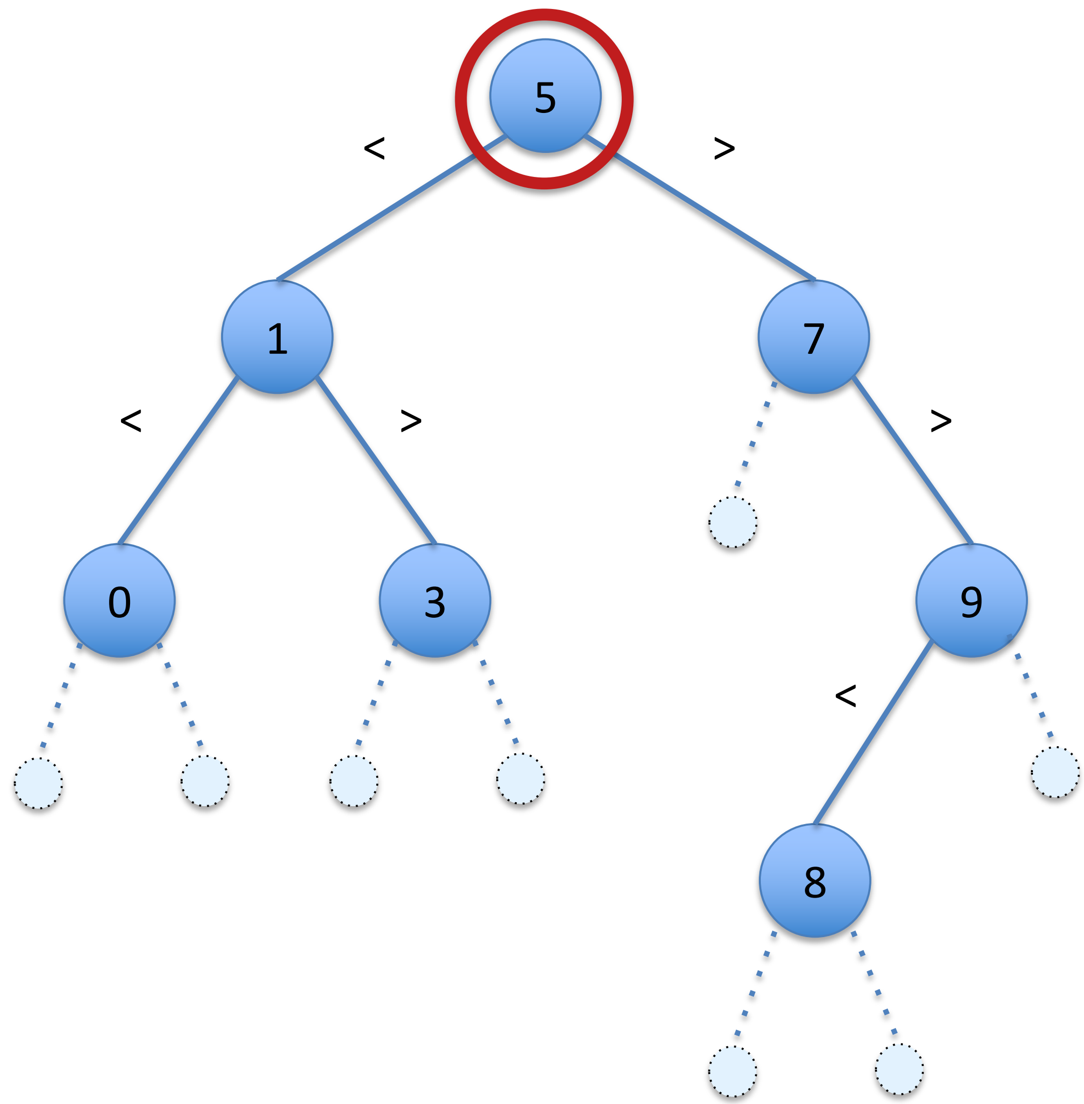


Deletion – One Child: `(delete t 7)`

If the node to be delete has one child, replace the deleted node by its child.

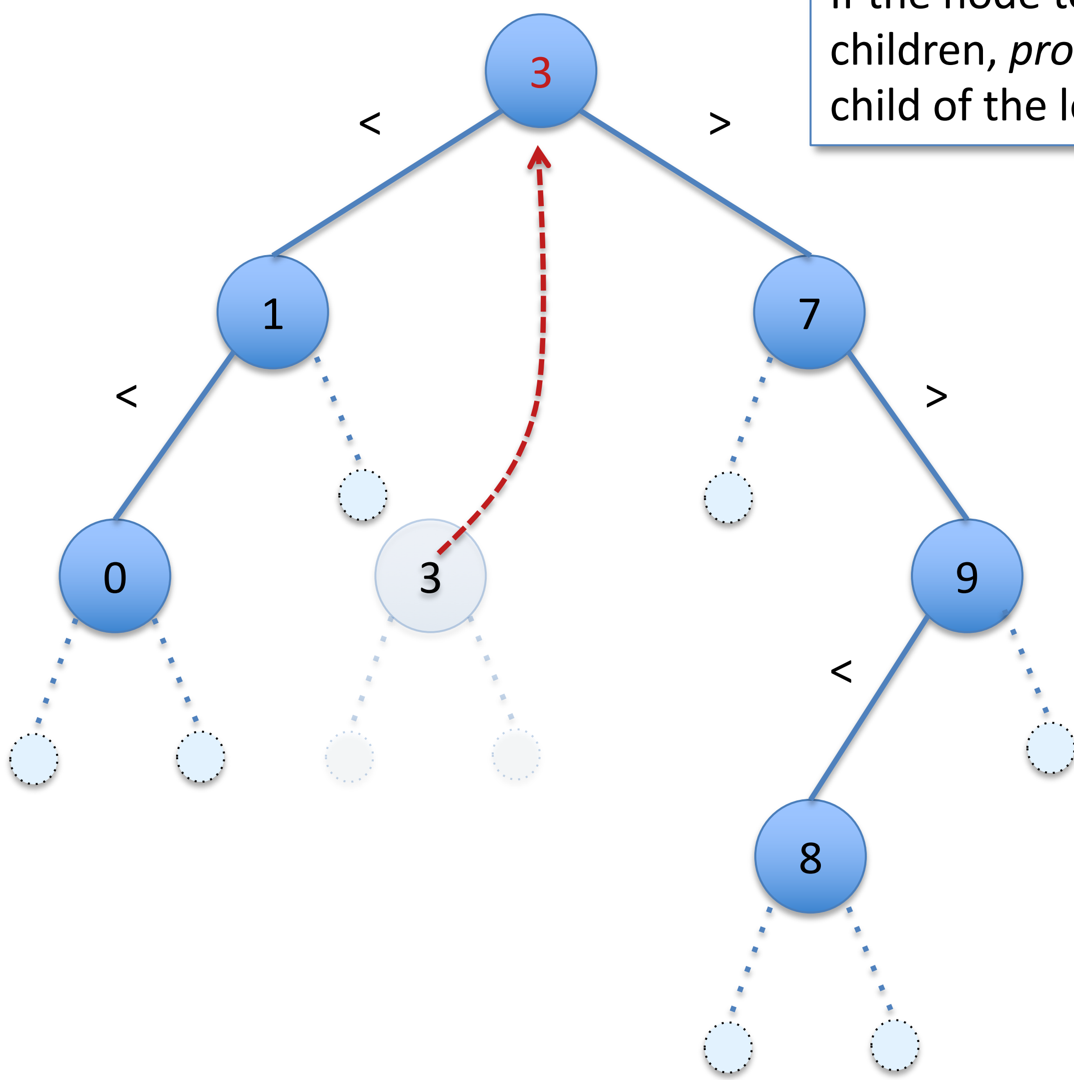


Deletion – Two Children: `(delete t 5)`

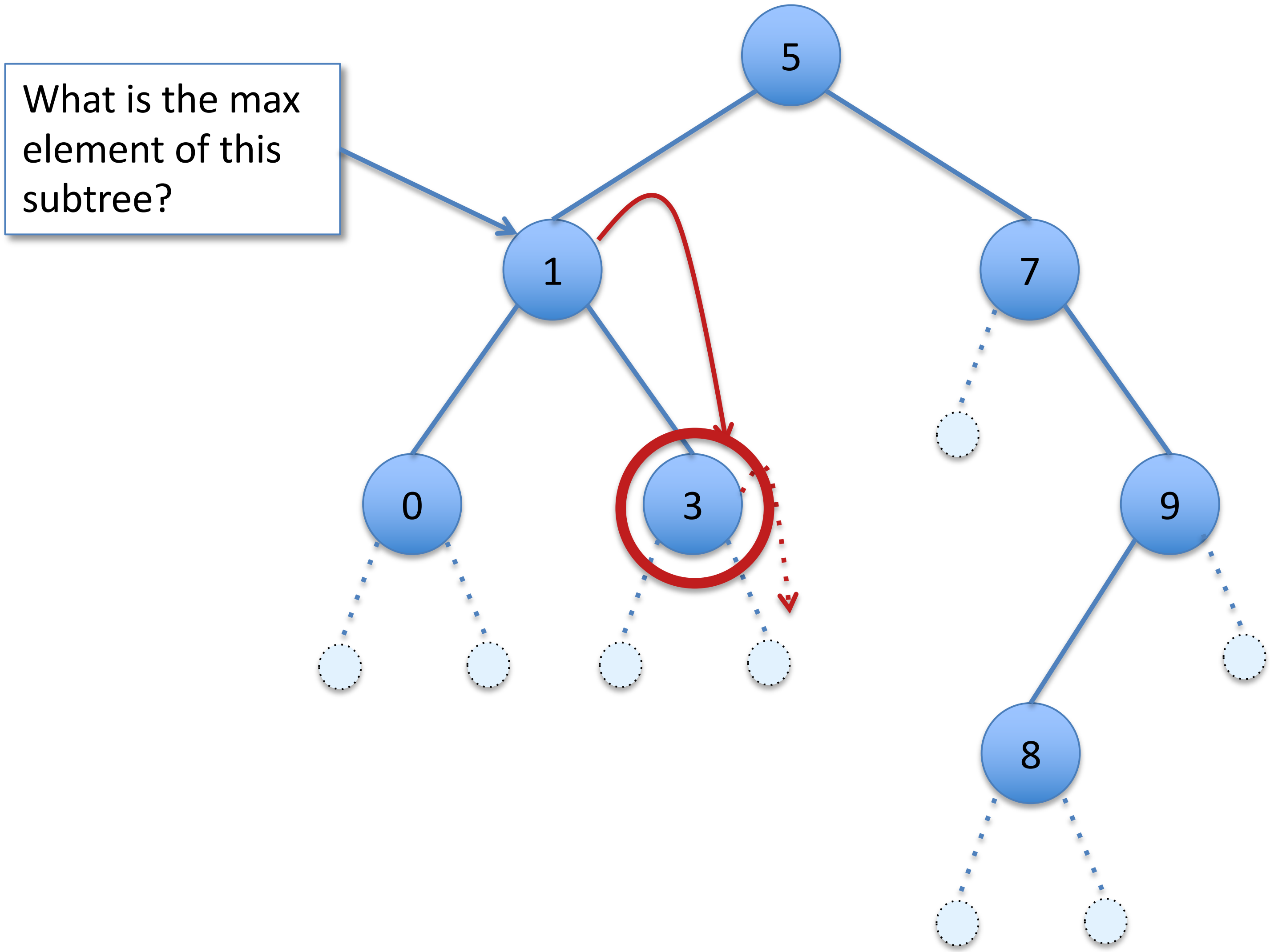


Deletion – Two Children: `(delete t 5)`

If the node to be delete has two children, *promote* the maximum child of the left tree.

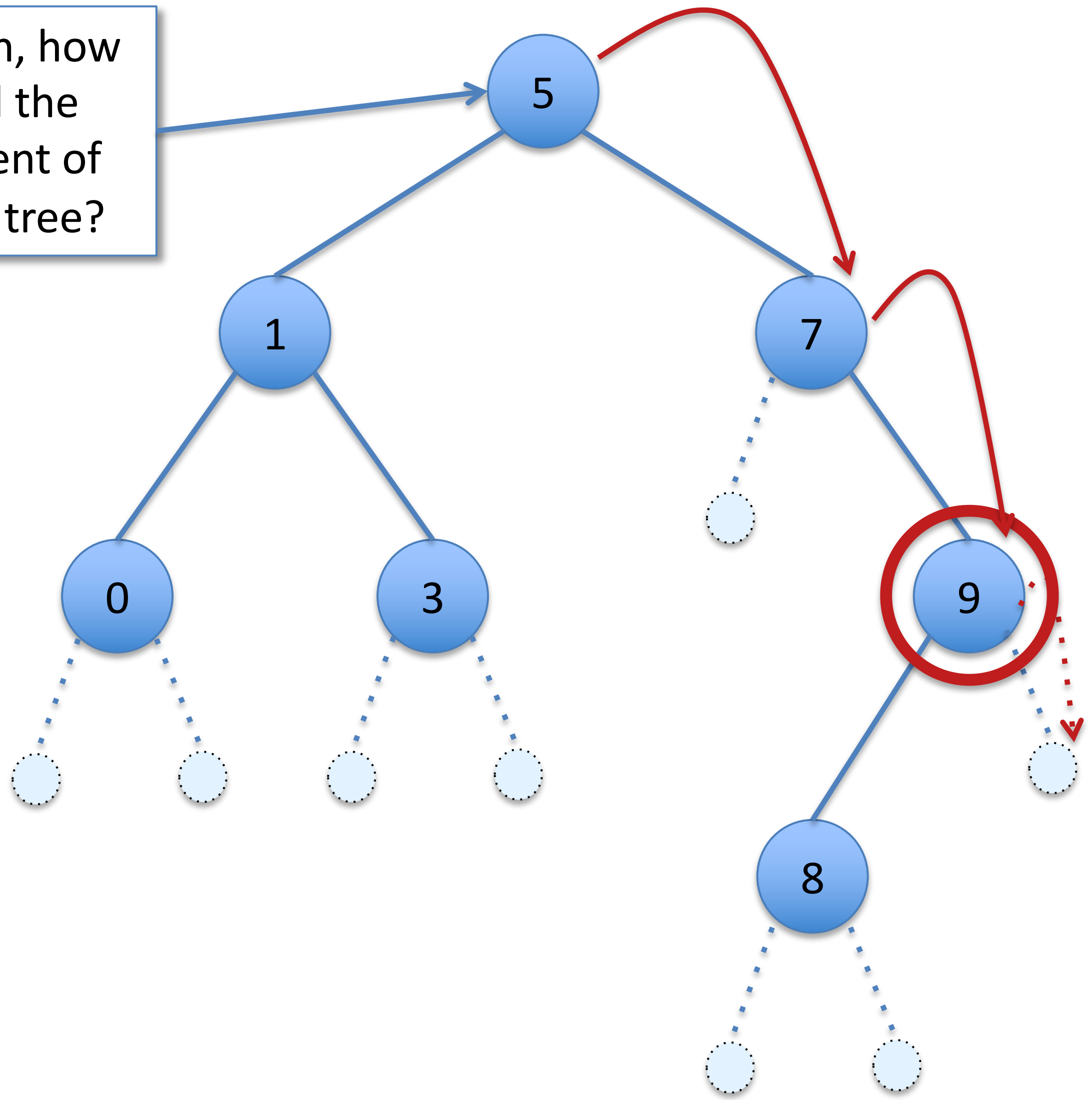


How to Find the Maximum Element?



How to Find the Maximum Element?

Just for fun, how do we find the max element of the whole tree?



Tree Max

```
let rec tree_max (t:tree) : int =
  begin match t with
  | Node(_,x,Empty) -> x
  | Node(_,_,rt) -> tree_max rt
  | _ -> failwith "tree_max called on Empty"
  end
```

- BST invariant guarantees that the maximum-value node is farthest to the right
- Note that `tree_max` is a *partial** function
  - Fails when called with an empty tree
- Fortunately, we never need to call tree_max on an empty tree
  - This is a consequence of the BST invariants and the case analysis done by the delete function

* Partial, in this context, means "not defined for all inputs".

Code for BST delete

bst.ml

Deleting From a BST

```
let rec delete (t: tree) (n: int) : tree =
  begin match t with
  | Empty -> Empty
  | Node(lt, x, rt) ->
    if x = n then
      begin match (lt, rt) with
      | (Empty, Empty) -> Empty
      | (Node _, Empty) -> lt
      | (Empty, Node _) -> rt
      | _ -> let m = tree_max lt in
        Node(delete lt m, m, rt)
    end
    else if n < x then Node(delete lt n, x, rt)
  else Node(lt, x, delete rt n)
end
```

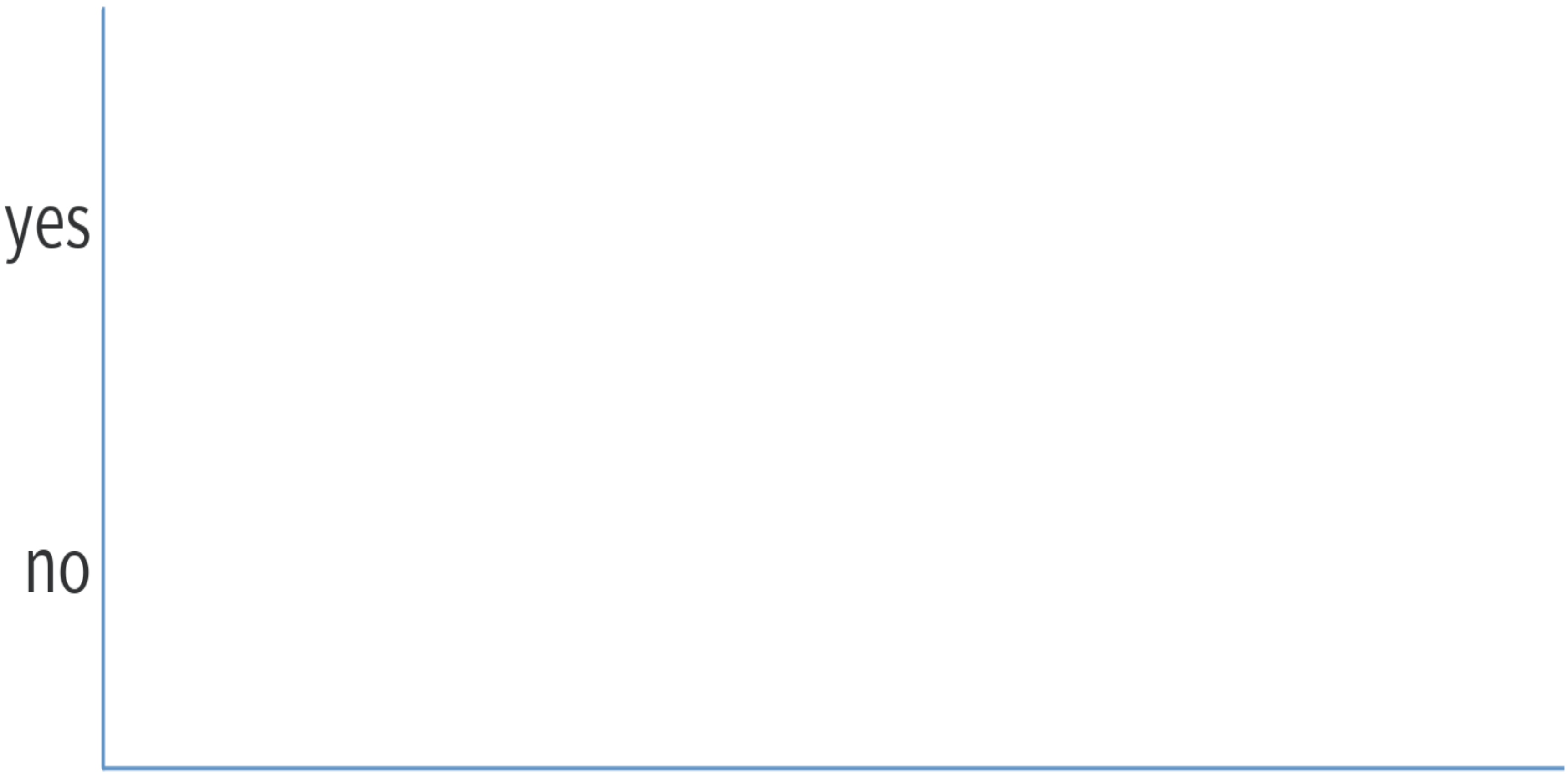
See bst.ml

Subtleties of the Two-Child Case

- Suppose Node(lt,x,rt) is to be deleted and lt and rt are both themselves nonempty trees.
- Then:
  1. There exists a maximum element, m, of lt (Why?)
  2. Every element of rt is greater than m (Why?)

- To promote m we replace the deleted node by:
  Node(delete lt m, m, rt)
  - I.e. we recursively delete m from lt and relabel the root node m
  - The resulting tree satisfies the BST invariants

If we insert a label n into a BST and then immediately delete n, do we always get back a tree of exactly the same shape?

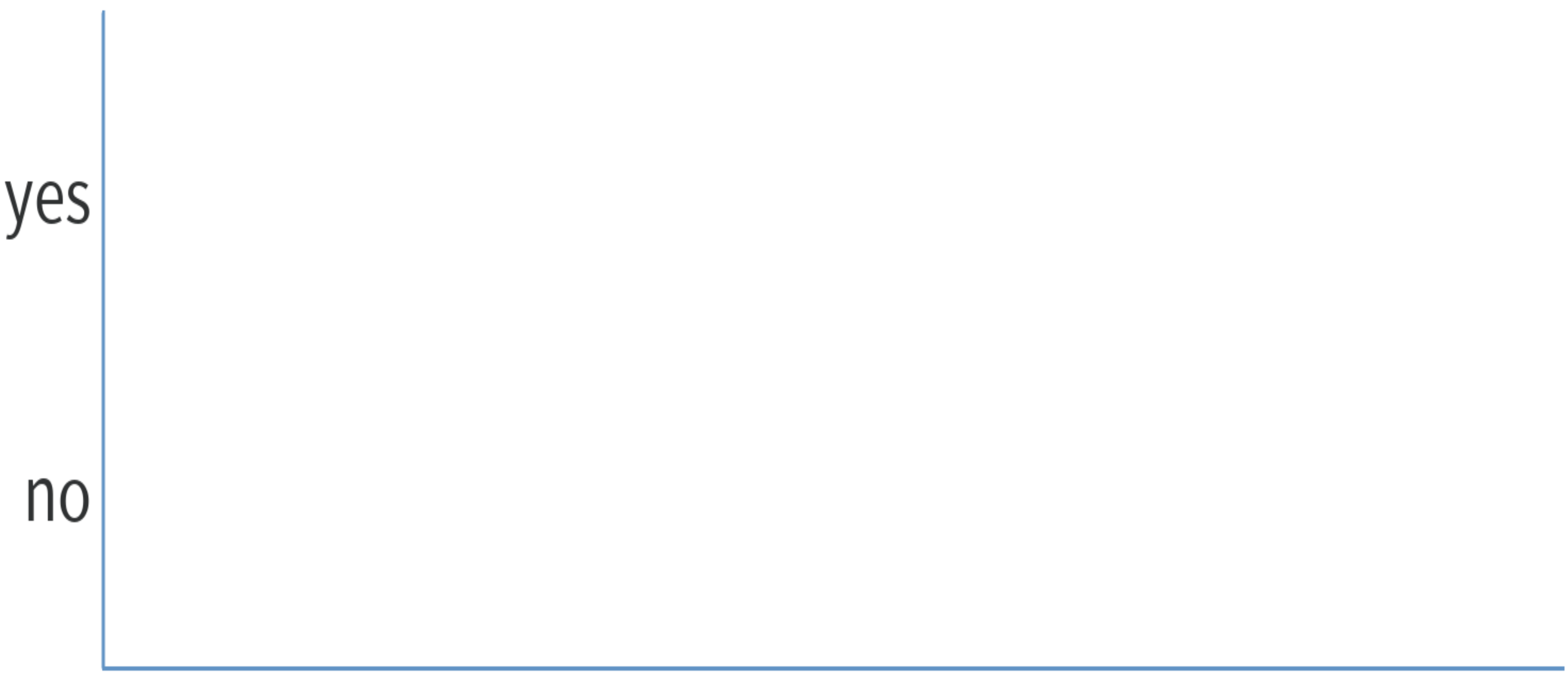


If we insert a label n into a BST and then immediately delete n, do we always get back a tree of exactly the same shape?

1. yes
2. no

Answer: no (what if the node was in the tree to begin with?)

If we insert a value n into a BST that *does not* already contain n and then immediately delete n, do we always get back a tree of exactly the same shape?

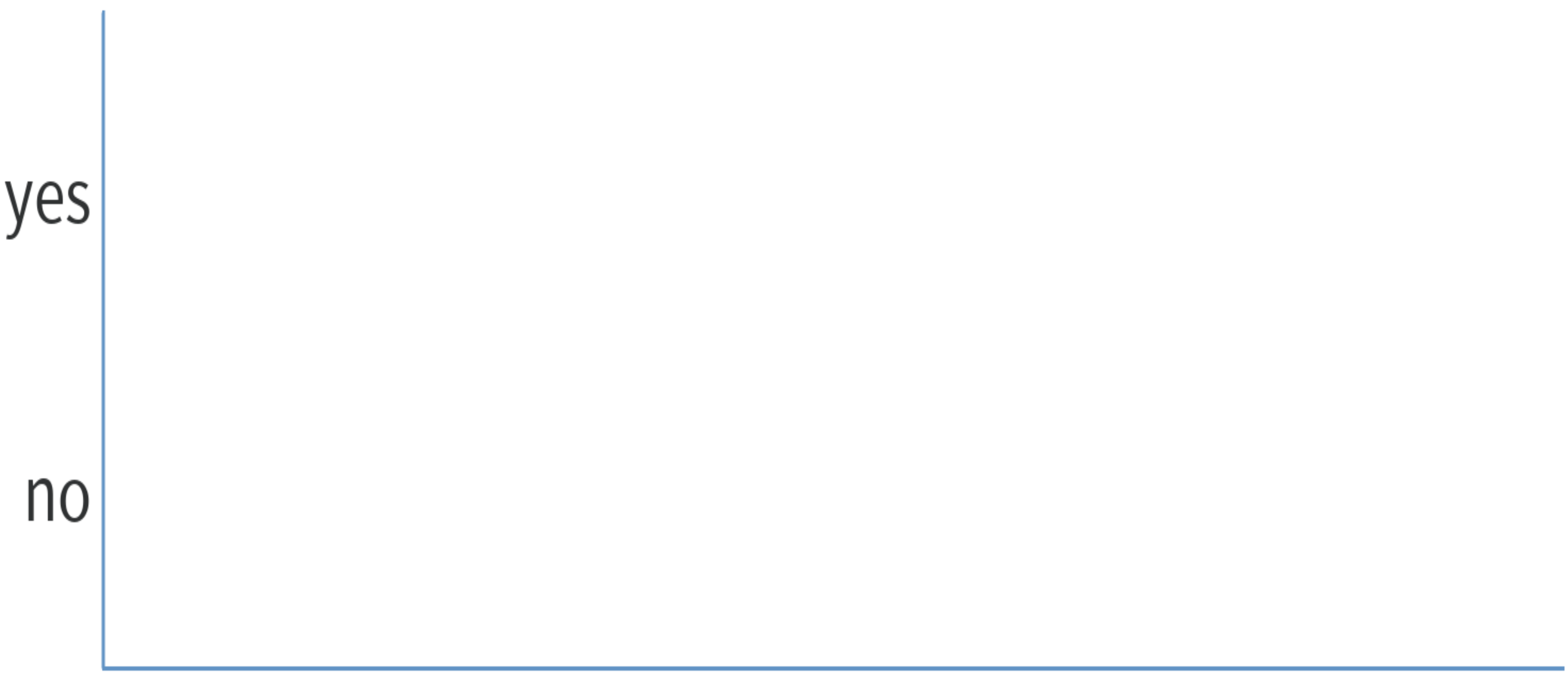


If we insert a value n into a BST *that does not already contain n* and then immediately delete n, do we always get back a tree of exactly the same shape?

1. yes
2. no

Answer: yes

If we delete n from a BST (containing n) and then immediately insert n again, do we always get back a tree of exactly the same shape?



If we delete n from a BST (containing n) and then immediately insert n again, do we always get back a tree of exactly the same shape?

1. yes
2. no

Answer: no (e.g., what if we delete the item at the root node?)