Programming Languages and Techniques (CIS120)

Lecture 14

ASM & Equality

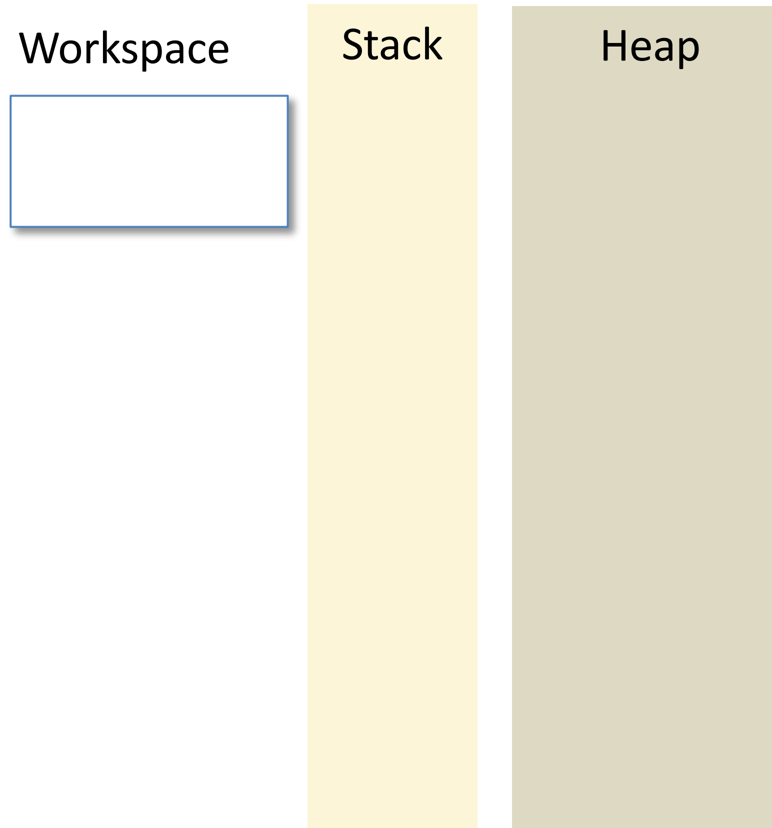
Lecture notes: Chapter 16

Announcements

- Midterm grading in progress
  - Scores will be released soon
- Homework 4
  - due Tuesday next week
- Lecture Section 002:
  - Dr. Sheth will be away Weds. & Fri.
  - Dr. Zdancewic will give those lectures

Review: Abstract Stack Machine

- Three “spaces”
  - workspace
    - the expression the computer is currently working on simplifying
  - stack
    - temporary storage for `let` bindings and partially simplified expressions
  - heap
    - storage area for large data structures
- Initial state:
  - workspace contains whole program
  - stack and heap are empty
- Machine operation:
  - In each step, choose next part of the workspace expression and simplify it
  - (Sometimes this will also involve changes to the stack and/or heap)
  - Stop when there are no more simplifications to be done



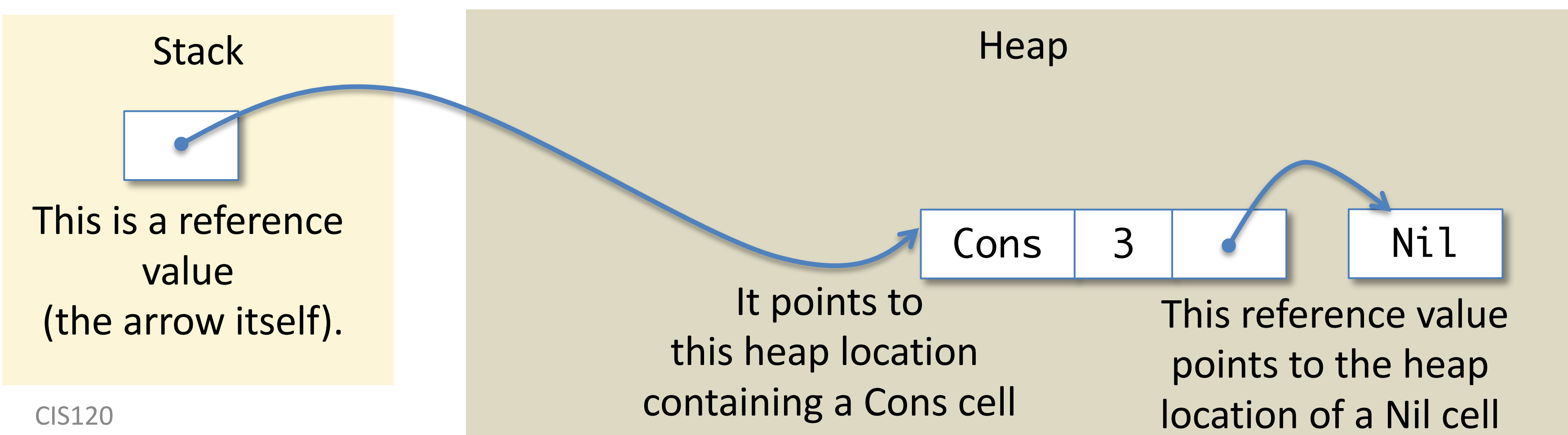
Review: Values and References

A *value* is either:

- a *primitive value* like an integer, or,
- a *reference* to a location in the heap

A reference is the *address* of a piece of data in the heap. We draw a reference value as an "arrow":

- The start of the arrow is the reference itself (i.e. the address).
- The arrow "points" to the value located at the reference's address.



Review: Example

Workspace

```
let p1 : point = {x=1; y=1;}
let p2 : point = p1
let ans : int =
    p2.x <- 17; p1.x
```

Stack

Heap

Allocate a Record

Workspace

```
let p1 : point = {x=1; y=1;}
let p2 : point = p1
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```

Stack

Heap

Allocate a Record

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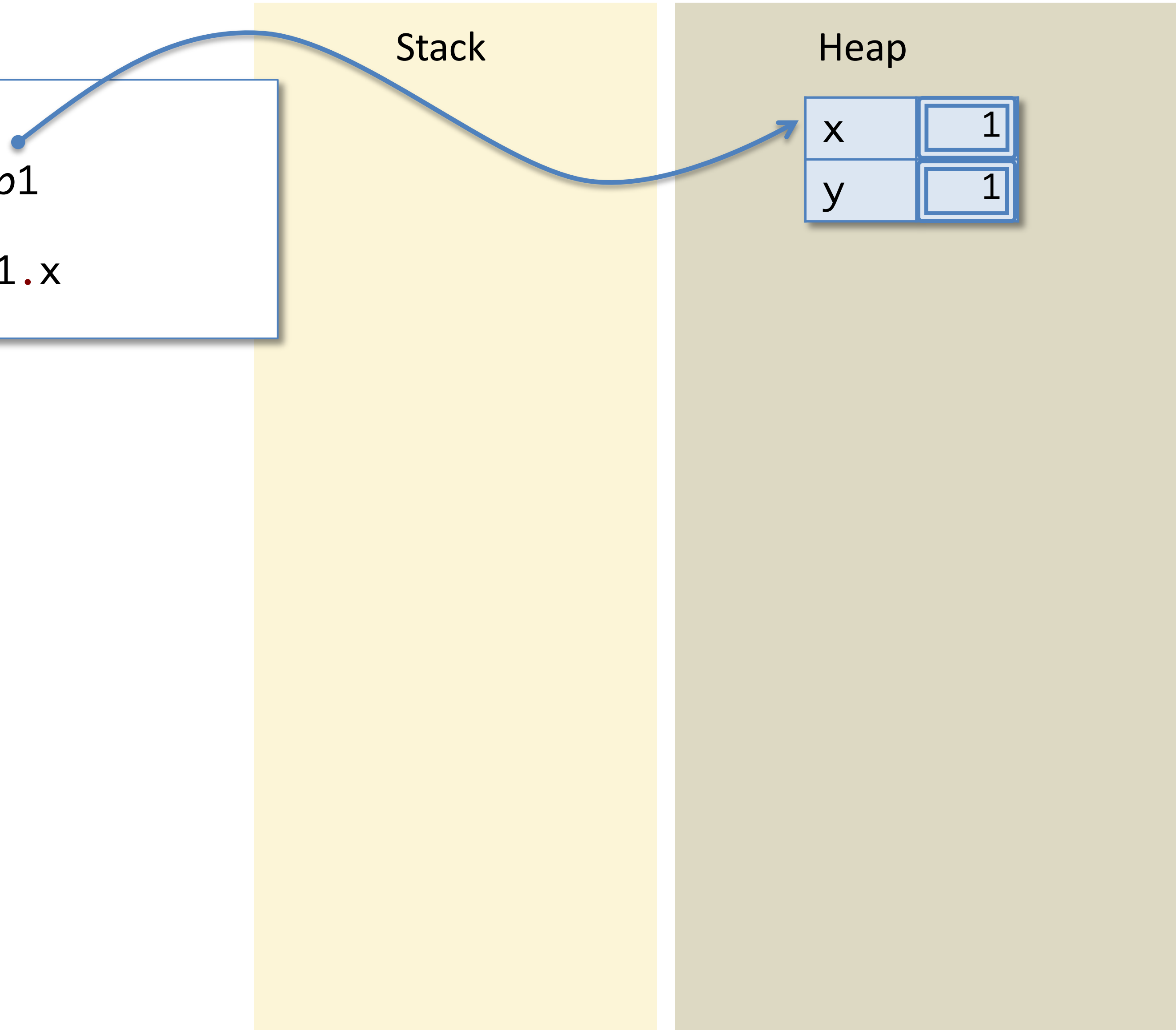
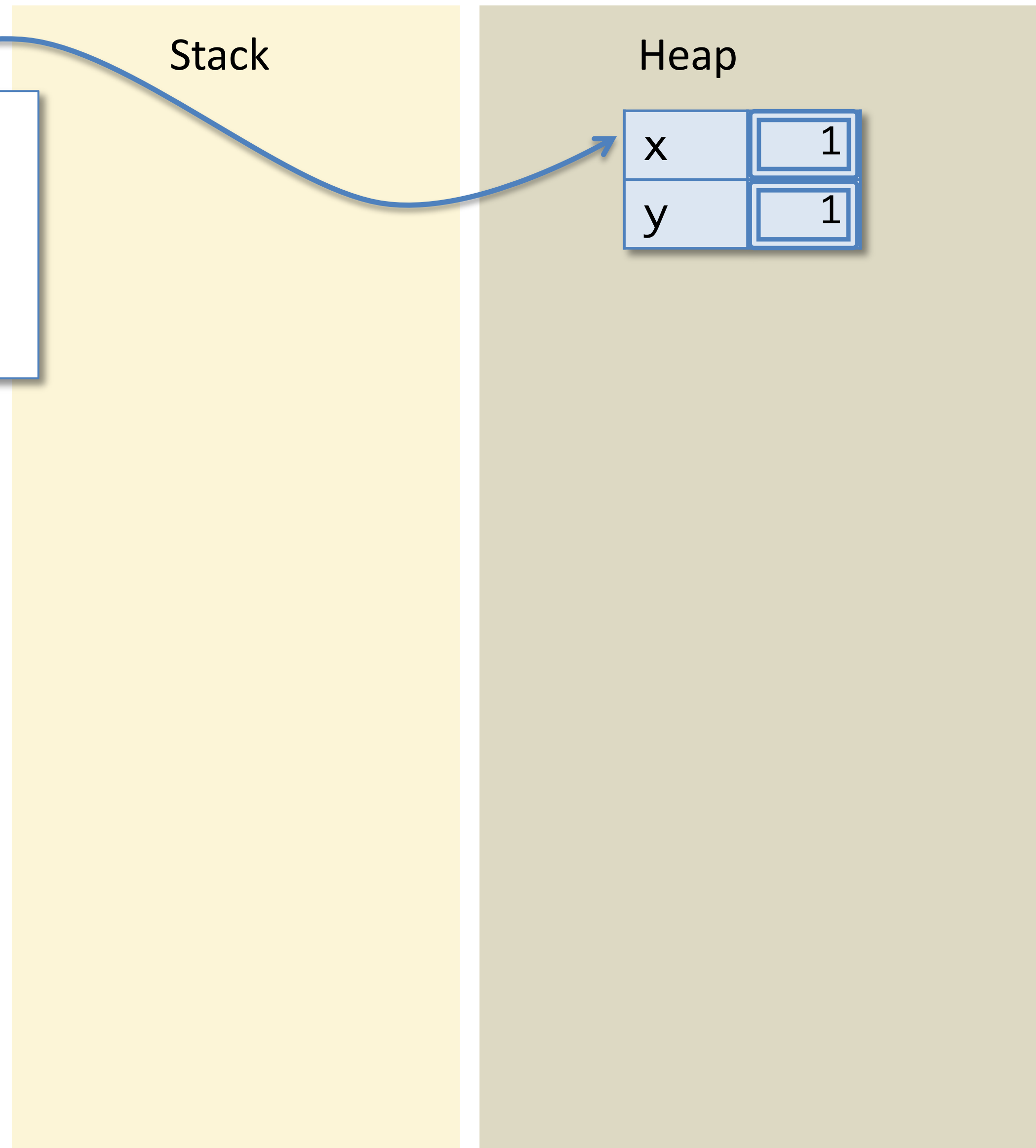


Let Expression

Workspace

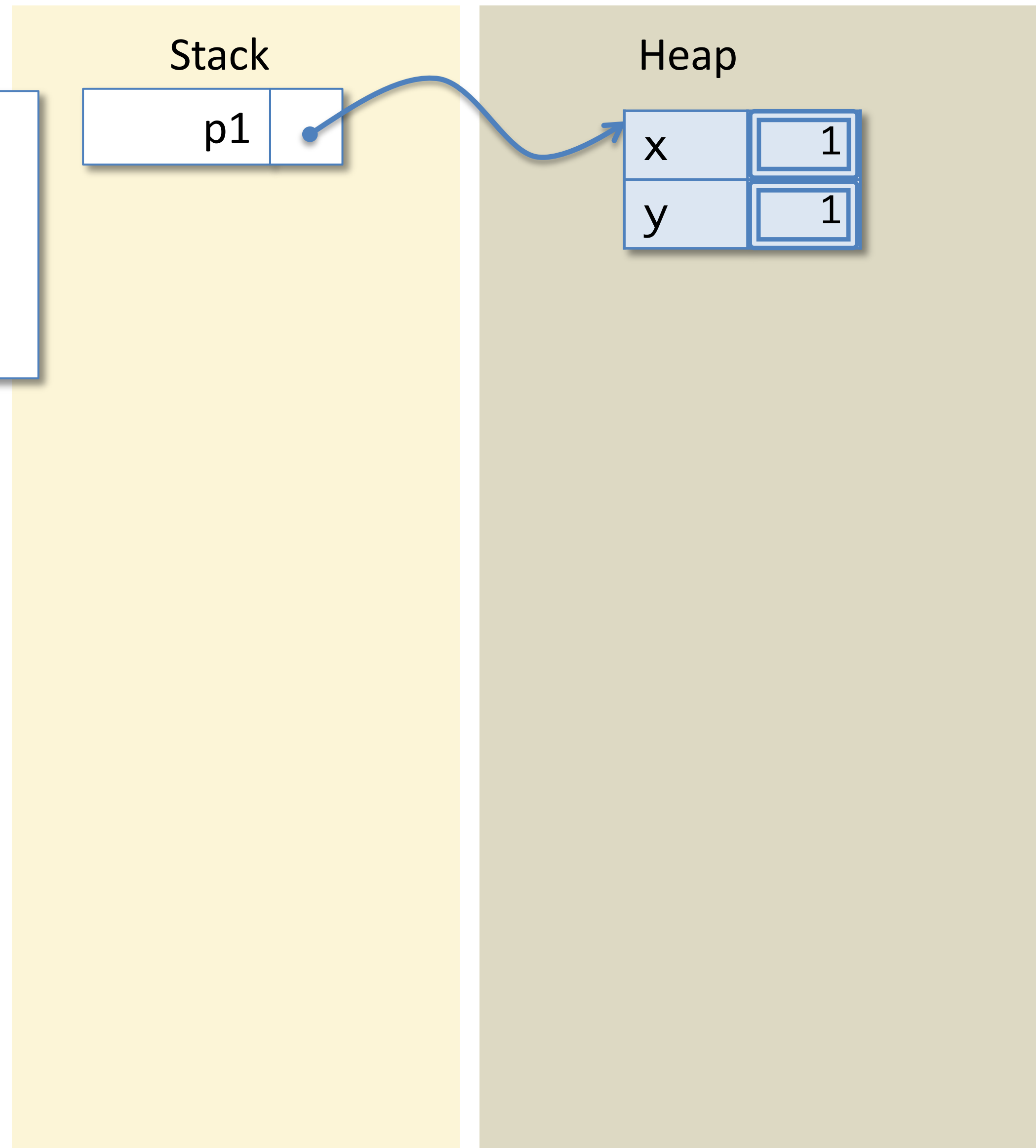
```
let p1 : point =   .
let p2 : point = p1
let ans : int =
    p2.x <- 17; p1.x
```



Push p1

Workspace

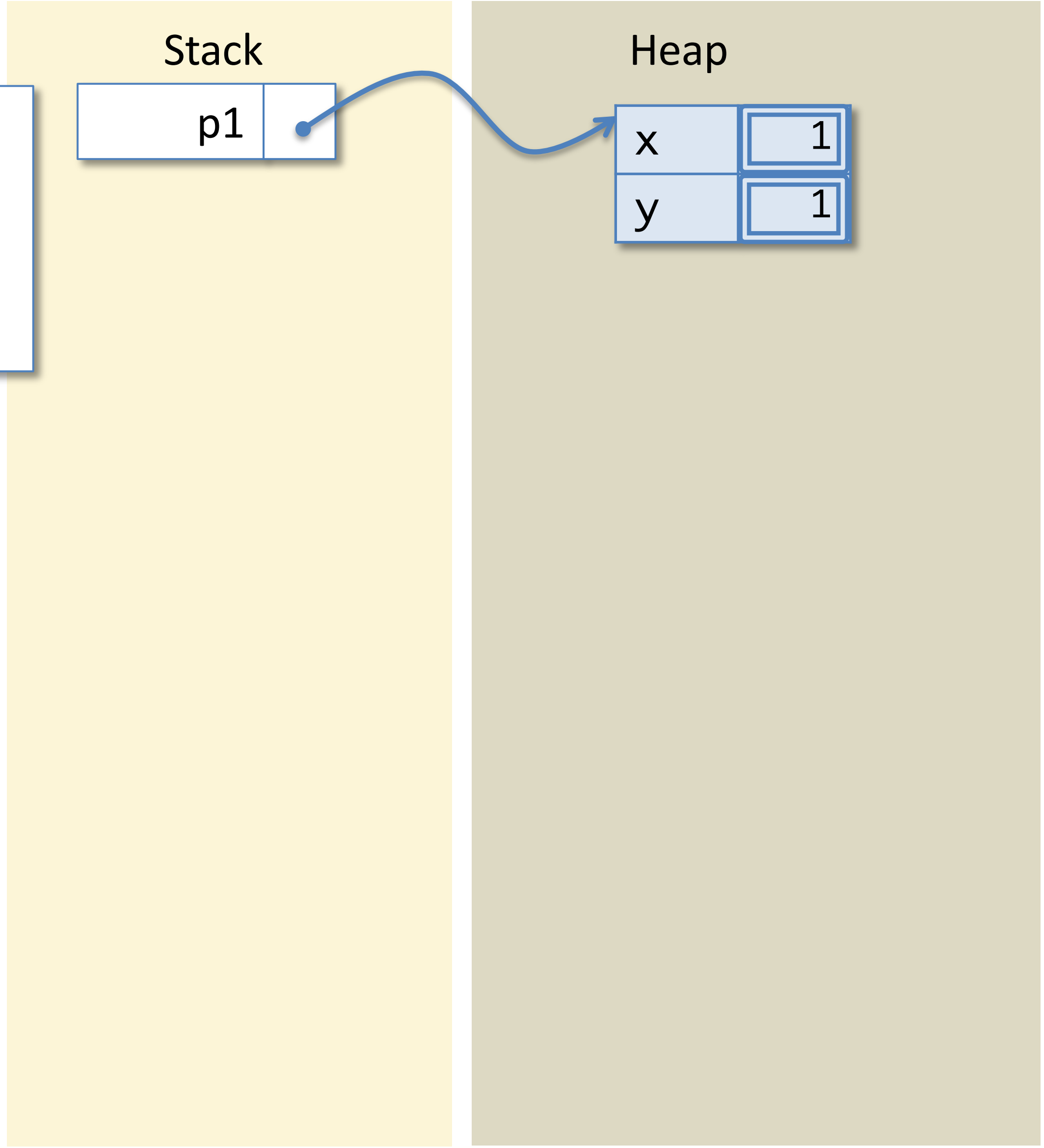
```
let p2 : point = p1
let ans : int =
    p2.x <- 17; p1.x
```



Look Up ‘p1’

Workspace

```
let p2 : point = p1
let ans : int =
    p2.x <- 17; p1.x
```



Look Up ‘p1’

Workspace

```
let p2 : point =
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    p2.x <- 17; p1.x
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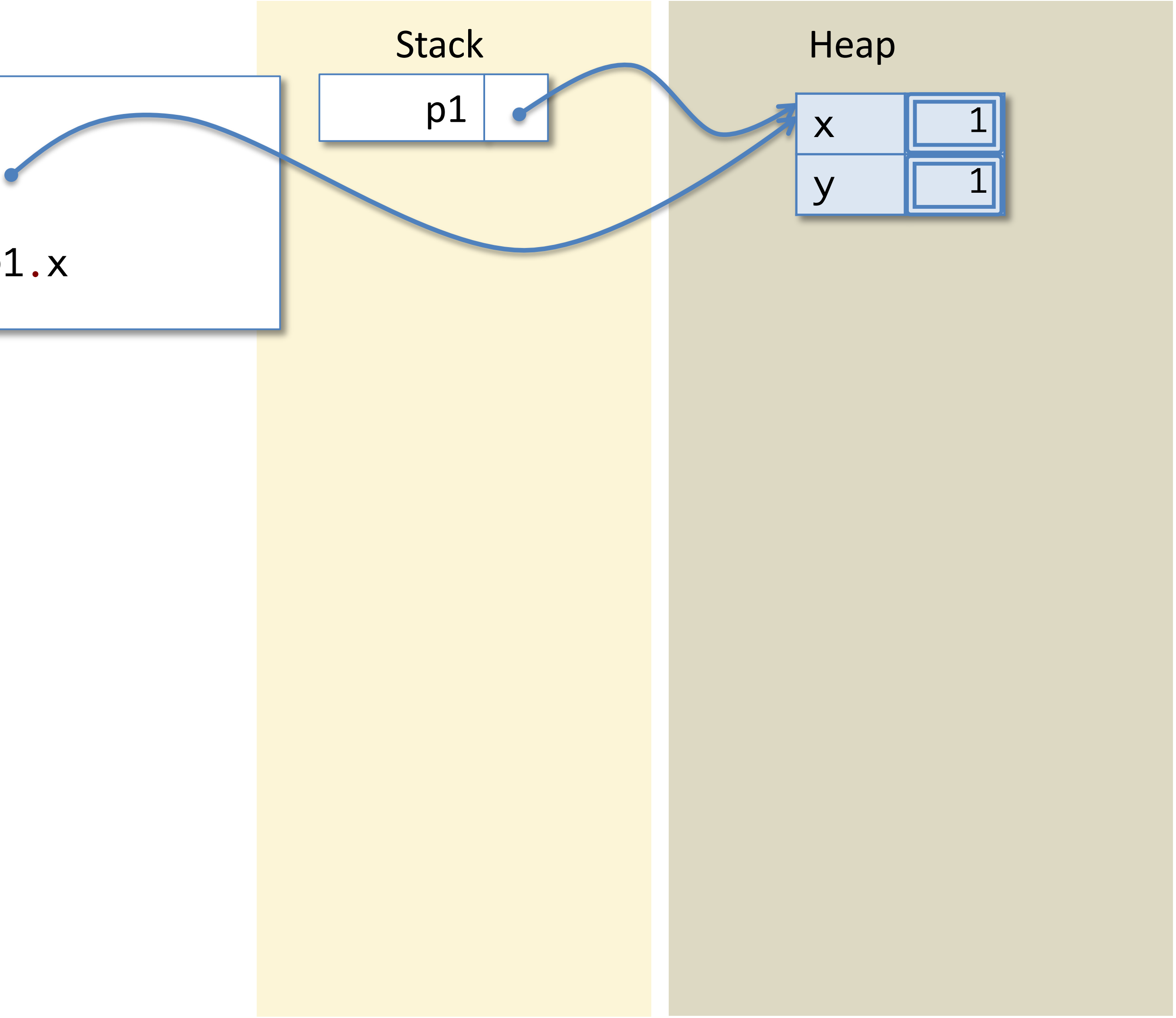
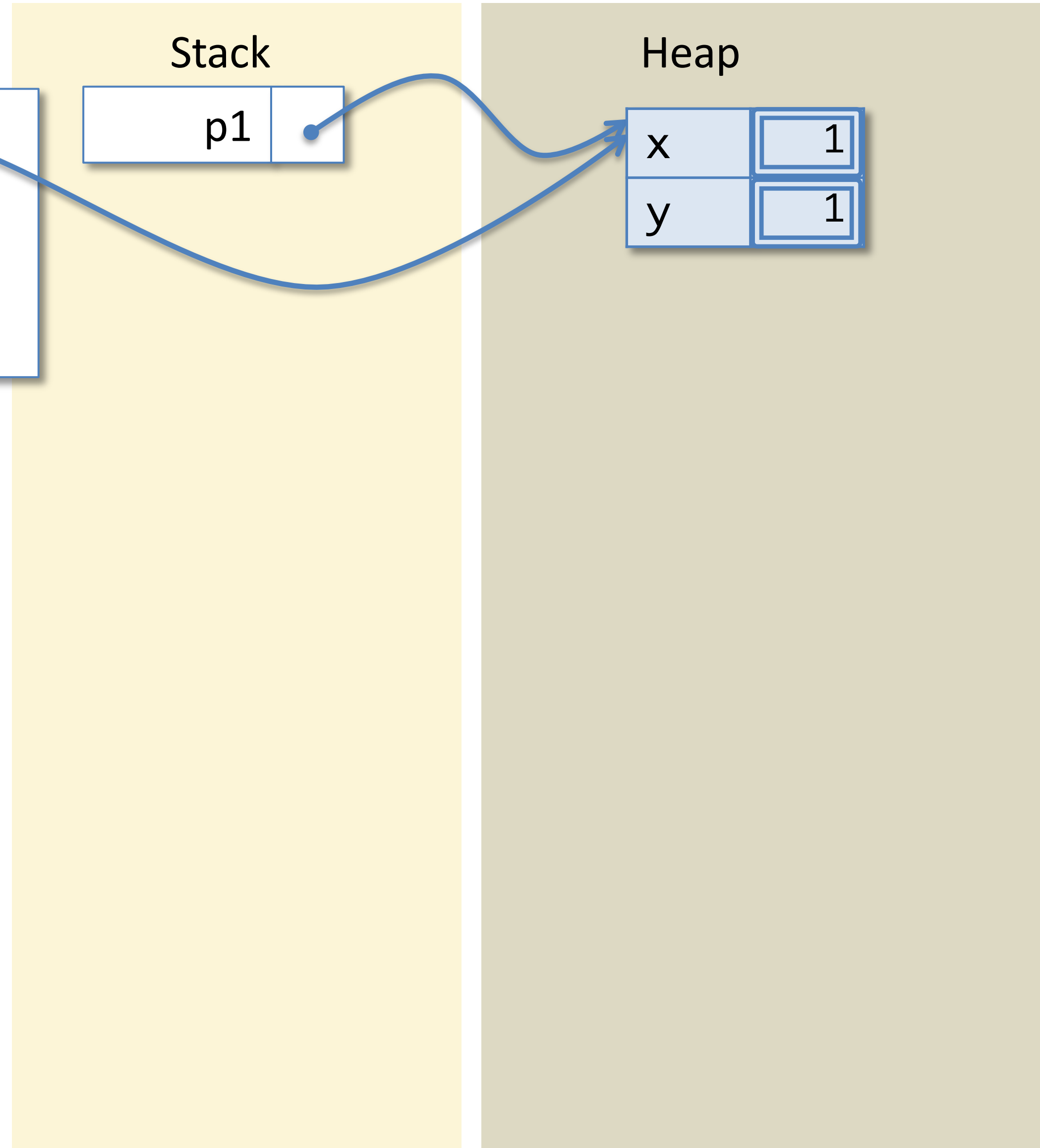


Let Expression

Workspace

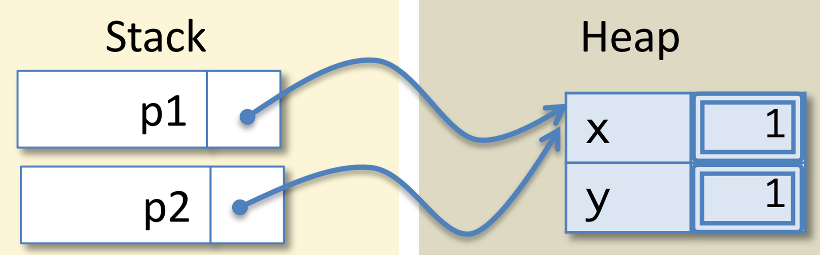
```
let p2 : point = .
let ans : int =
    p2.x <- 17; p1.x
```



Push p2

Workspace

```
let ans : int =
    p2.x <- 17; p1.x
```

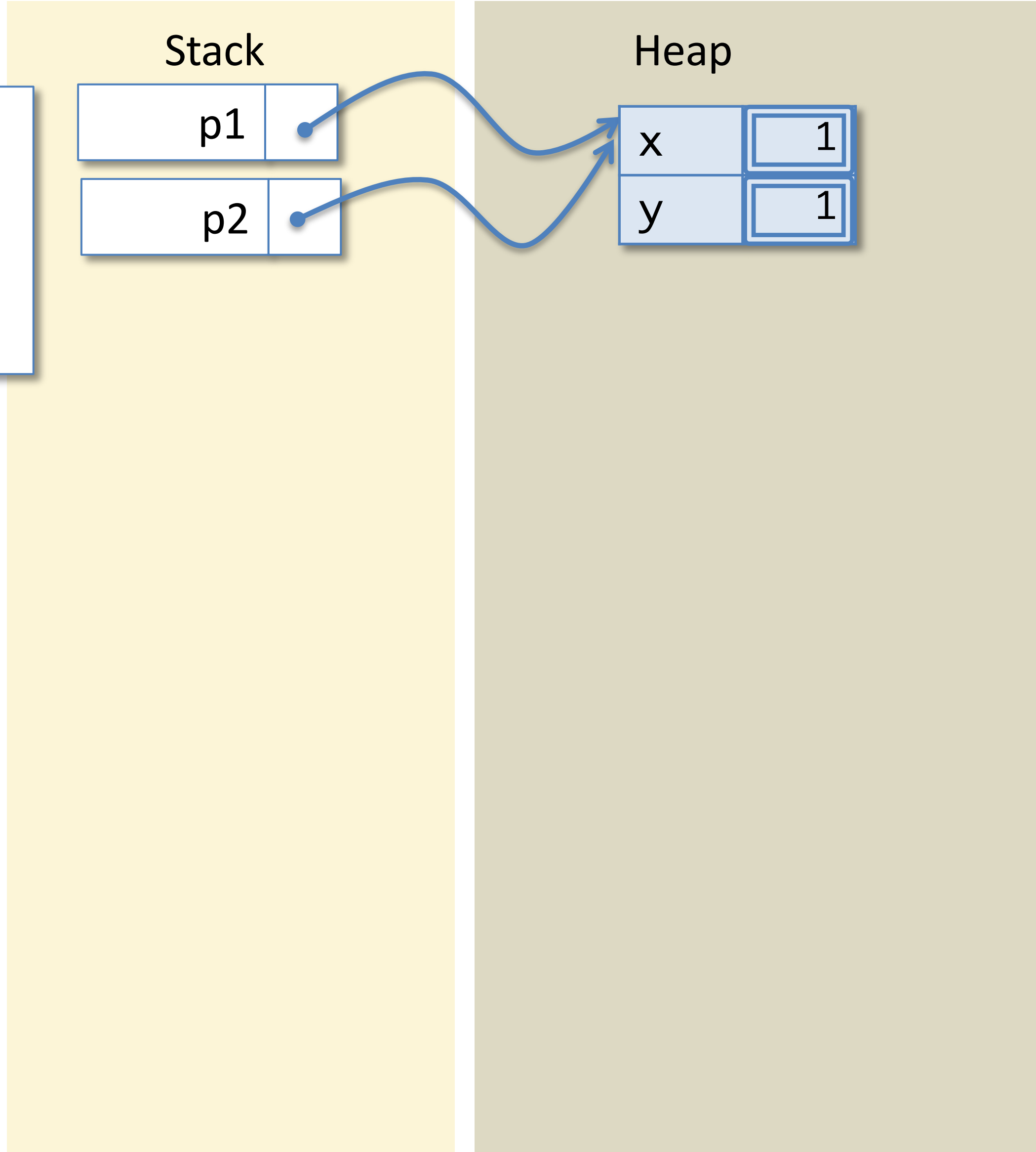


Note: p1 and p2 are references to the *same* heap record. They are *aliases* – two different names for the *same thing*.

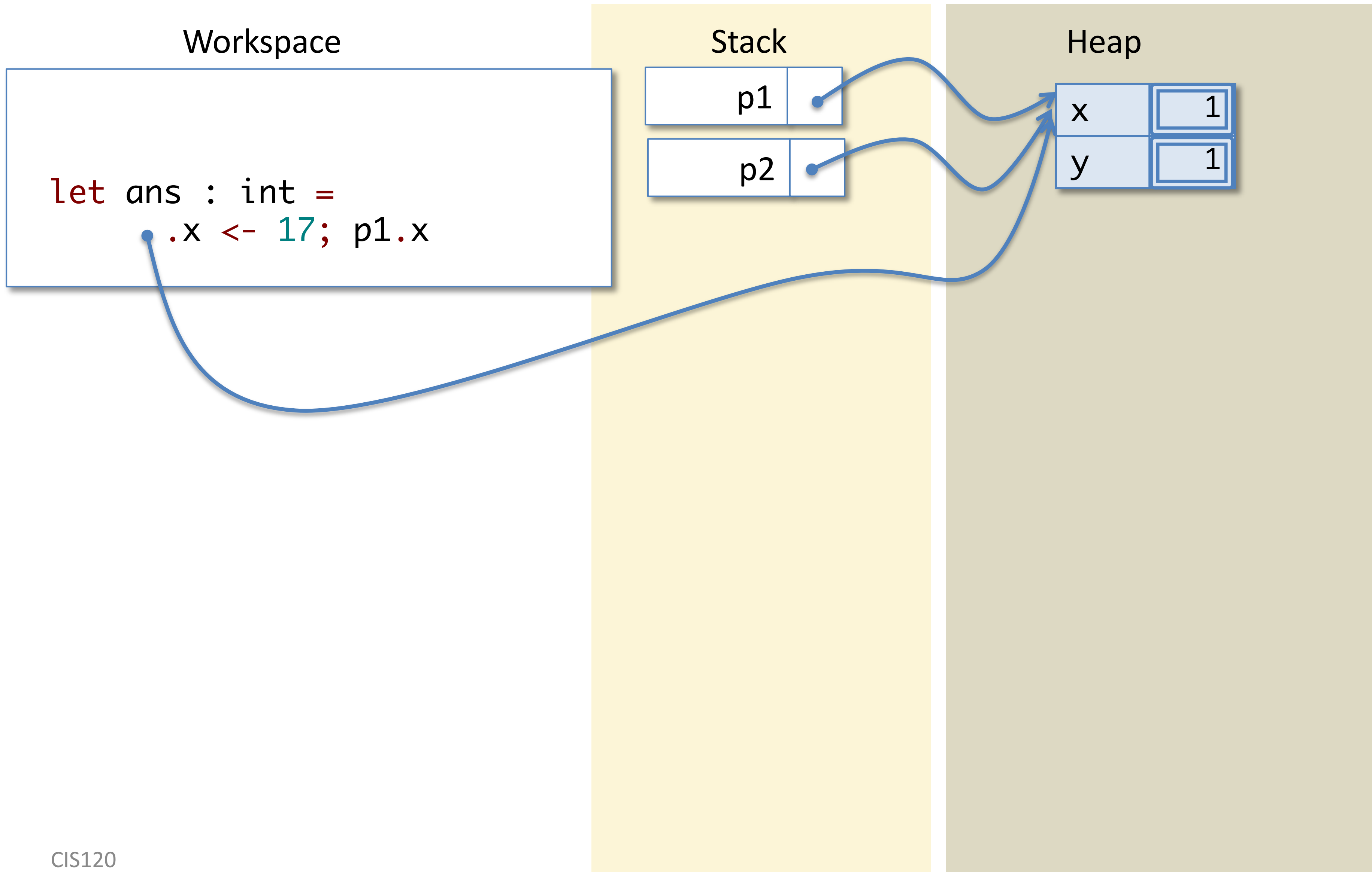
Look Up ‘p2’

Workspace

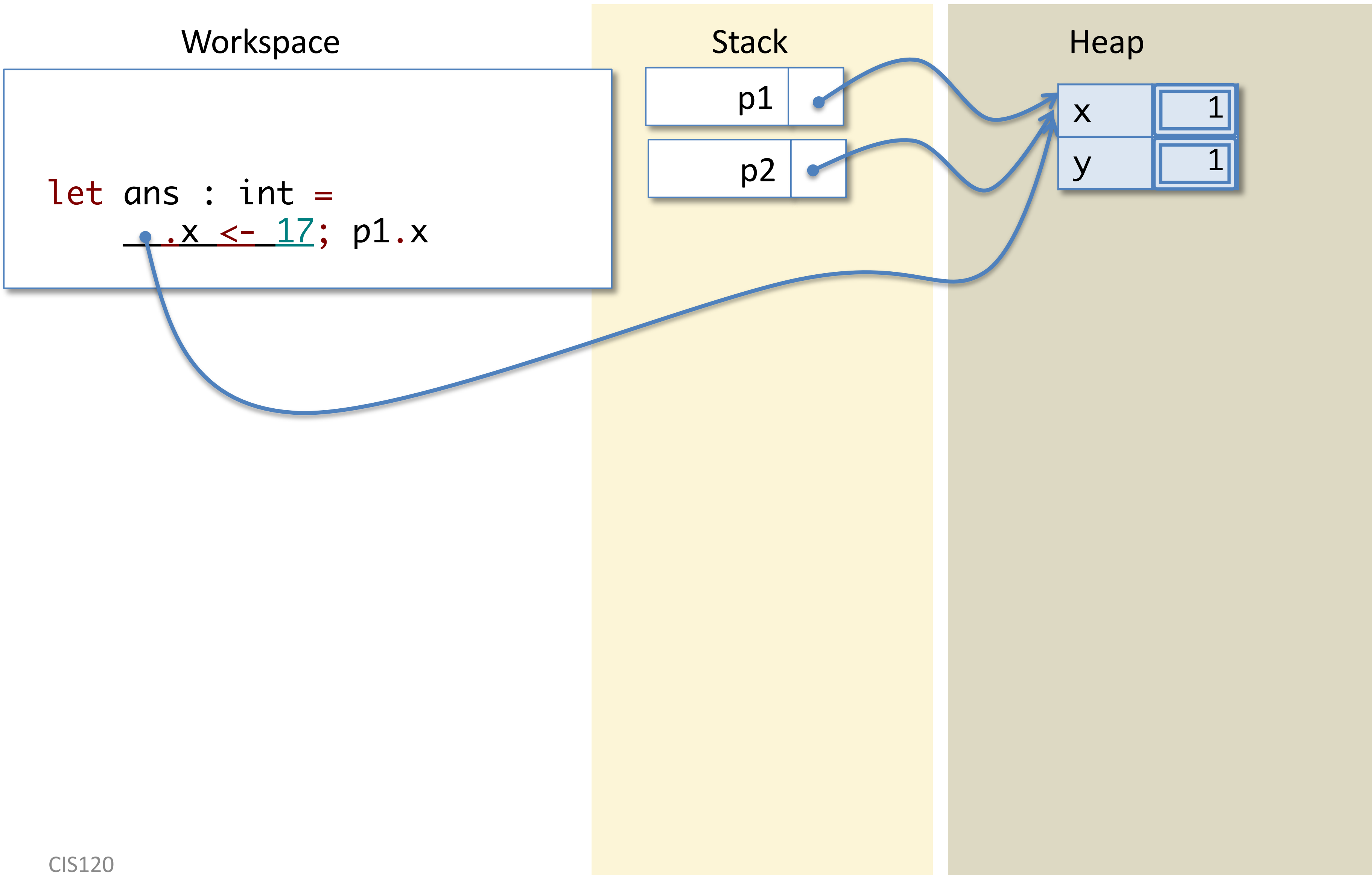
```
let ans : int =
    p2.x <- 17; p1.x
```



Look Up ‘p2’



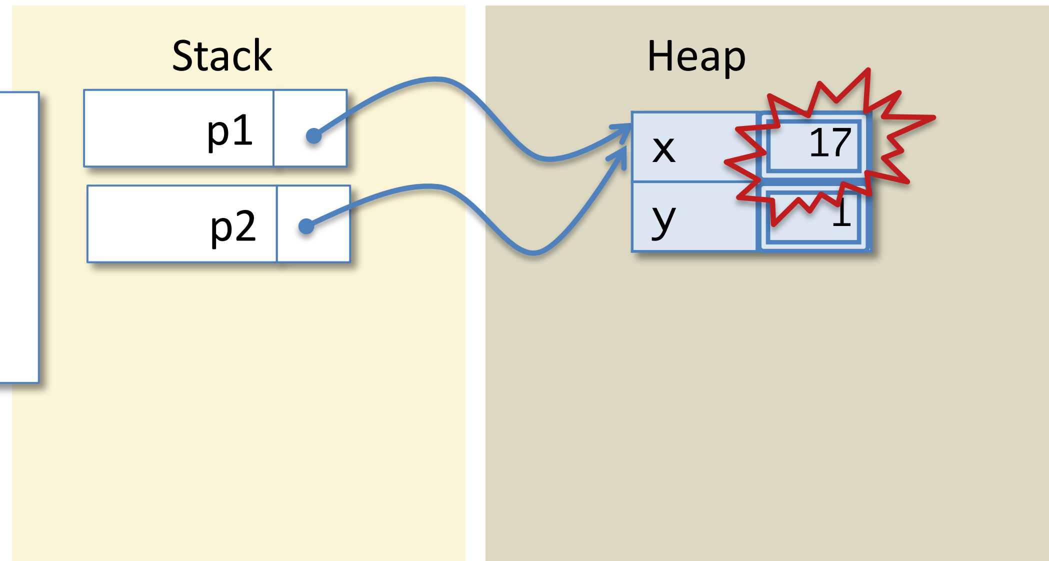
Assign to x field



Assign to x field

Workspace

```
let ans : int =
    (); p1.x
```

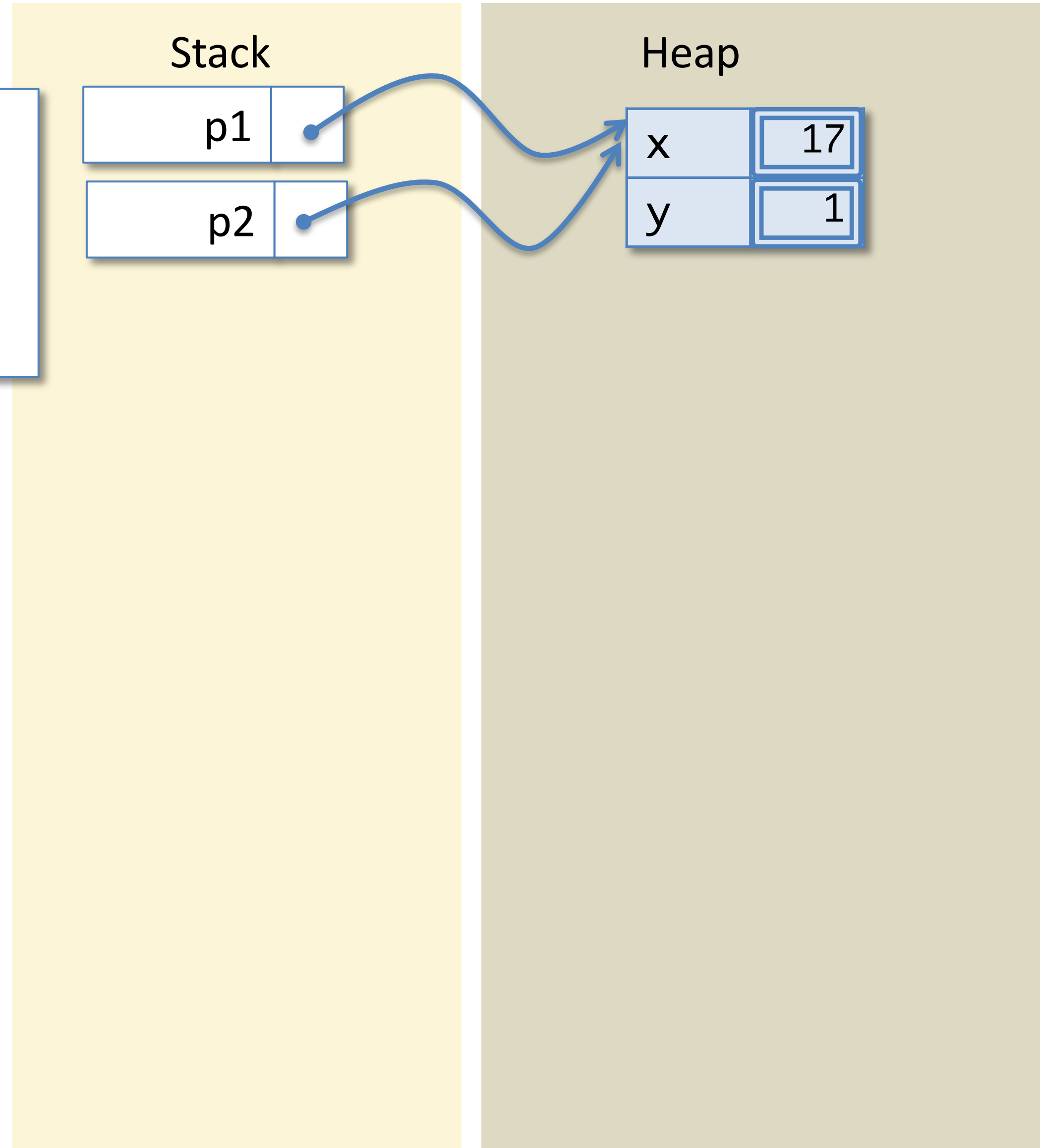


This is the step in which the ‘imperative’ update occurs. The mutable field x has been modified in place to contain the value 17.

Sequence ';' Discards Unit

Workspace

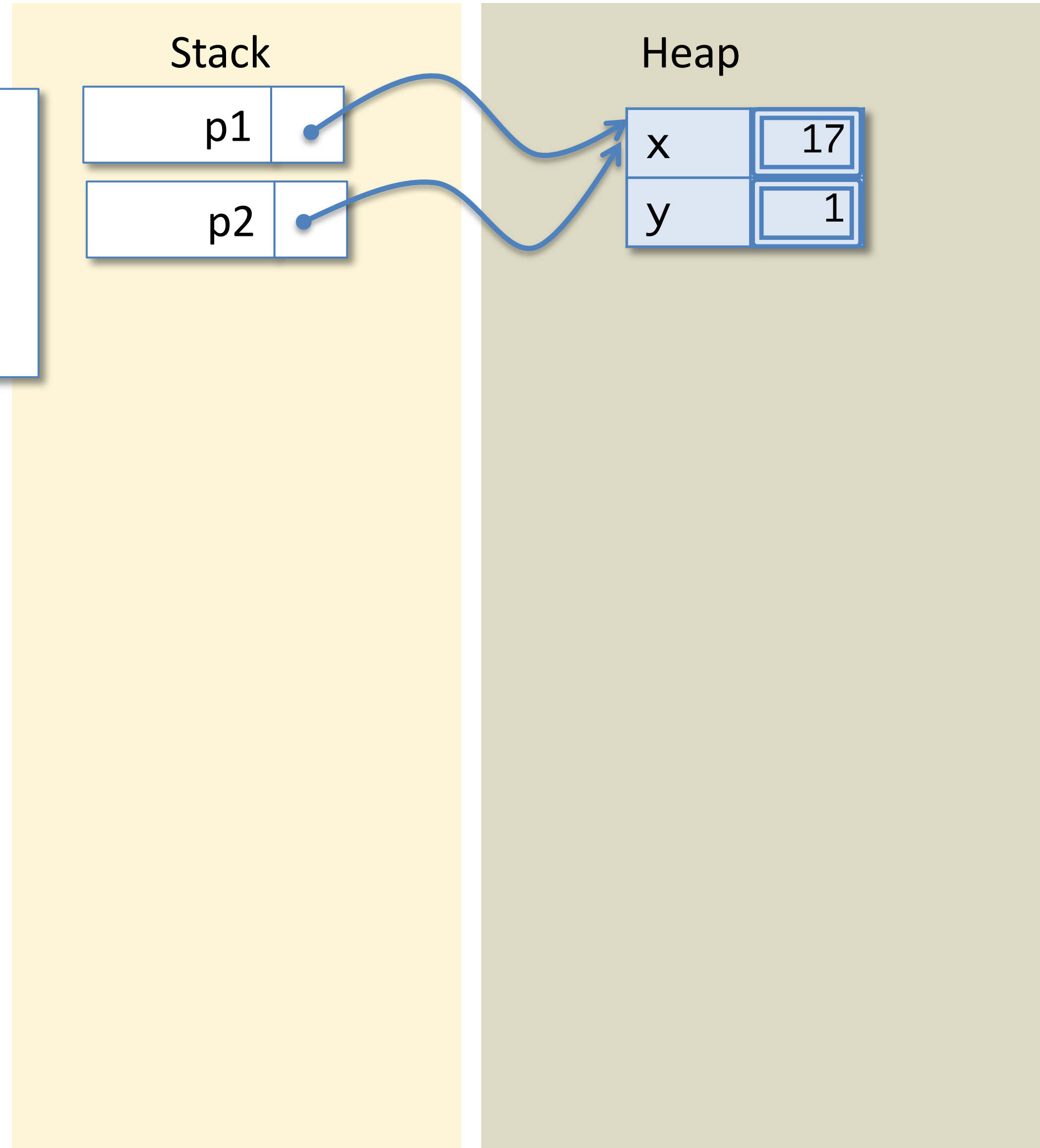
```
let ans : int =
    (); p1.x
```



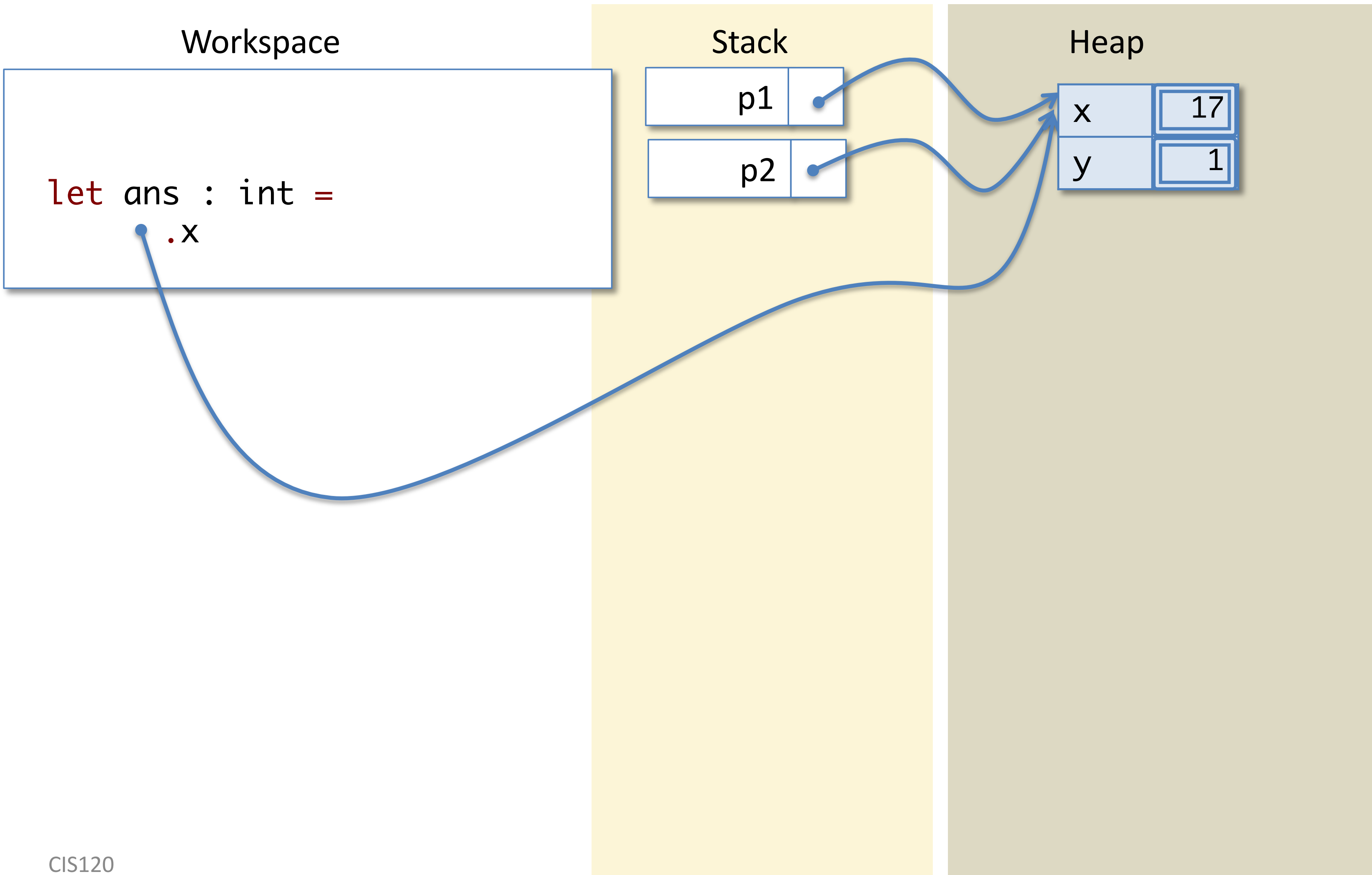
Look Up ‘p1’

Workspace

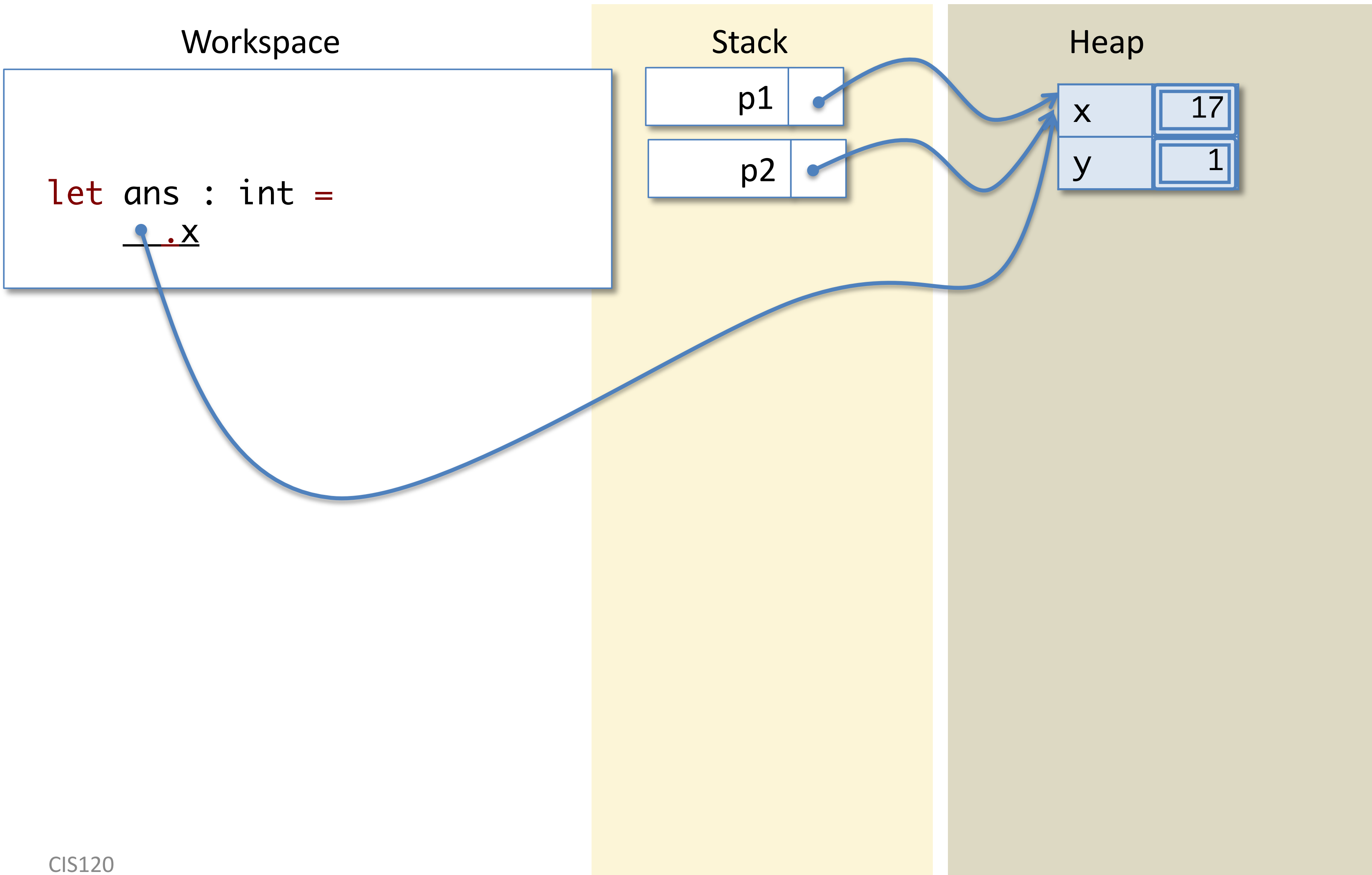
```
let ans : int =
    p1.x
```



Look Up 'p1'



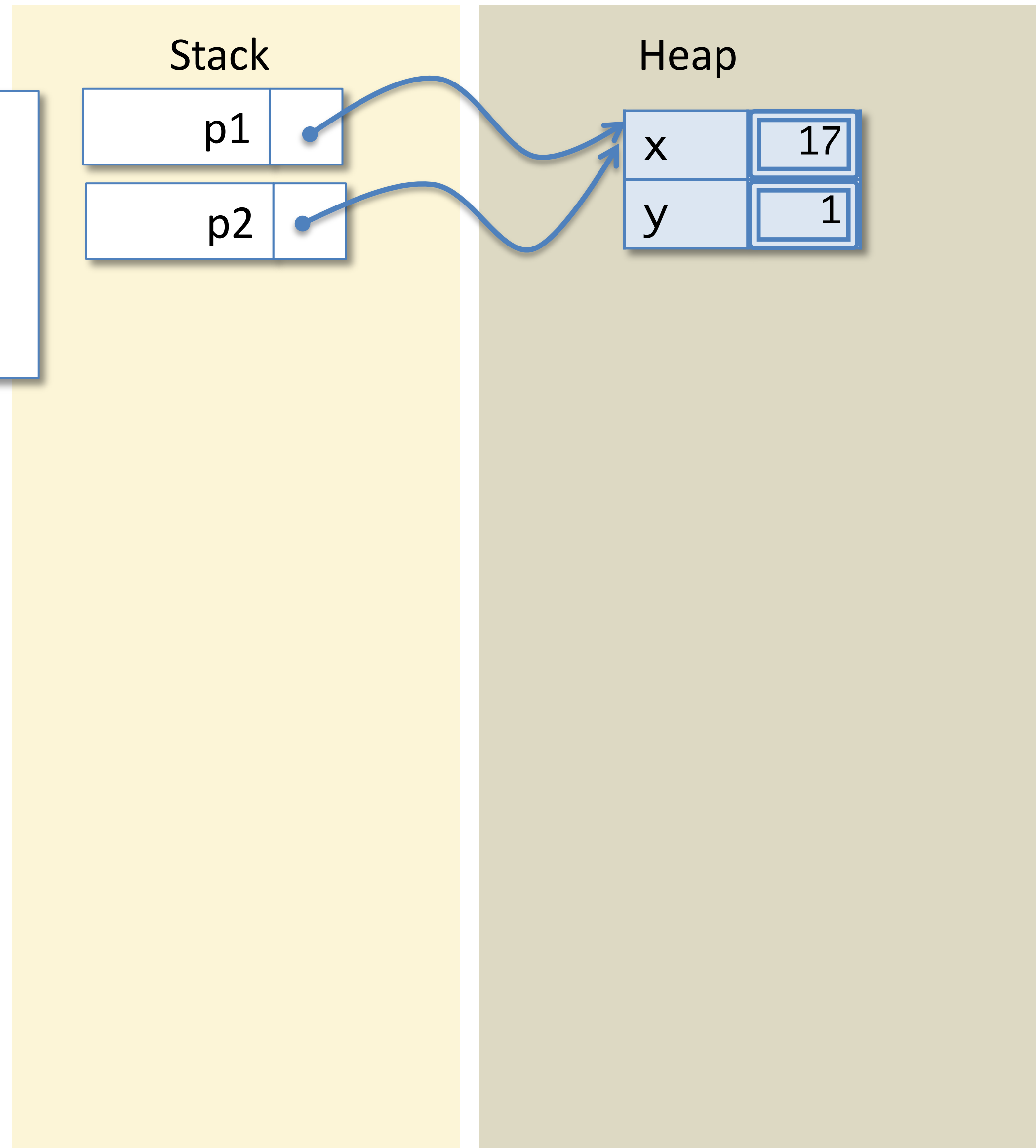
Project the ‘x’ field



Project the 'x' field

Workspace

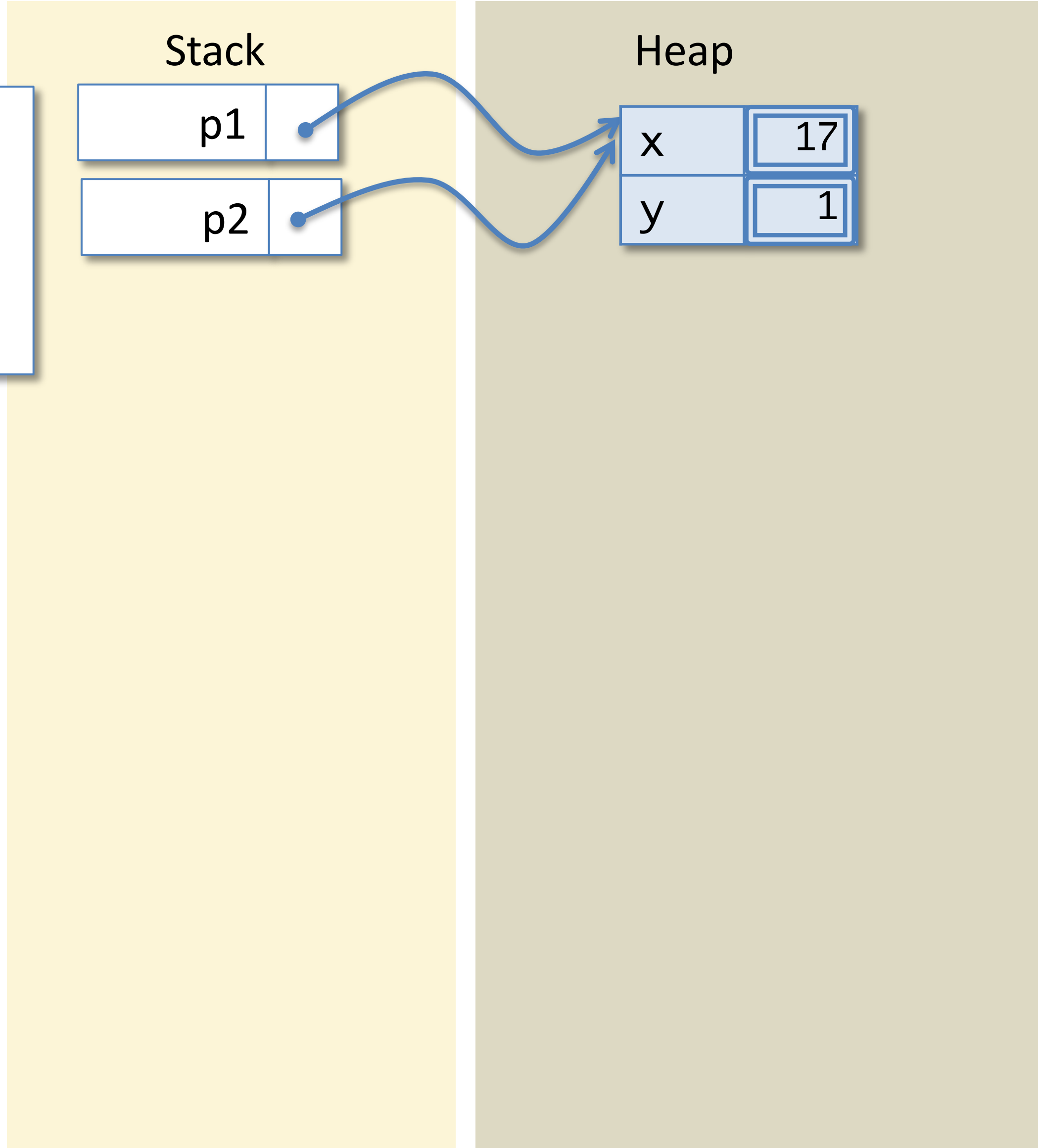
```
let ans : int =
    17
```



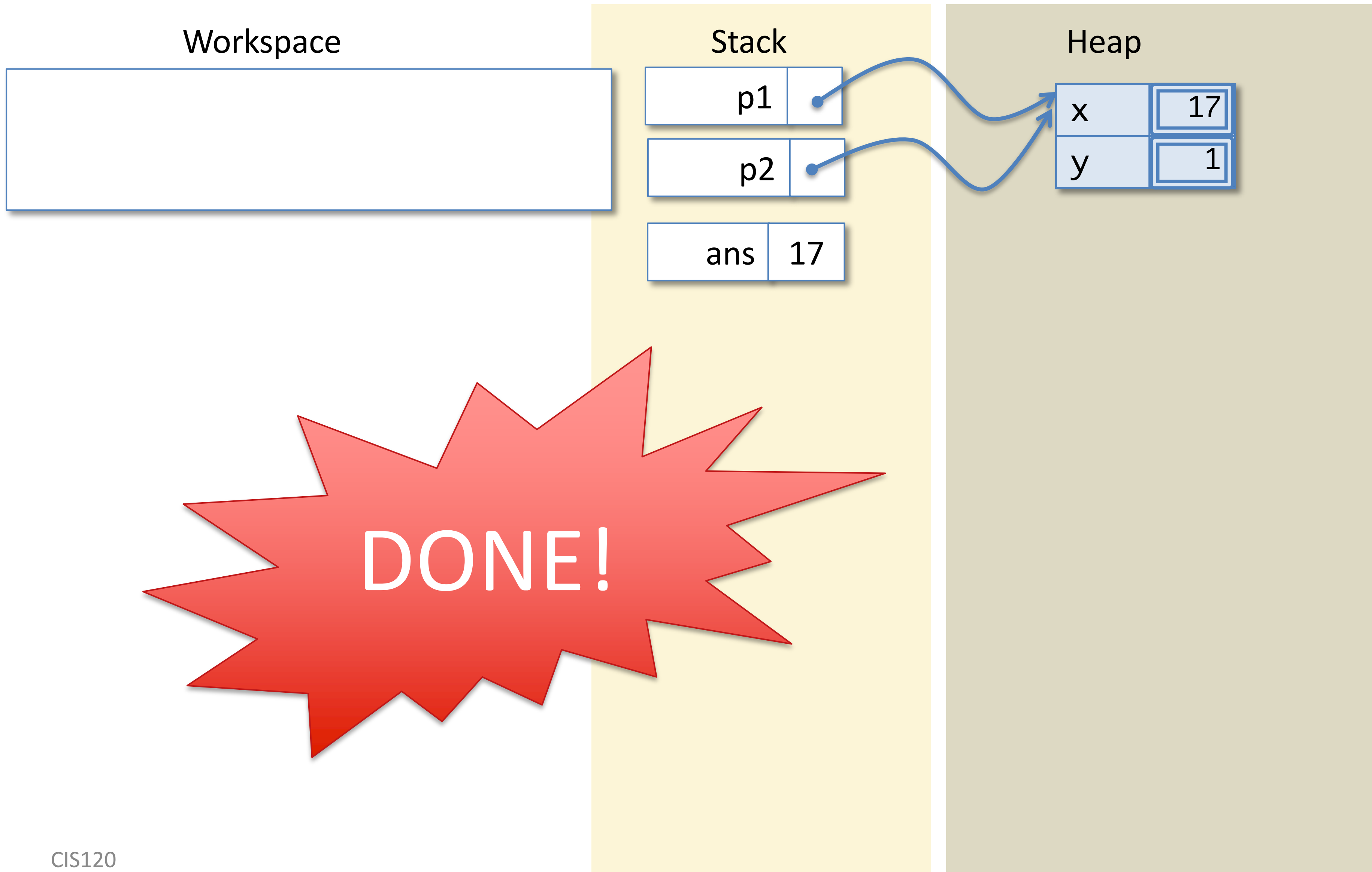
Let Expression

Workspace

```
let ans : int =
    17
```



Push ans

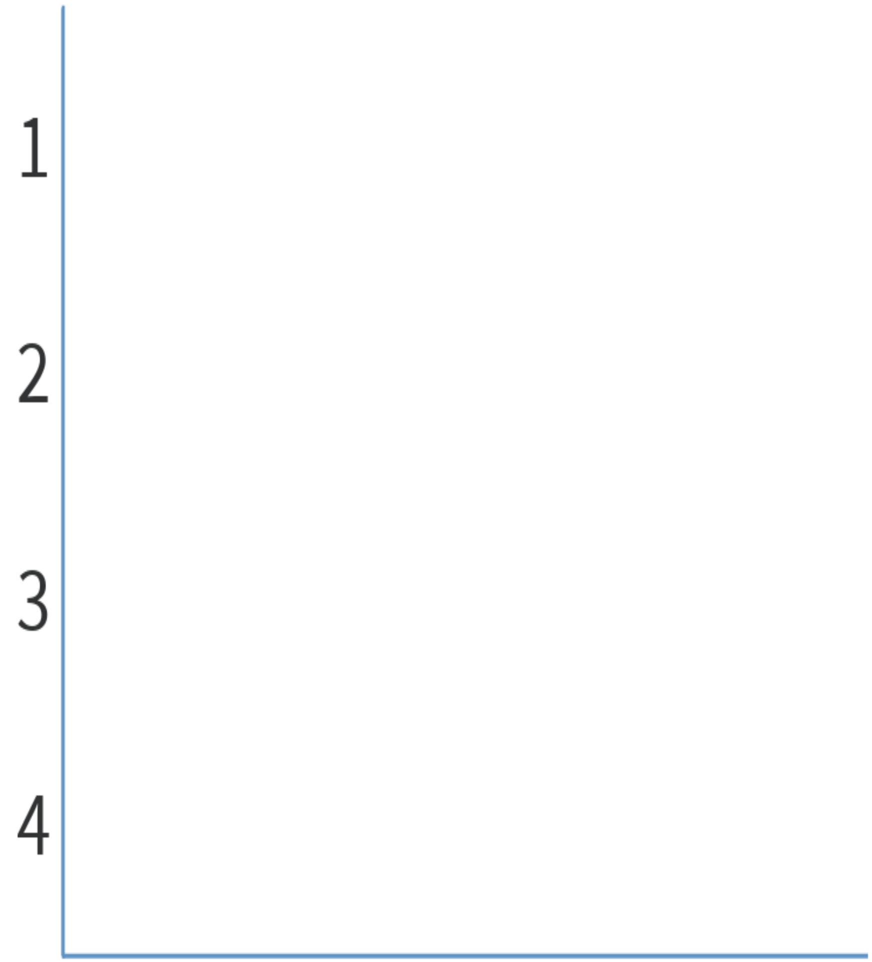


Simplifying code on the ASM

What does the Stack look like after simplifying the following code on the workspace?

```
let z = 20 in
let z = 2 + z in
  z
```

1. Stack	2. Stack	3. Stack	4. Stack
z 22	z 20	z 22	z 22
z 20	z 22		z 22



What does the Stack look like after simplifying the following code on the workspace?

```
let z = 20 in
let z = 2 + z in
  z
```

1. Stack

z	22
z	20

2. Stack

z	20
z	22

3. Stack

z	22

4. Stack

z	22
z	22

ANSWER: 2

References and Equality

= vs. ==

Reference Equality

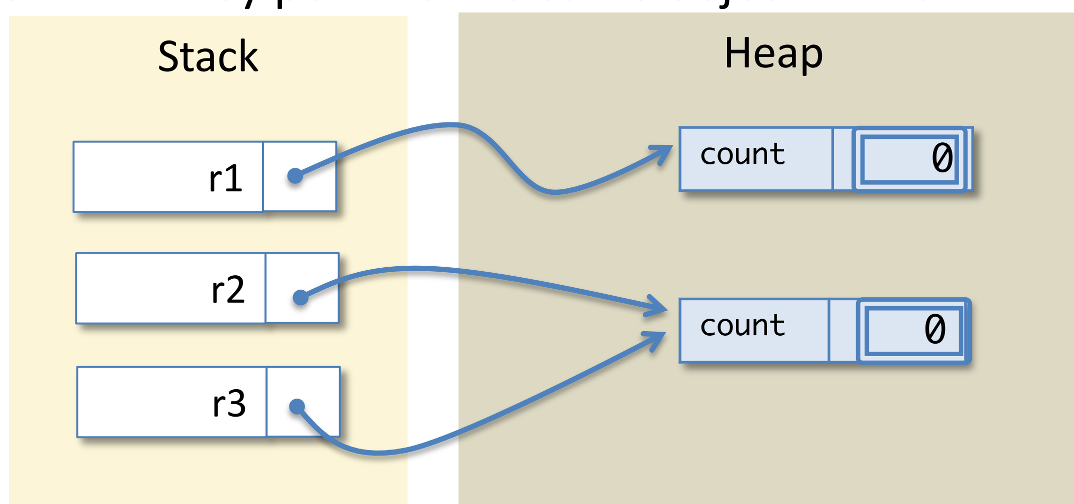
- Suppose we have two counters. How do we know whether they share the same internal state?
  - `type counter = { mutable count : int }`
  - We could increment one and see whether the other's value changes.
  - But we could also just test whether the references alias directly.

- Ocaml uses '==' to mean *reference* equality:
  - two reference values are '==' if they point to the same object in the heap; so:

```
r2 == r3

not (r1 == r2)

r1 = r2
```



Structural vs. Reference Equality

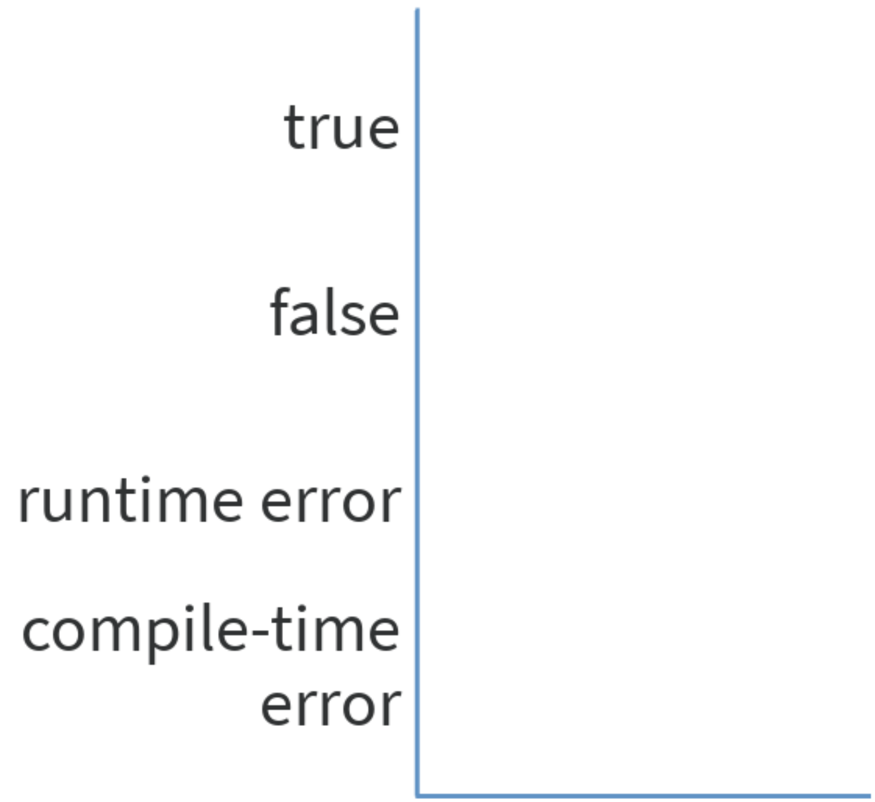
- *Structural (in)equality*: `v1 = v2` `v1 <> v2`
  - recursively traverses over the *structure* of the data, comparing the two values' components for structural equality
  - function values are never structurally equivalent to anything
  - structural equality can go into an infinite loop (on cyclic structures)
  - appropriate for comparing *immutable* datatypes

- *Reference (in)equality*: `v1 == v2` `v1 != v2`
  - Only looks at where the two references point in the heap
  - function values are only equal to themselves
  - equates strictly fewer things than structural equality
  - appropriate for comparing *mutable* datatypes

What is the result of evaluating the following expression?

```
let p1 : point = { x = 0; y = 0; } in
let p2 : point = p1 in

p1 = p2
```



What is the result of evaluating the following expression?

```
let p1 : point = { x = 0; y = 0; } in
let p2 : point = p1 in

p1 = p2
```

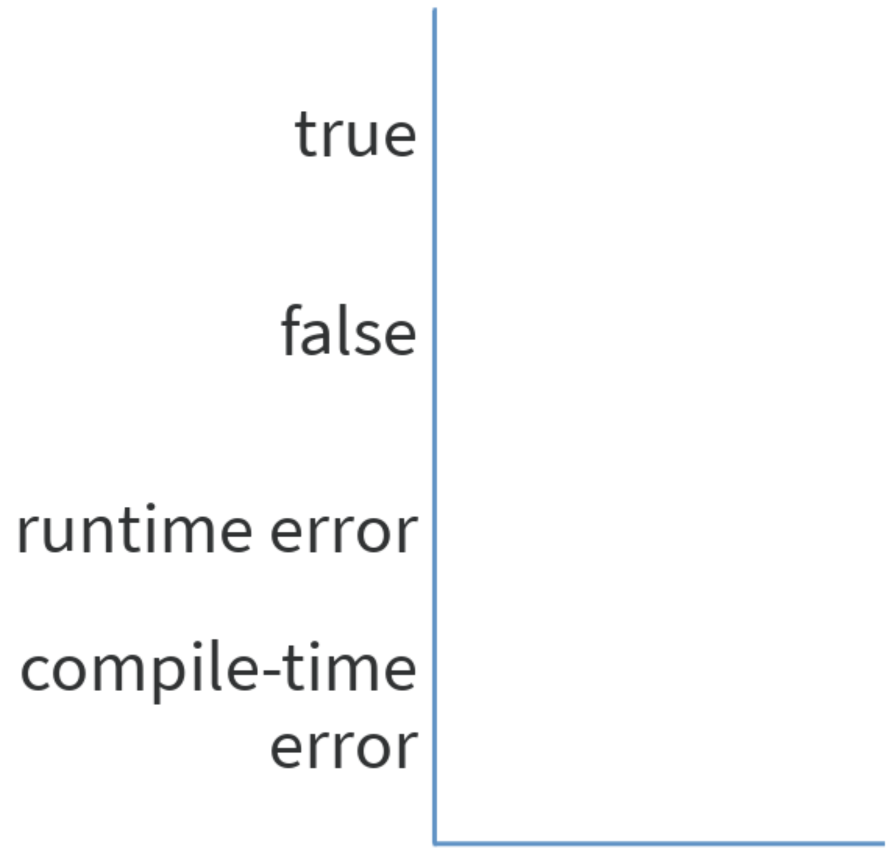
1. `true`
2. `false`
3. runtime error
4. compile-time error

Answer: true

What is the result of evaluating the following expression?

```
let p1 : point = { x = 0; y = 0; } in
let p2 : point = p1 in

p1 == p2
```



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p1 == p2
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Answer: true

What is the result of evaluating the following expression?

```
let p1 : point = { x = 0; y = 0; } in
let p2 : point = { x = 0; y = 0; } in

p1 == p2
```

1. `true`
2. `false`
3. runtime error
4. compile-time error

Answer: false

What is the result of evaluating the following expression?

```
let p1 : point = { x = 0; y = 0; } in
let p2 : point = { x = 0; y = 0; } in
let l1 : point list = [p1] in
let l2 : point list = [p2] in

l1 = l2
```

1. `true`
2. `false`
3. runtime error
4. compile-time error

Answer: true

What is the result of evaluating the following expression?

```
let p1 : point = { x = 0; y = 0; } in
let p2 : point = p1 in
let l1 : point list = [p1] in
let l2 : point list = [p2] in

l1 == l2
```

1. `true`
2. `false`
3. runtime error
4. compile-time error

Answer: false

Ah… Refs!

OCaml provides syntax for working with updatable *references*:

```
type 'a ref = {mutable contents:'a}
```

OCaml "syntactic sugar"		equivalent expressions	type constraints
`ref e`	means	`{contents = e}`	has type `t ref` when `(e : t)`
`e1 := e2`	means	`(e1).contents <- e2`	has type `unit` when `(e1 : t ref)` and `(e2 : t)`
`!e`	means	`(e).contents`	has type `t` when `(e : t ref)`

"means": "is defined to be" (not Ocaml syntax)

Comparison To Java (or C, C++, …)

Java

```
int f() {
  int x = 3;
  x = x + 1;
  return x;
}
```

- x has type int
- meaning on left of = different than on right
- *implicit* dereference

OCaml

```
let f () : int =
  let x = ref 3 in
  x := !x + 1;
  !x
```

- x has type (int ref)
- use := for update
- *explicit* dereference

ASM:
Simplifying lists and user-defined datatypes using the heap

Simplification

Workspace

```
1::2::3::[]
```

Stack

Heap

For uniformity, we'll pretend lists are declared like this:

```
type ‘a list =
  | Nil
  | Cons of ‘a * ‘a list
```

Simplification

Workspace

```
Cons (1,Cons (2,Cons (3,Nil)))
```

Stack

Heap

For uniformity, we'll pretend lists are declared like this:

```
type ‘a list =
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Simplification

Workspace

```
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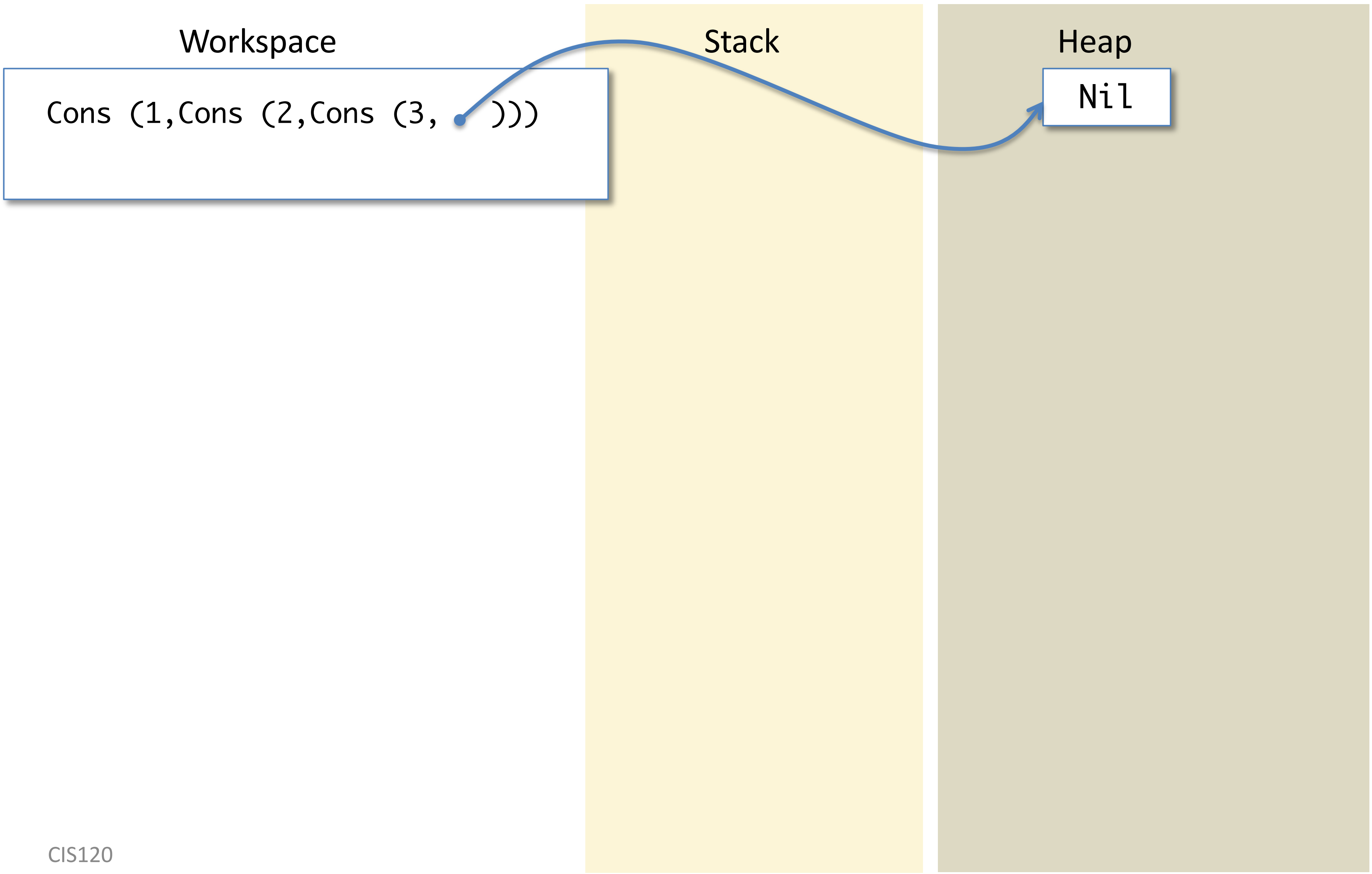
```

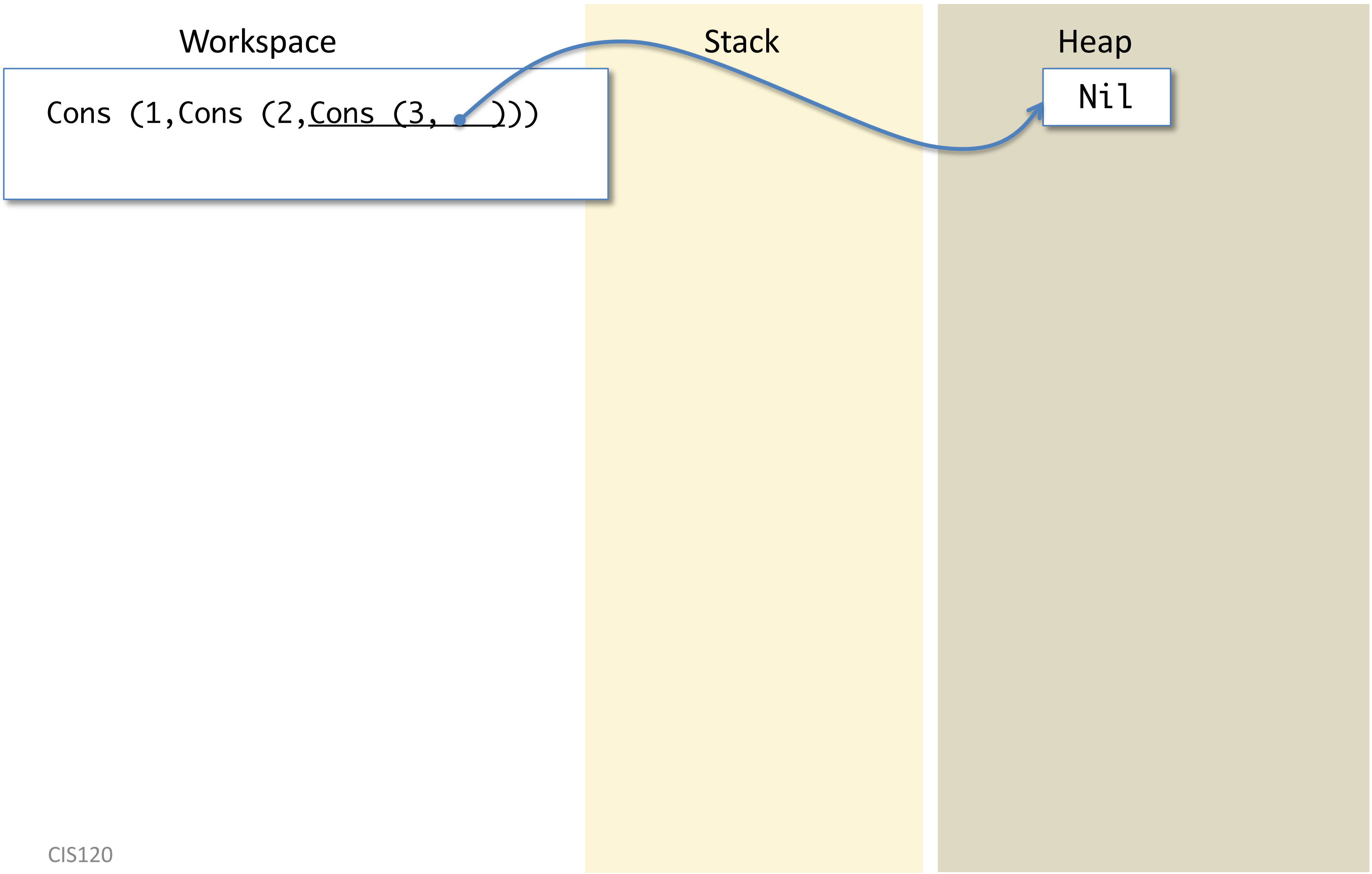
Stack

Heap

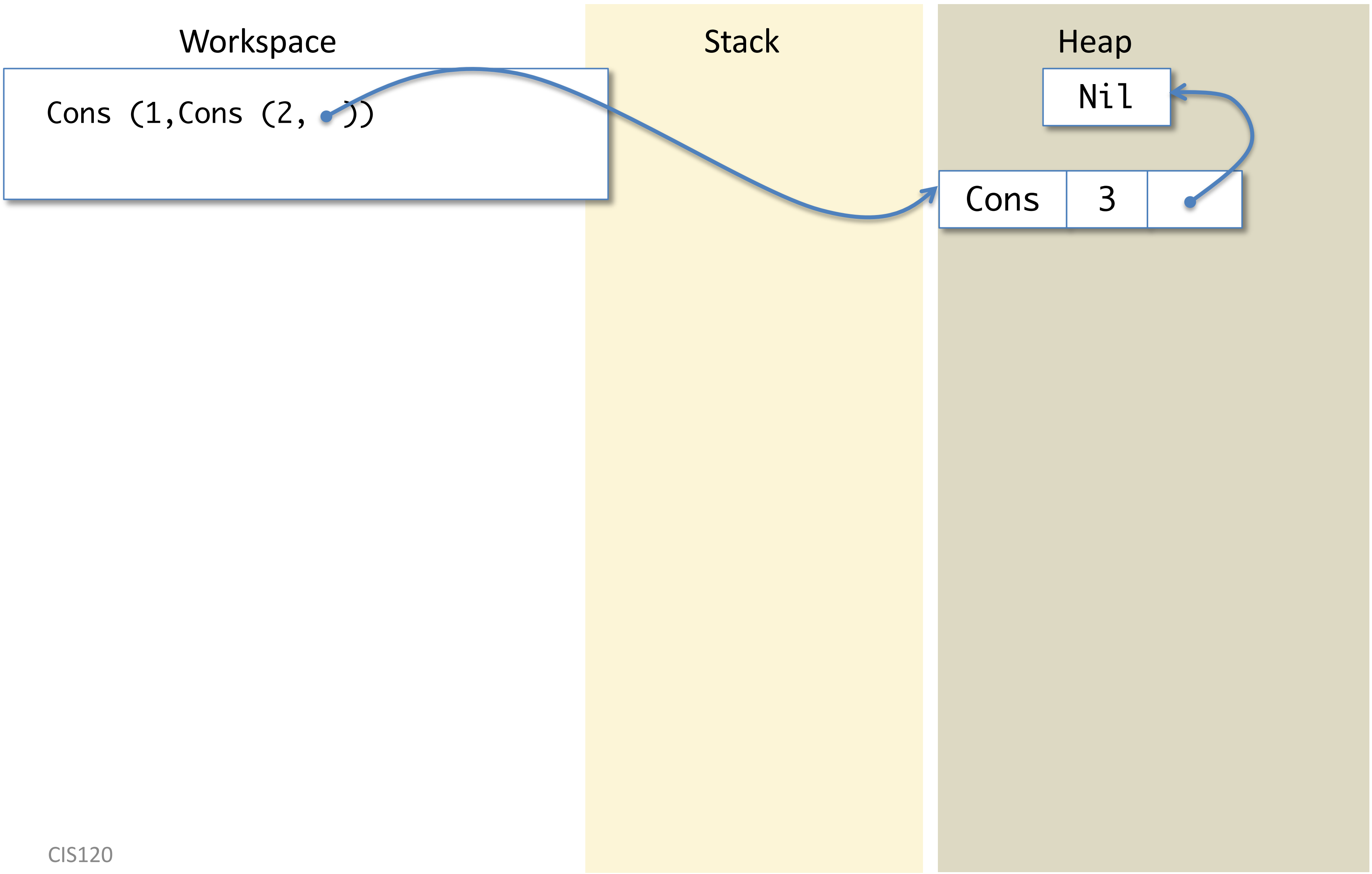
Simplification



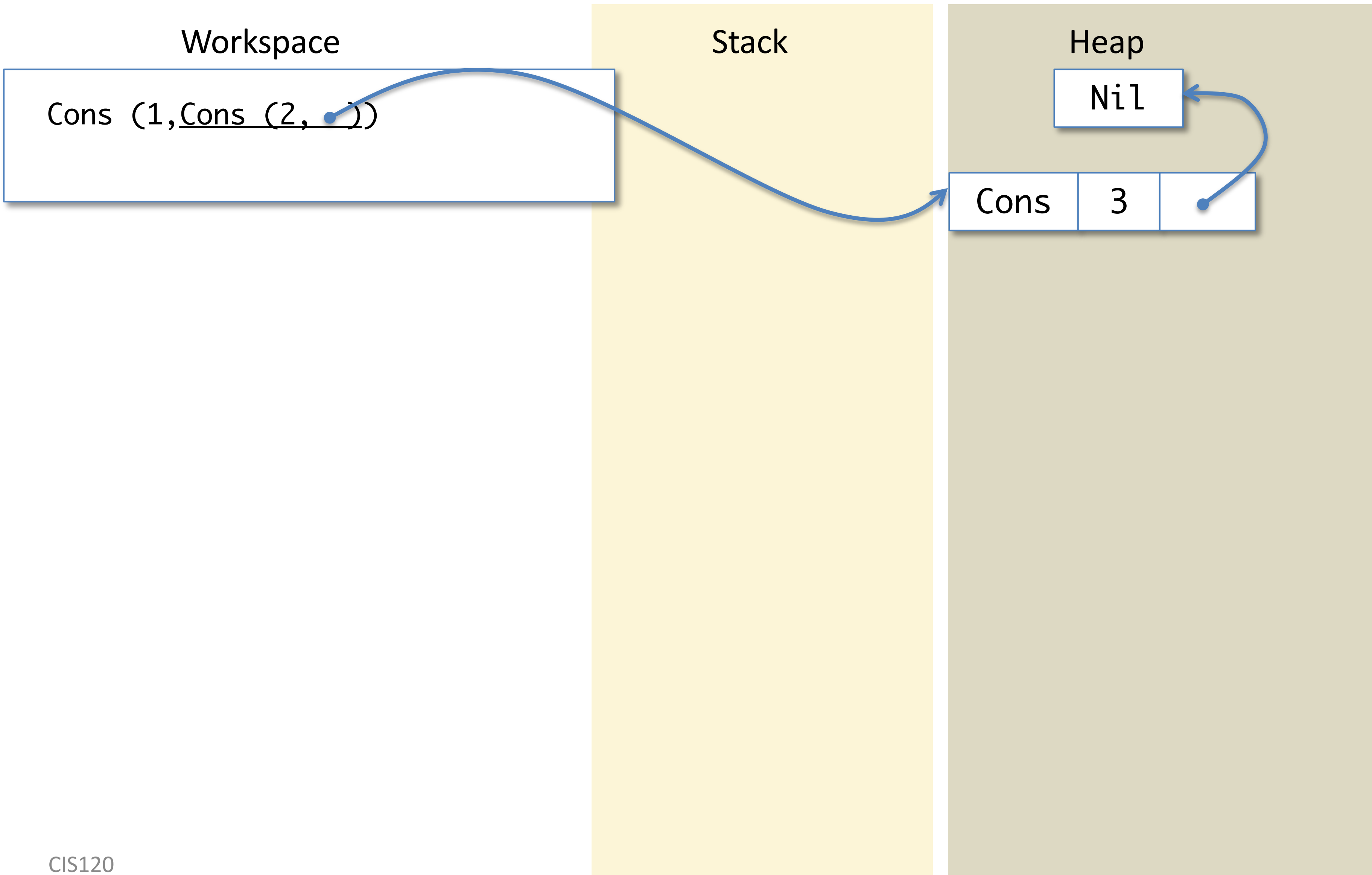
Simplification



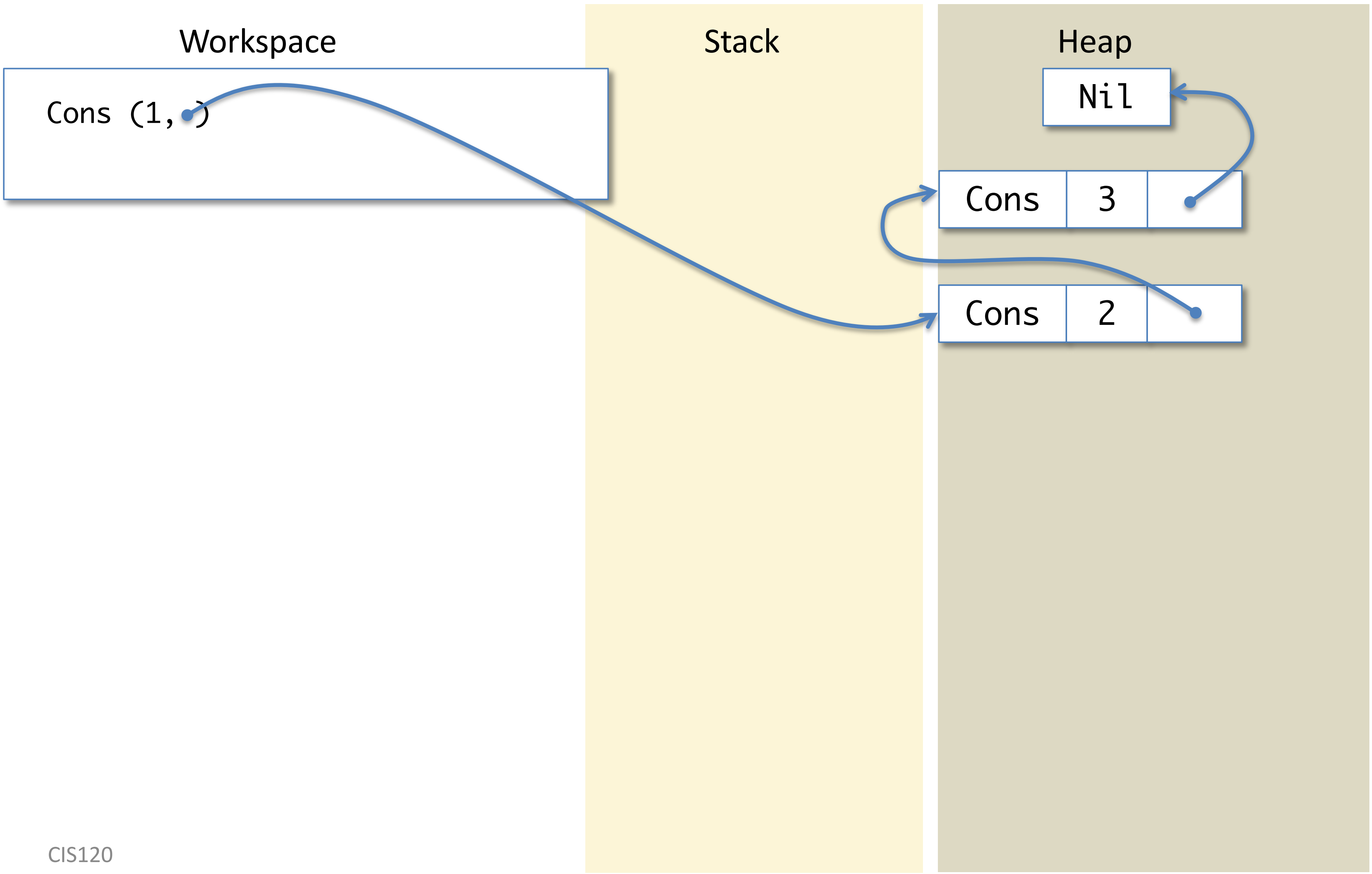
Simplification



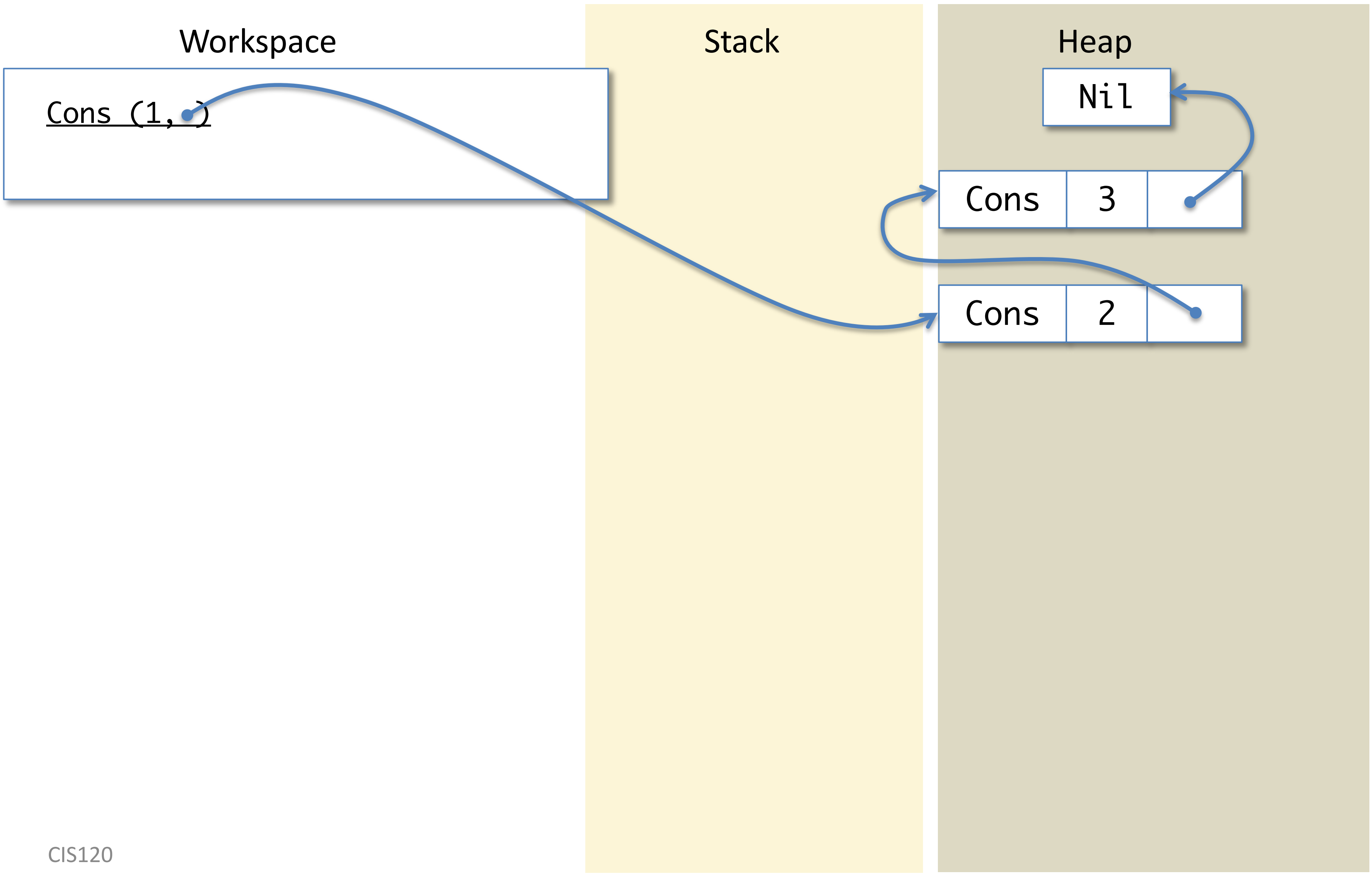
Simplification



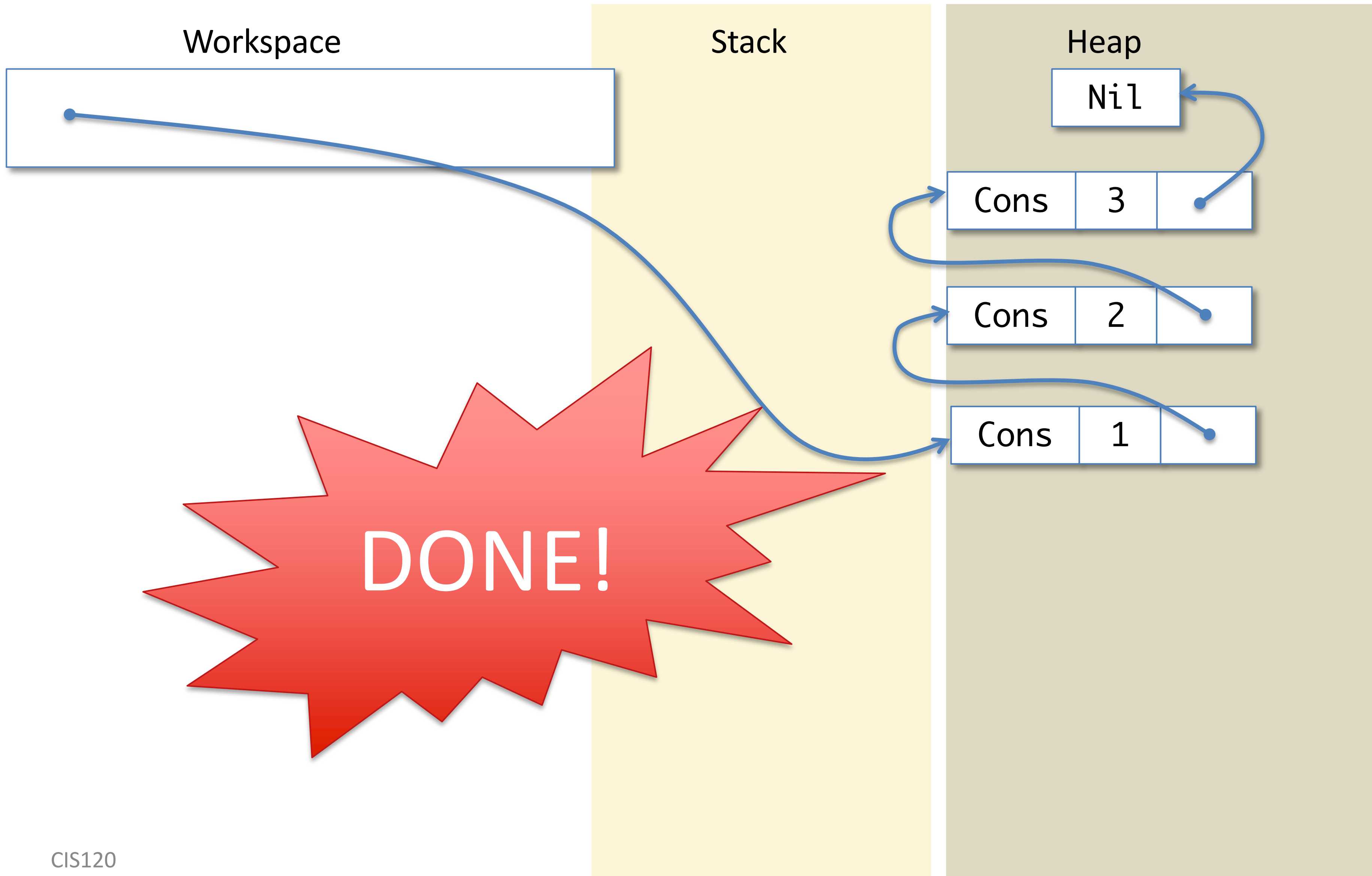
Simplification



Simplification



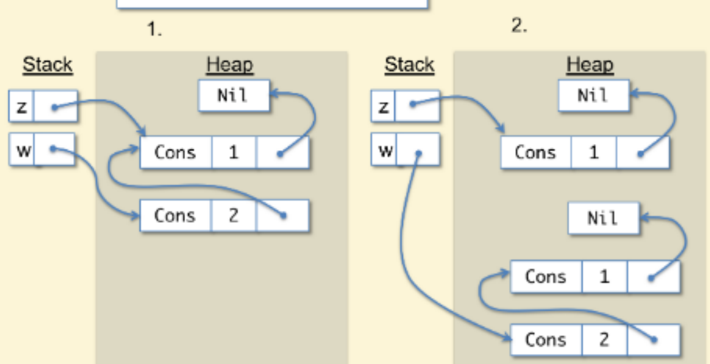
Simplification

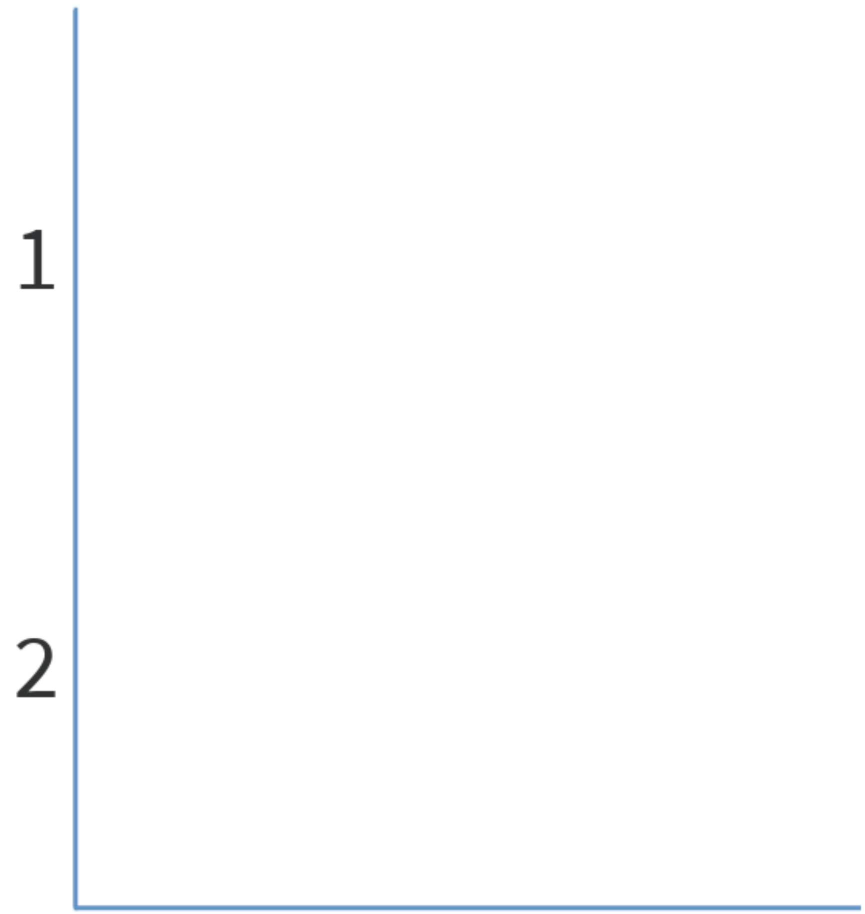


Simplifying code on the ASM

What do the Stack and Heap look like after simplifying the following code on the workspace?

```
let z = Cons (1, Nil) in
let w = Cons (2, z) in
  w
```





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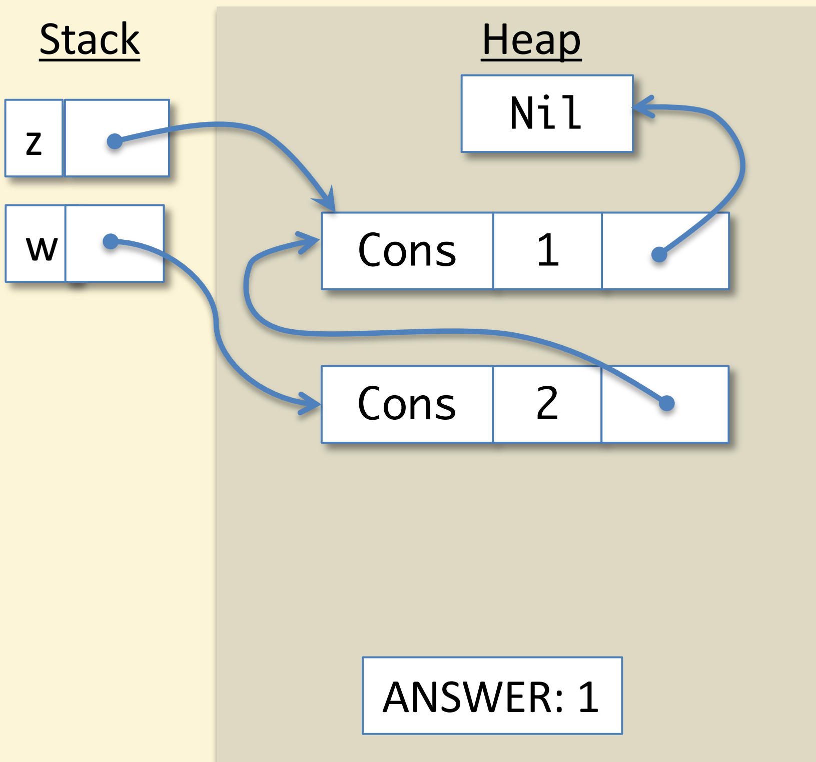
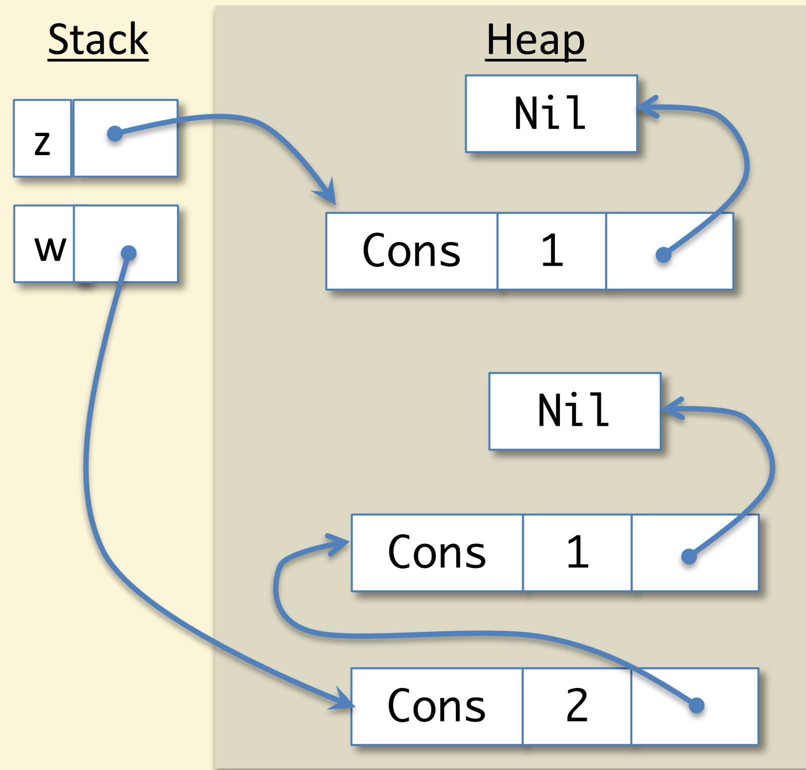
    w
```

1.



2.



ASM:
Simplifying functions

Function Simplification

Workspace

```
let add1 (x : int) : int =
  x + 1 in
add1 (add1 0)
```

Stack

Heap

Function Simplification

Workspace

```
let add1 (x : int) : int =
   x + 1 in
add1 (add1 0)
```

First step: replace declaration of add1 with more primitive version

Stack

Heap

Function Simplification

Workspace

```
let add1 : int ->  int =
  fun (x:int) -> x + 1 in
add1 (add1 0)
```

Stack

Heap

Function Simplification

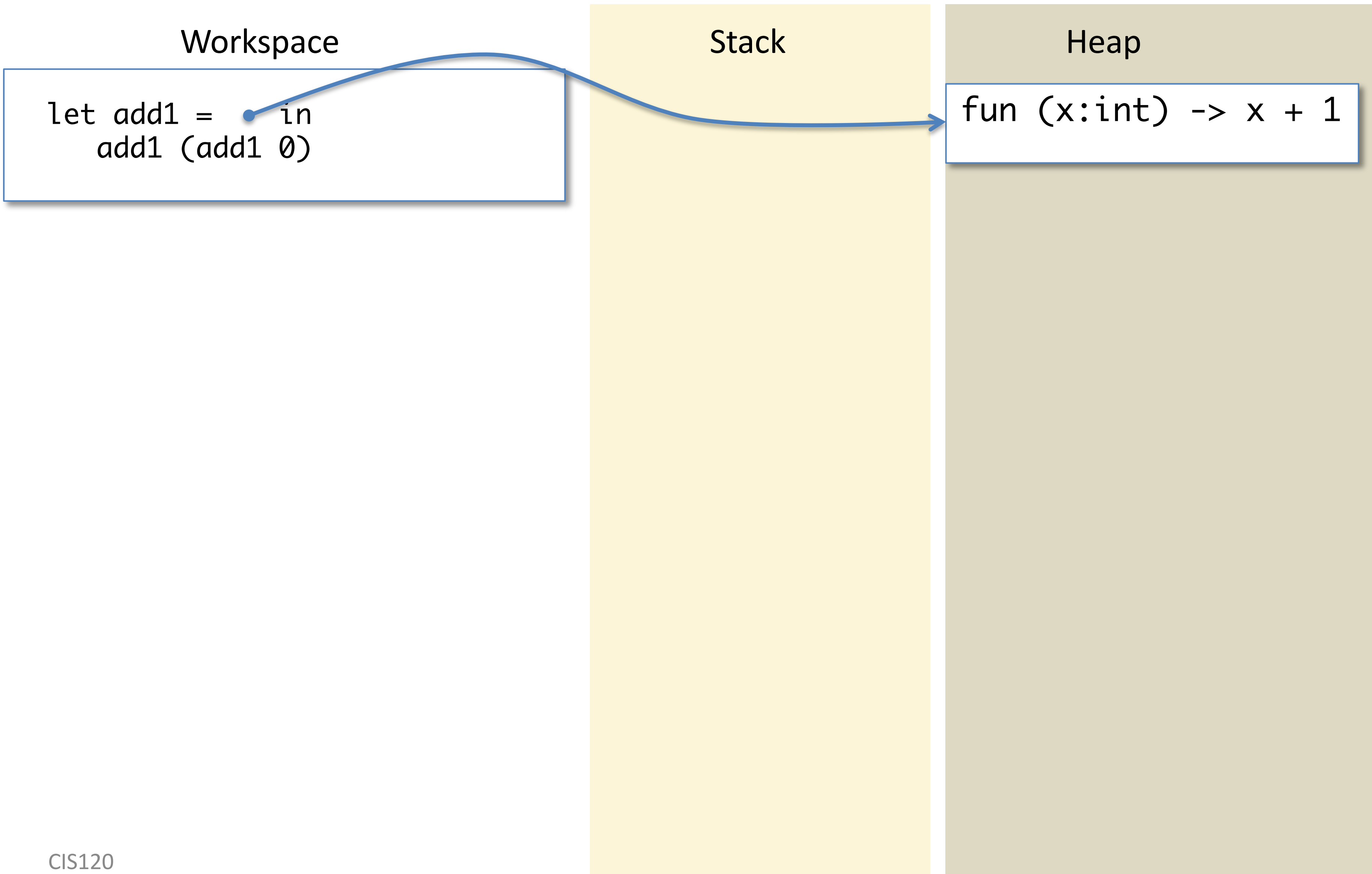
Workspace

```
let add1 : int ->  int =
  fun (x:int) -> x + 1 in
add1 (add1 0)
```

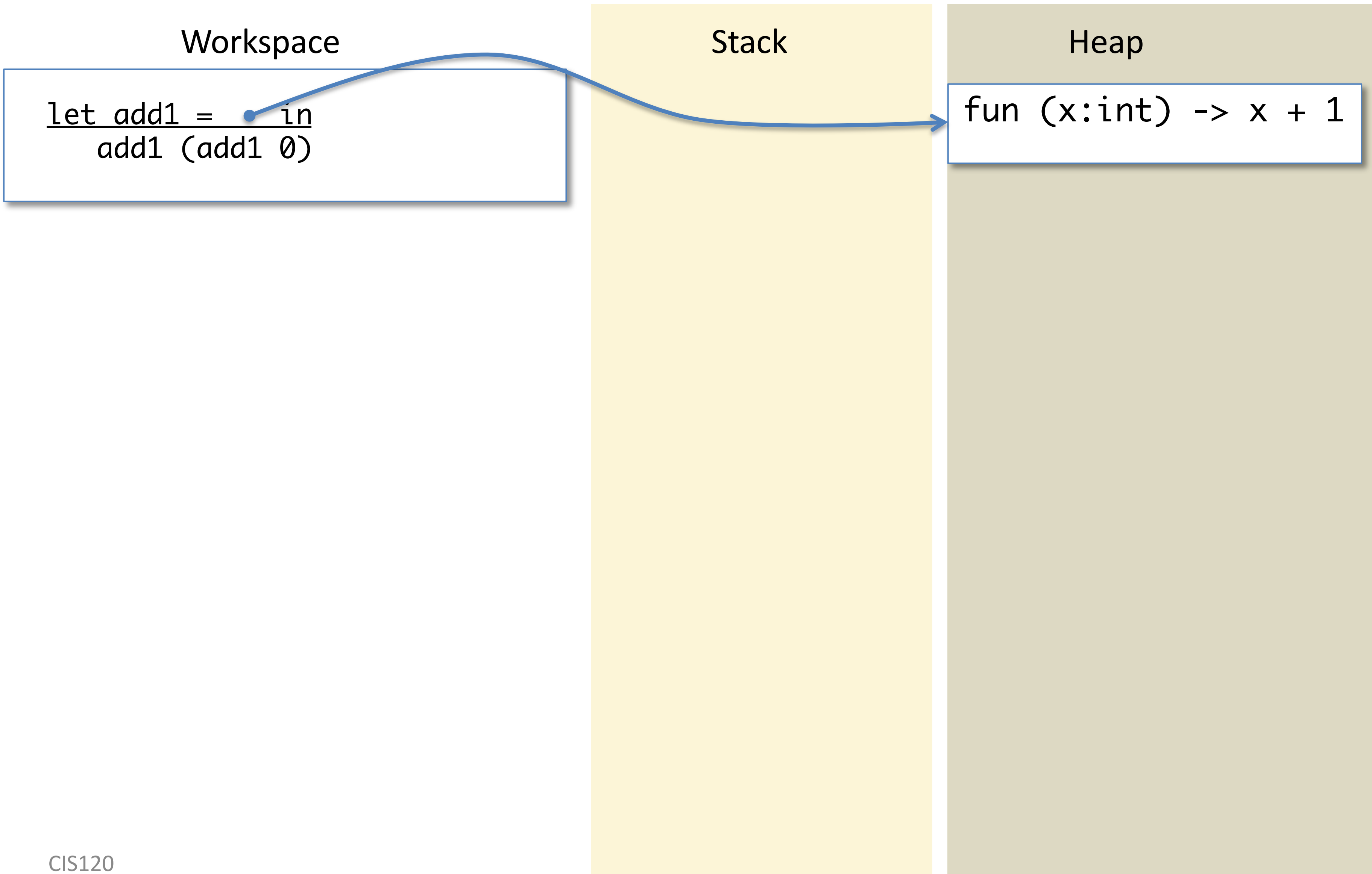
Stack

Heap

Function Simplification



Function Simplification



Function Simplification

Workspace

```
add1 (add1 0)
```

Stack

add1 | •

Heap

```
fun (x:int) -> x + 1
```

Function Simplification

Workspace

`add1 (add1 0)`

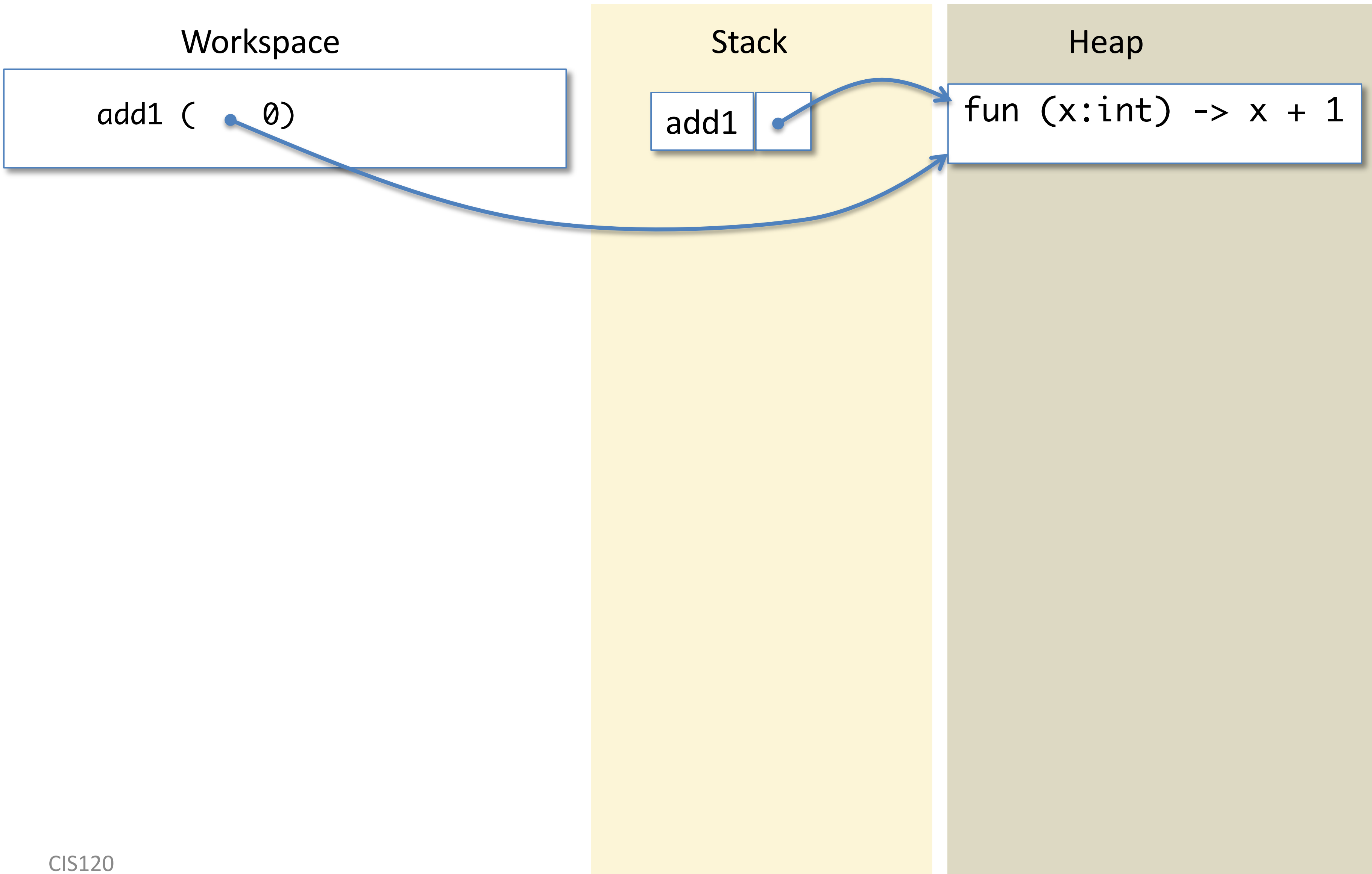
Stack

add1	•

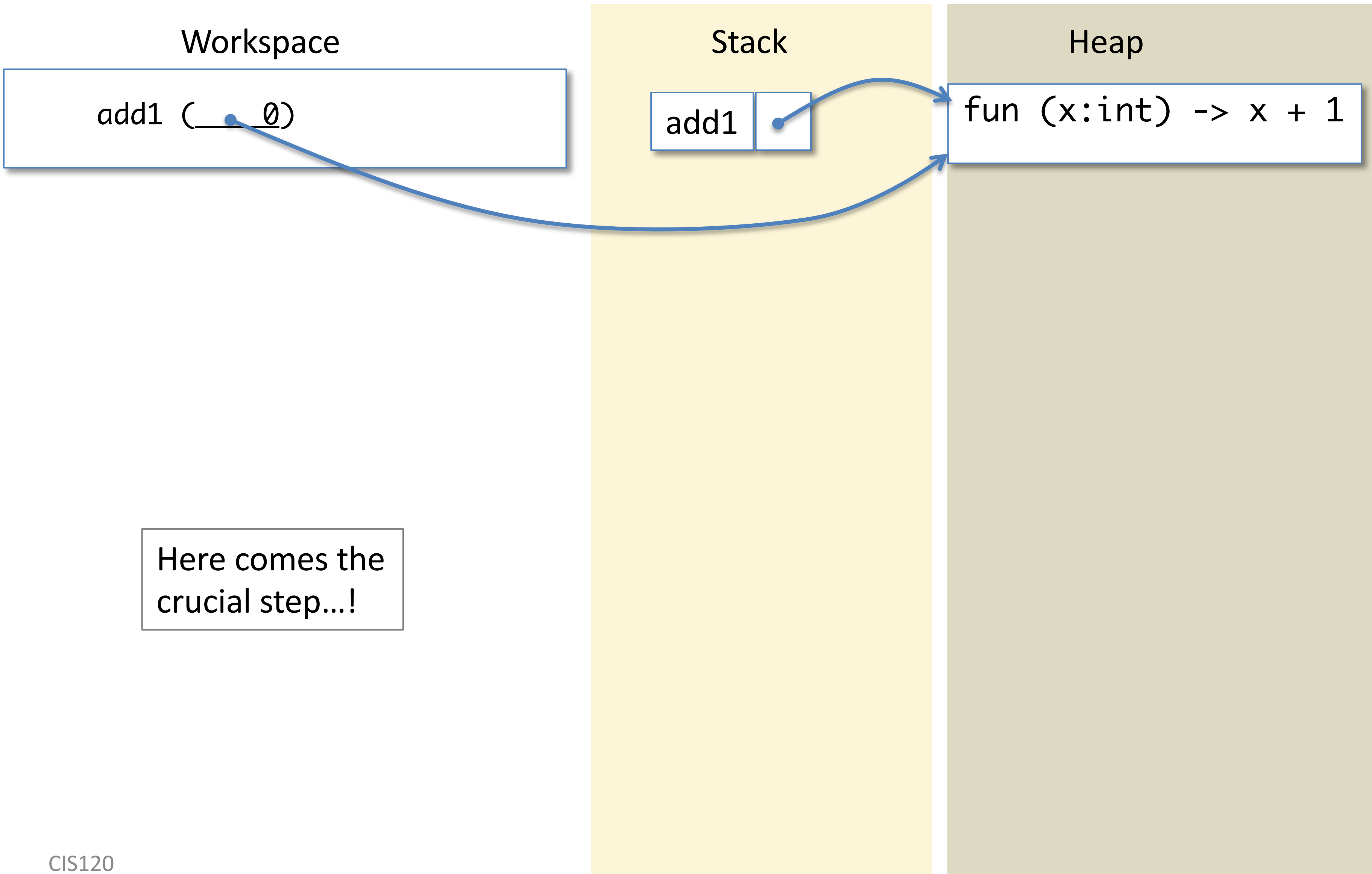
Heap

`fun (x:int) -> x + 1`

Function Simplification

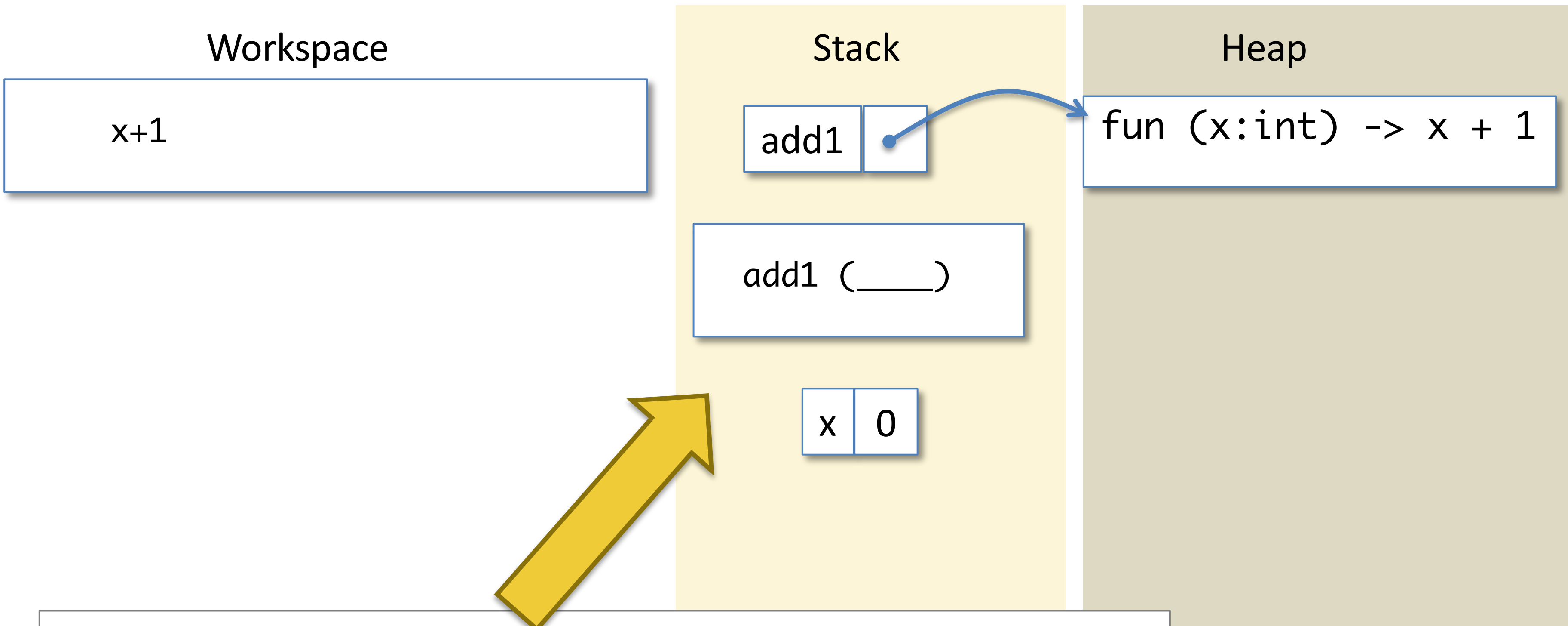


Function Simplification



Here comes the crucial step…!

Do the Call, Saving the Workspace

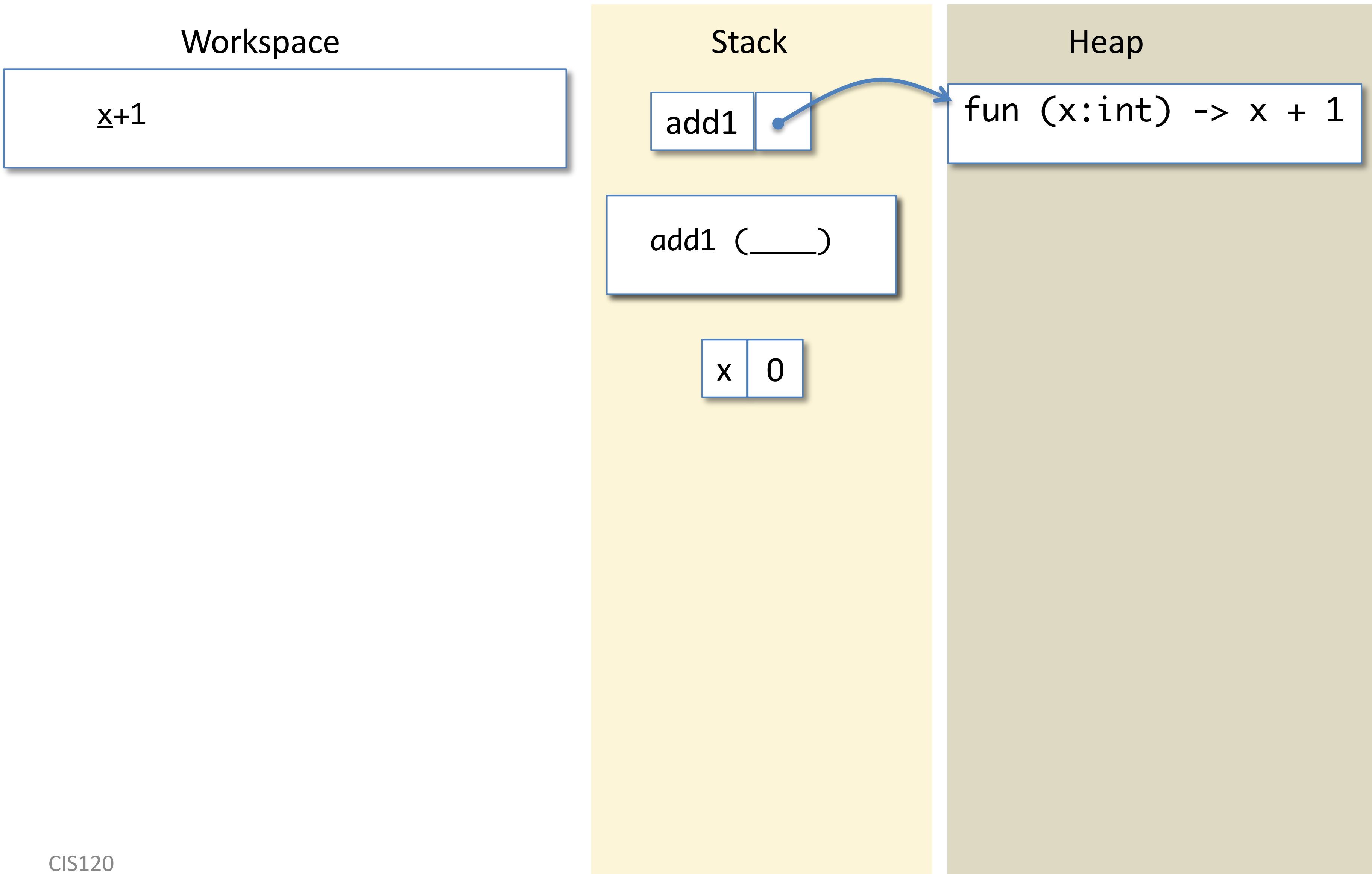


Note the saved workspace and pushed function argument

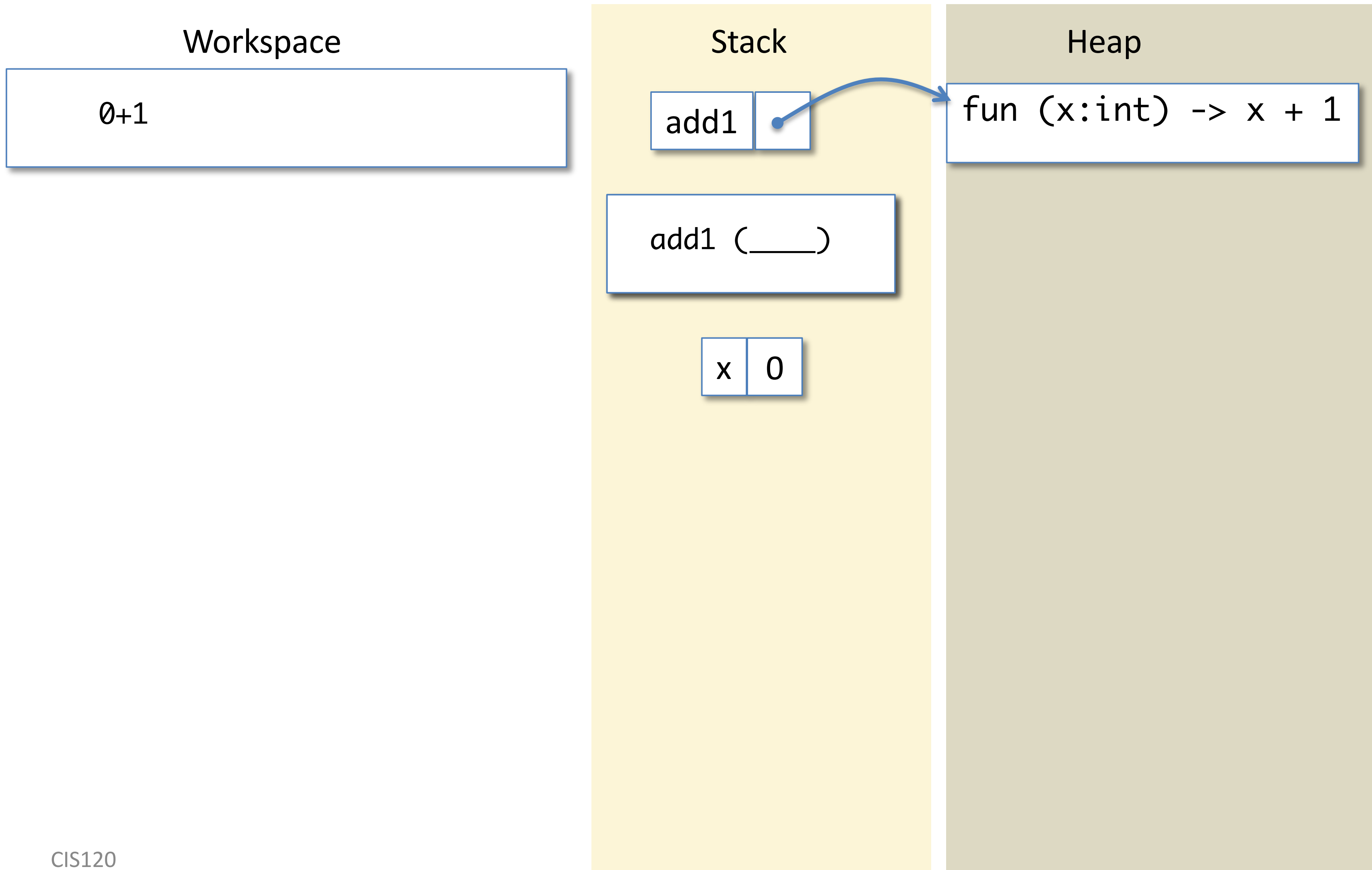
- compare with the workspace on the previous slide
- the name ‘x’ comes from the *parameter* name in the heap

The new workspace contains the *body* of the function

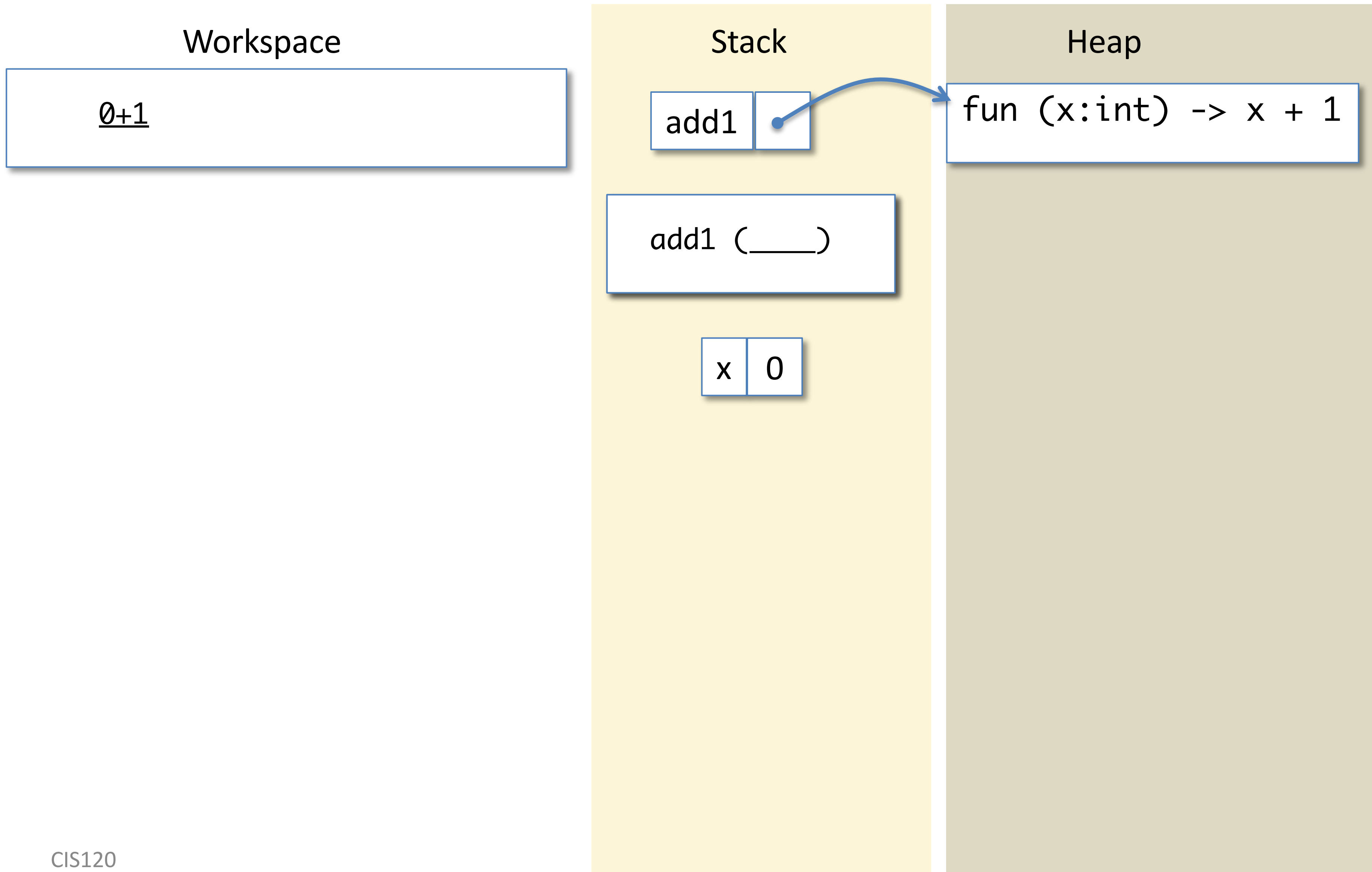
Function Simplification



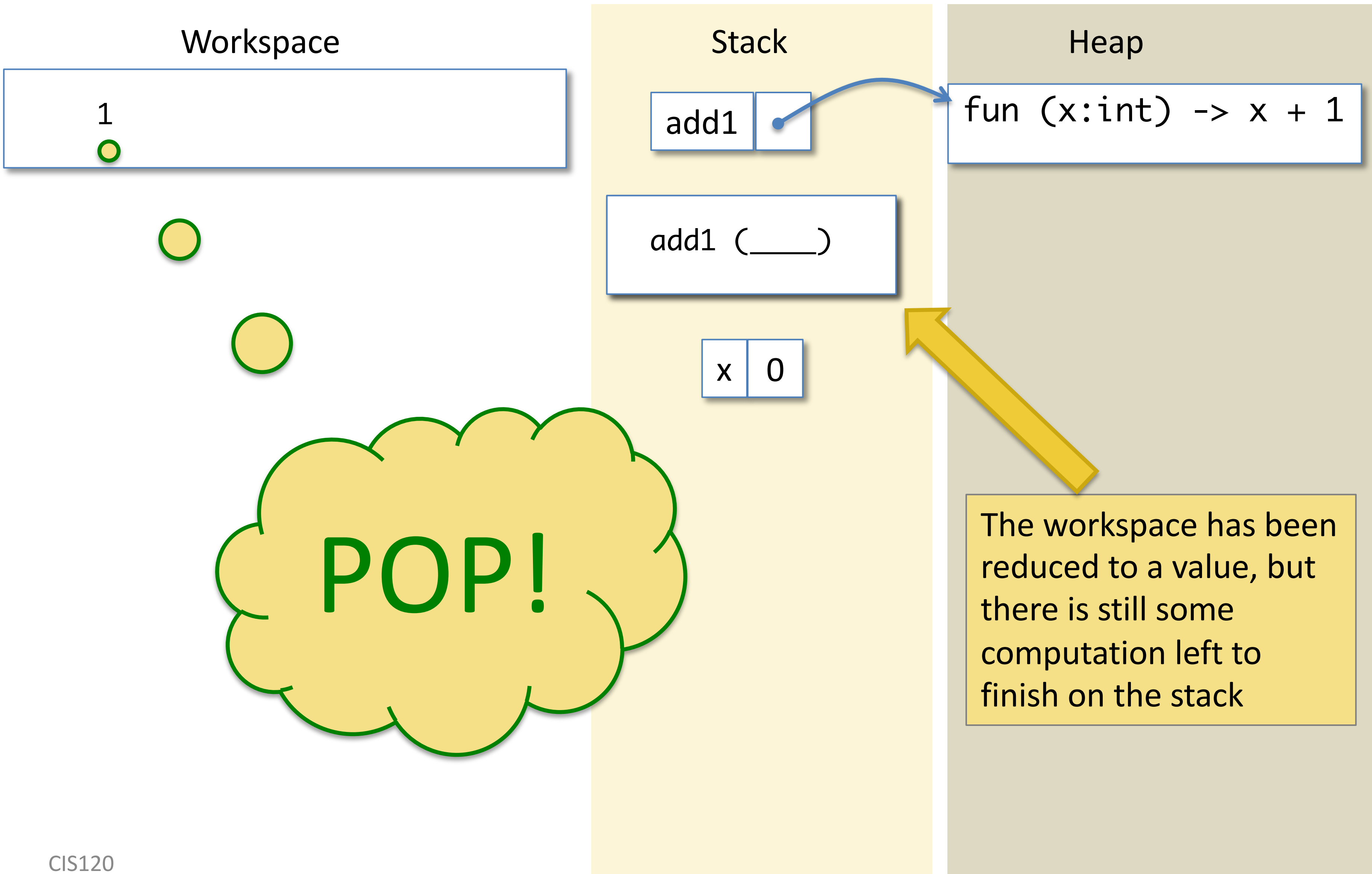
Function Simplification



Function Simplification



Function Simplification



Function Simplification

Workspace

```
add1 1
```

Stack

add1 | •

Heap

```
fun (x:int) -> x + 1
```

See how the ASM *restored* the saved workspace, replacing its `hole’ with the value computed into the old workspace. (Compare with previous slide.)

Function Simplification

Workspace

add1 1

Stack

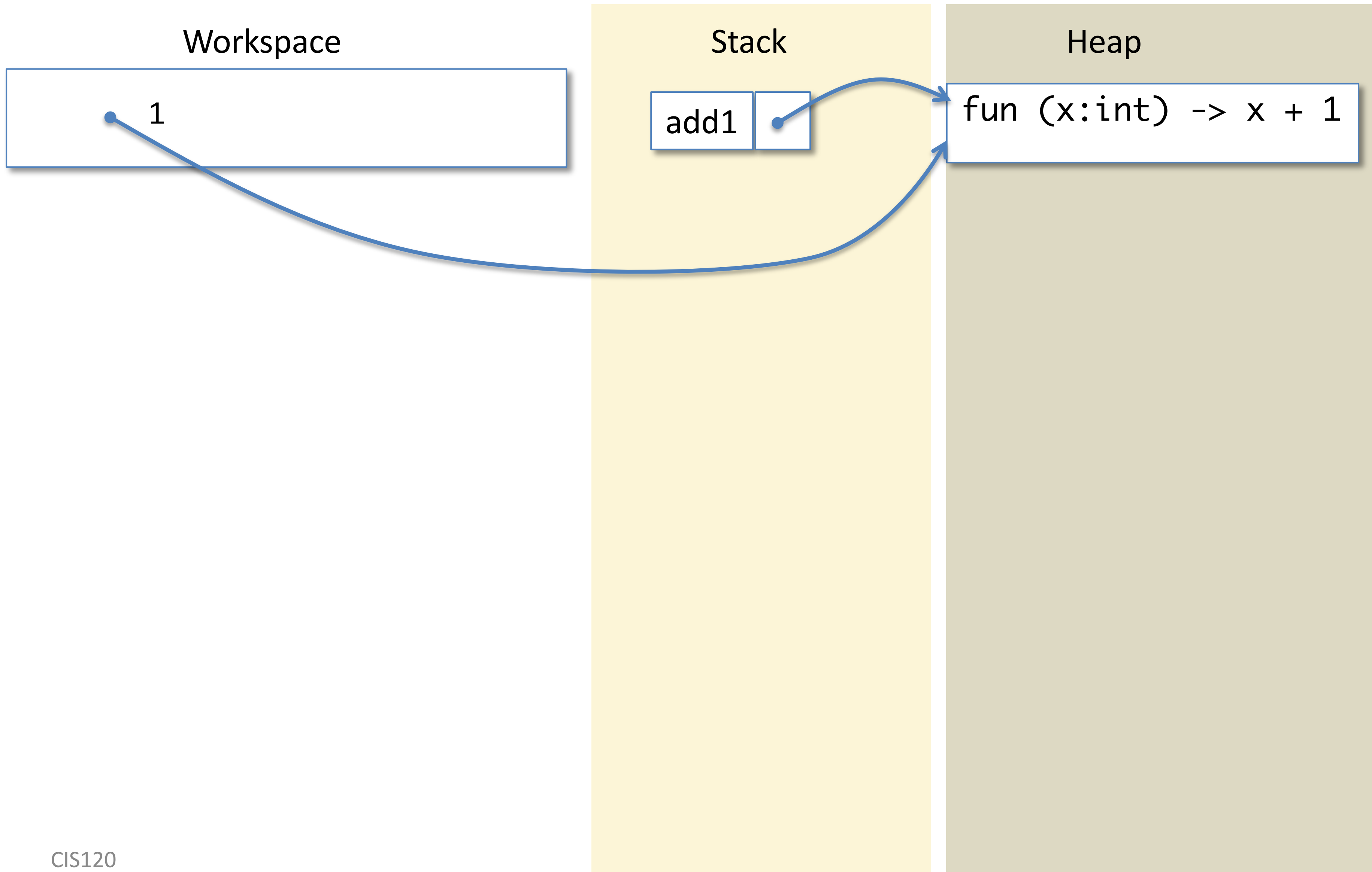
add1

Heap

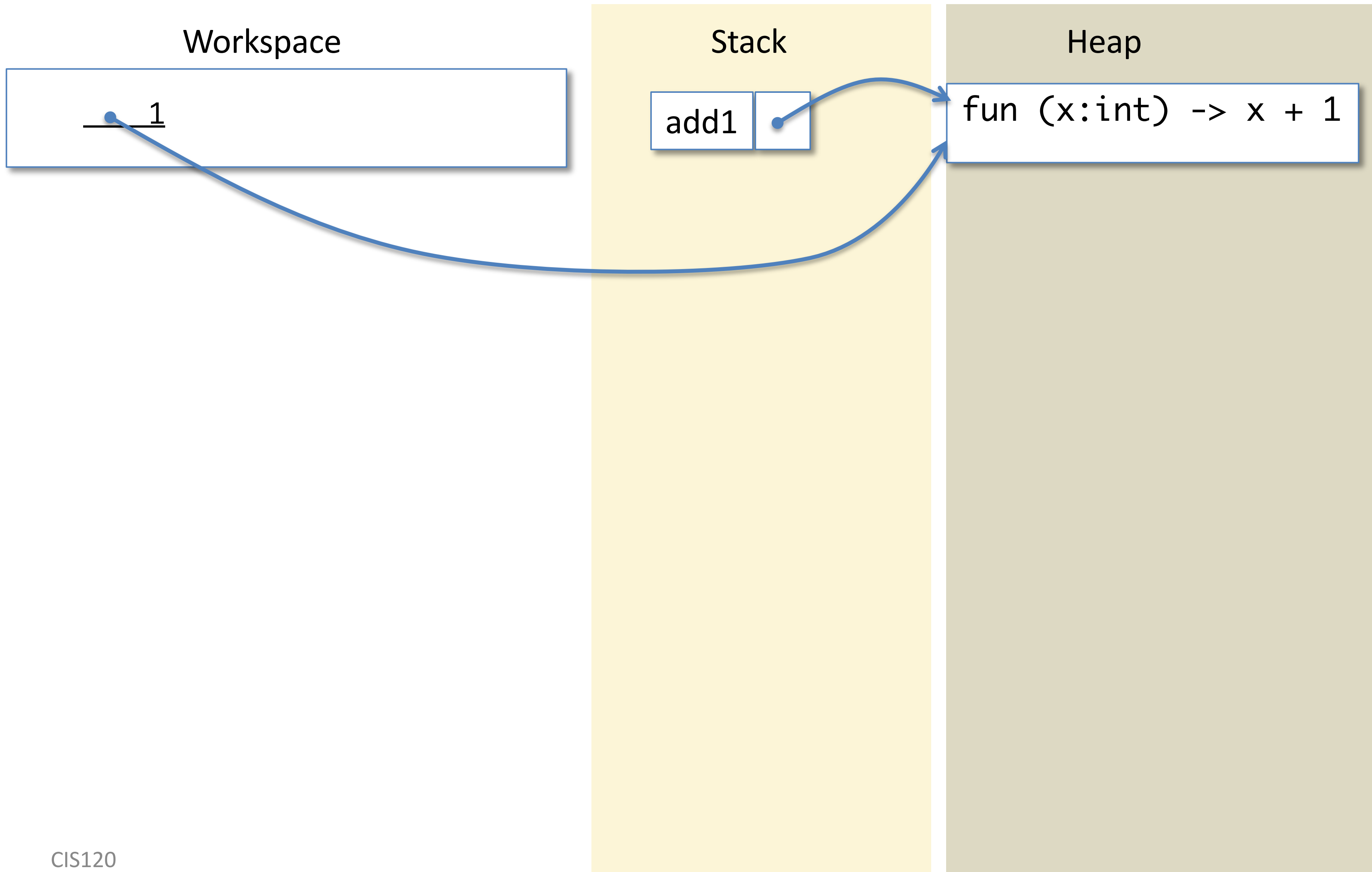
`fun (x:int) -> x + 1`

Now we have to do it all over again for the second invocation of add1…

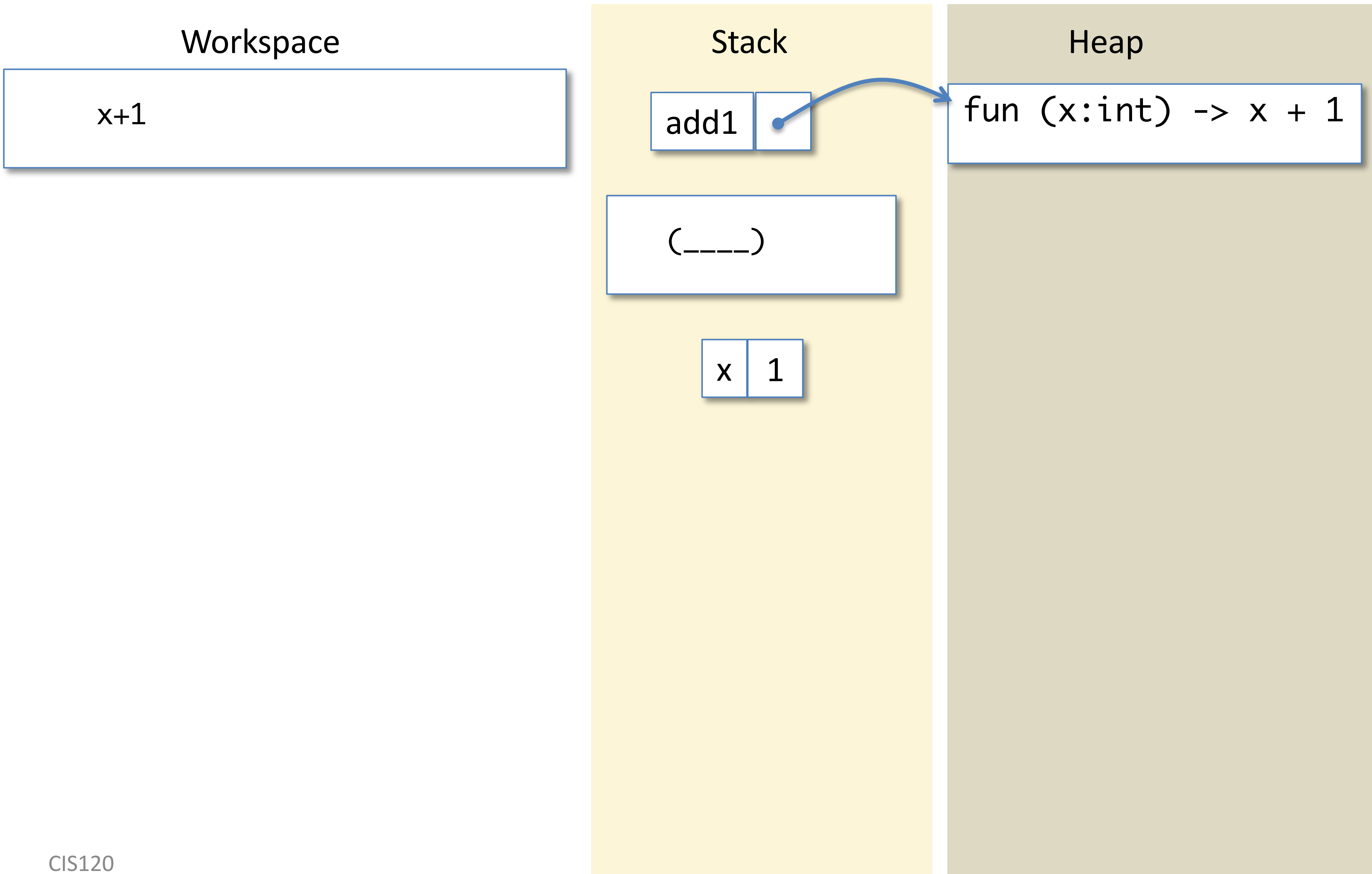
Function Simplification



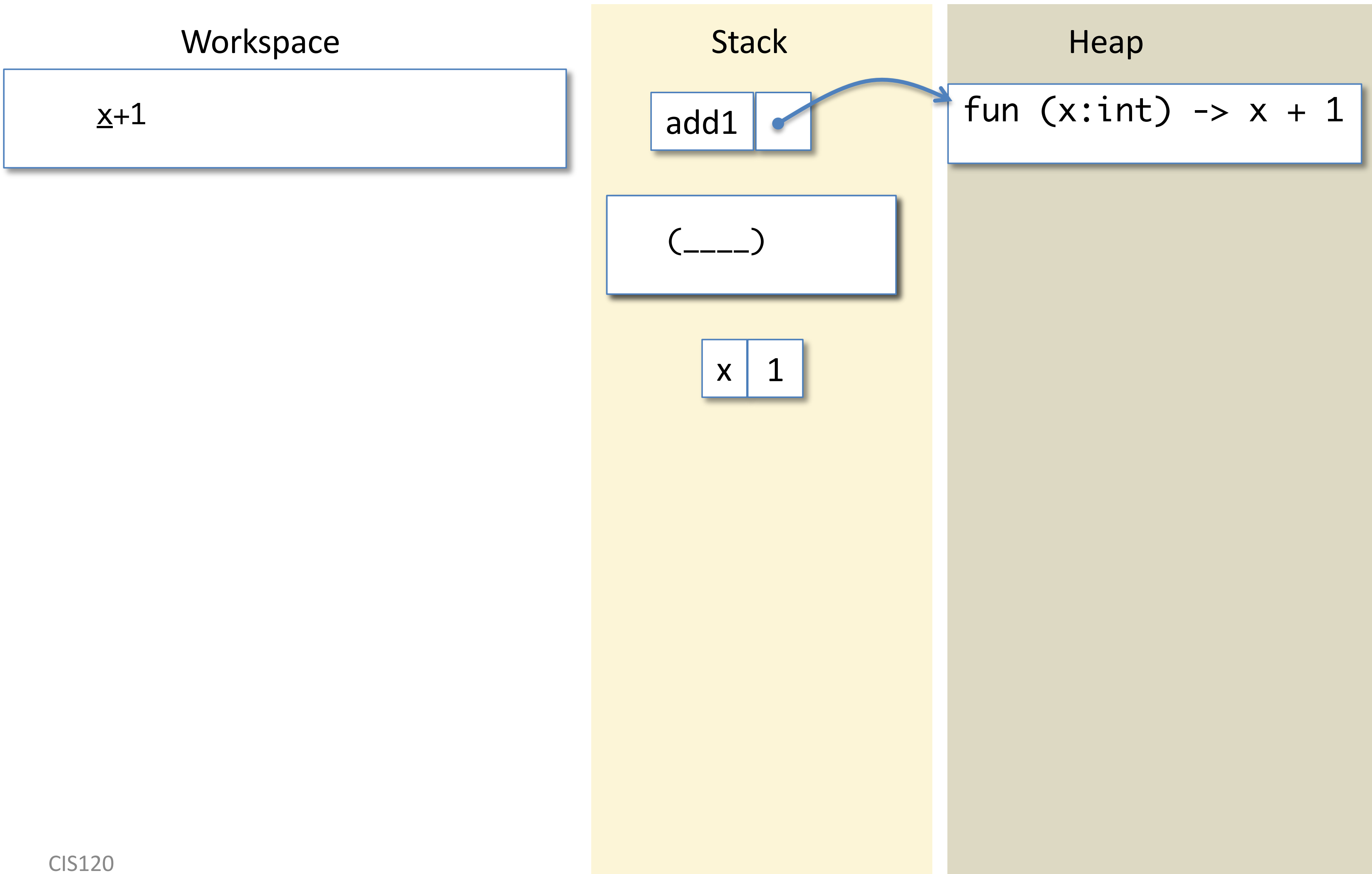
Function Simplification



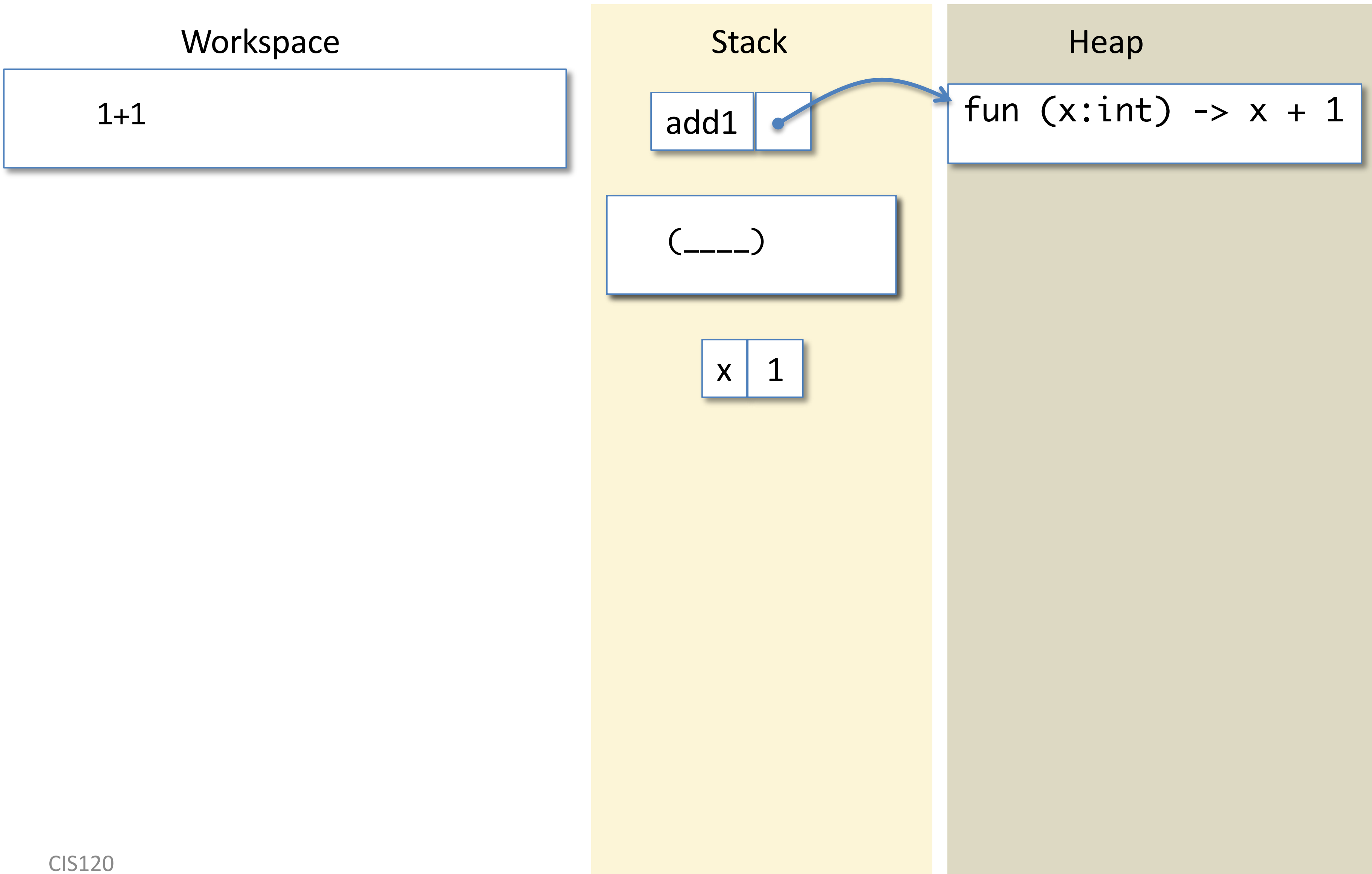
Function Simplification



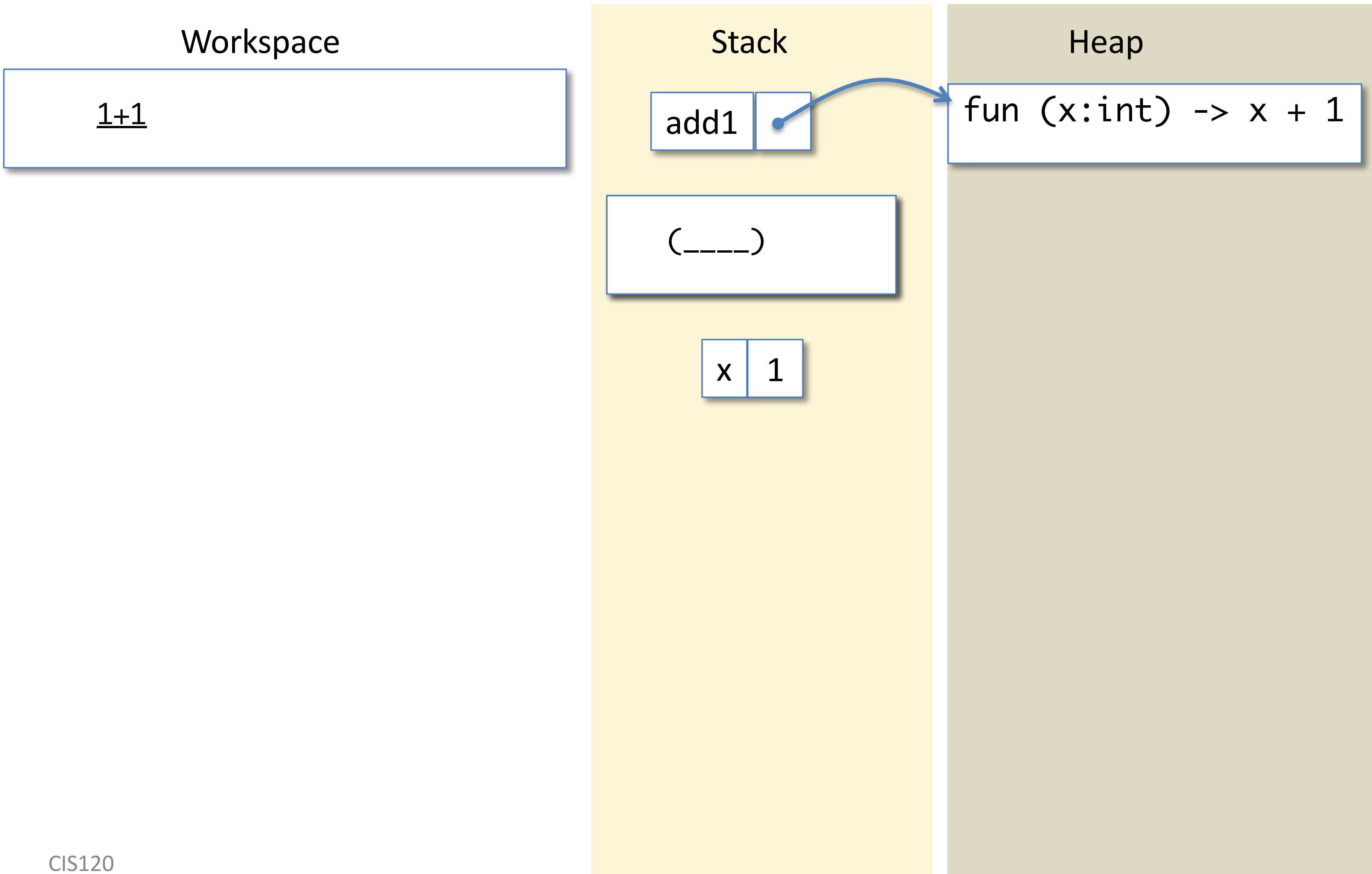
Function Simplification



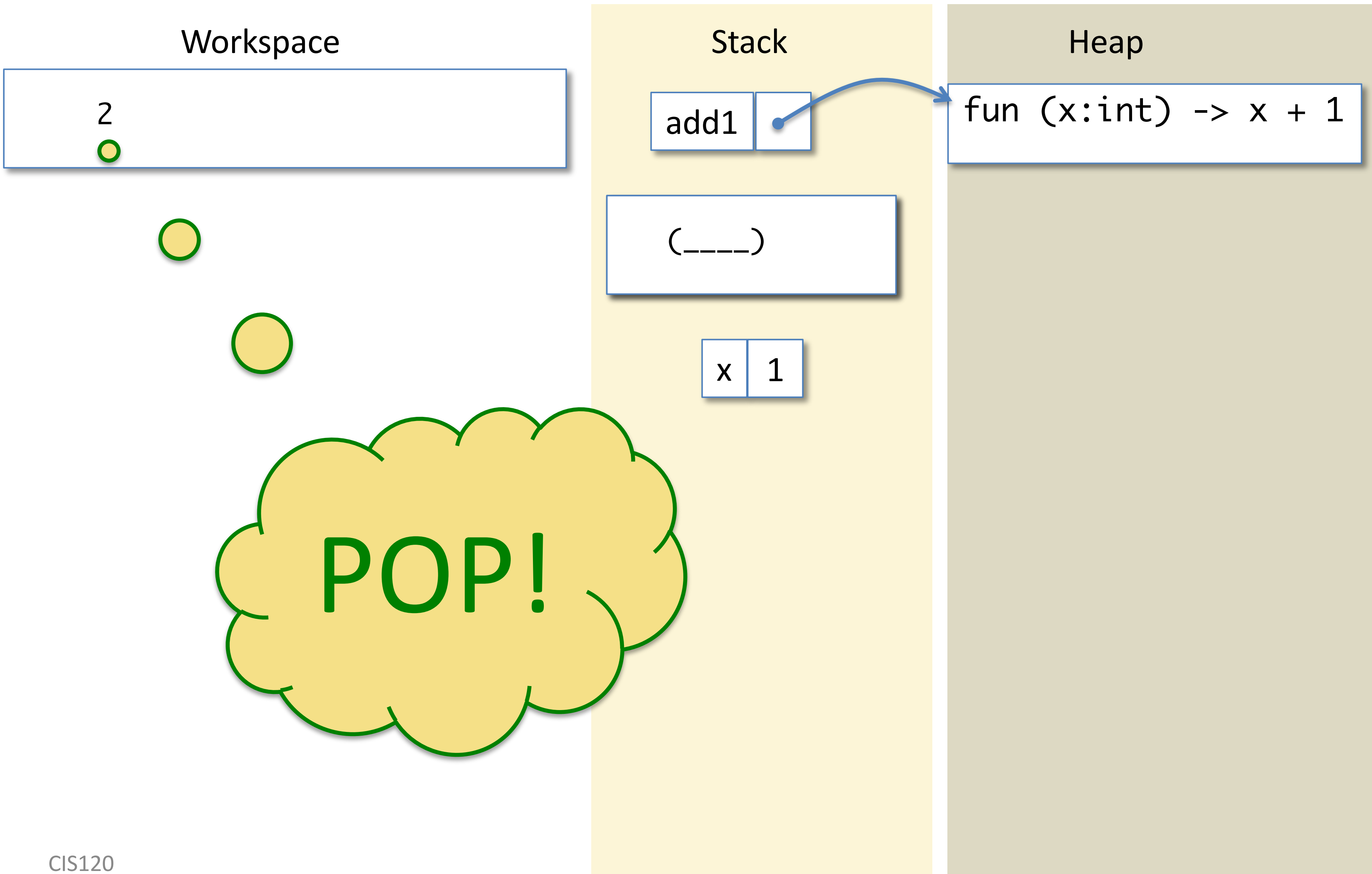
Function Simplification



Function Simplification



Function Simplification



Function Simplification

Simplifying Functions

- A function definition “let f $(x_1{:}t_1)\ldots(x_n{:}t_n)$ = e in body” is always ready.
  - It is simplified by replacing it with “let f = fun $(x{:}t_1)\ldots(x{:}t_n)$ = e in body”

- A function “fun $(x_1{:}t_1)\ldots(x_n{:}t_n)$ = e” is always ready.
  - It is simplified by moving the function to the heap and replacing the function expression with a pointer to that heap data.

- A function *call* is ready if the function and its arguments are all values
  - it is simplified by
    - saving the current workspace contents on the stack
    - adding bindings for the function’s parameter variables (to the actual argument values) to the end of the stack
    - copying the function’s body to the workspace

Function Completion

When the workspace contains just a single value, we *pop the stack* by removing everything back to (and including) the last saved workspace contents.

The value currently in the workspace is substituted for the function application expression in the saved workspace contents, which are put back into the workspace.

If there aren't any saved workspace contents in the stack, the whole computation is finished and the value in the workspace is its final result.

ASM: Simplifying pattern matching and recursion

Example

```
let rec append (l1: 'a list) (l2: 'a list) : 'a list =
  begin match l1 with
  | Nil -> l2
  | Cons(h, t) -> Cons(h, append t l2)
  end in

let a = Cons(1, Nil) in
let b = Cons(2, Cons(3, Nil)) in

append a b
```

Simplification

Workspace

```
let rec append (l1: 'a list)
    (l2: 'a list) : 'a list =
  begin match l1 with
  | Nil -> l2
  | Cons(h, t) ->
       Cons(h, append t l2)
  end in
let a = Cons(1, Nil) in
let b = Cons(2, Cons(3, Nil))
in
append a b
```

Stack

Heap

Function Definition

Workspace

```
let rec append (l1: 'a list)
    (l2: 'a list) : 'a list =
  begin match l1 with
  | Nil -> l2
  | Cons(h, t) ->
       Cons(h, append t l2)
  end in
let a = Cons(1, Nil) in
let b = Cons(2, Cons(3, Nil))
in
append a b
```

Stack

Heap

Rewrite to a “fun”

Workspace

```
let rec append =
  fun (l1: 'a list)
    (l2: 'a list) ->
  begin match l1 with
  | Nil -> l2
  | Cons(h, t) ->
       Cons(h, append t l2)
  end in
let a = Cons(1, Nil) in
let b = Cons(2, Cons(3, Nil))
in
append a b
```

Stack

Heap

Function Expression

Workspace

```
let rec append =
  fun (l1: 'a list)
    (l2: 'a list) ->
  begin match l1 with
  | Nil -> l2
  | Cons(h, t) ->
       Cons(h, append t l2)
  end in
let a = Cons(1, Nil) in
let b = Cons(2, Cons(3, Nil))
in
append a b
```

Stack

Heap

Copy to the Heap, Replace w/Reference

Workspace

```
let append =
   in
let a = Cons(1, Nil) in
let b = Cons(2, Cons(3, Nil))
in
append a b
```

Stack

Heap

```
fun (l1: 'a list)
    (l2: 'a list) ->
  begin match l1 with
  | Nil -> l2
  | Cons(h, t) ->
       Cons(h,         t l2)
  end
```

NOTE: The heap structure that we build for the recursive function replaces the the use of append in the body with a reference.

This *backpatching* is enabled by the '`rec`' keyword. The code for this function refers to itself.

Let Expression

Workspace

```
let append = ______
   in
let a = Cons(1, Nil) in
let b = Cons(2, Cons(3, Nil))
in
append a b
```

Stack

Heap

```
fun (l1: 'a list)
    (l2: 'a list) ->
  begin match l1 with
  | Nil -> l2
  | Cons(h, t) ->
       Cons(h,       t l2)
  end
```

Note that the reference to a function in the heap is a *value*.

Create a Stack Binding

Workspace

```
let a = Cons(1, Nil) in
let b = Cons(2, Cons(3, Nil))
in
append a b
```

Stack

append	

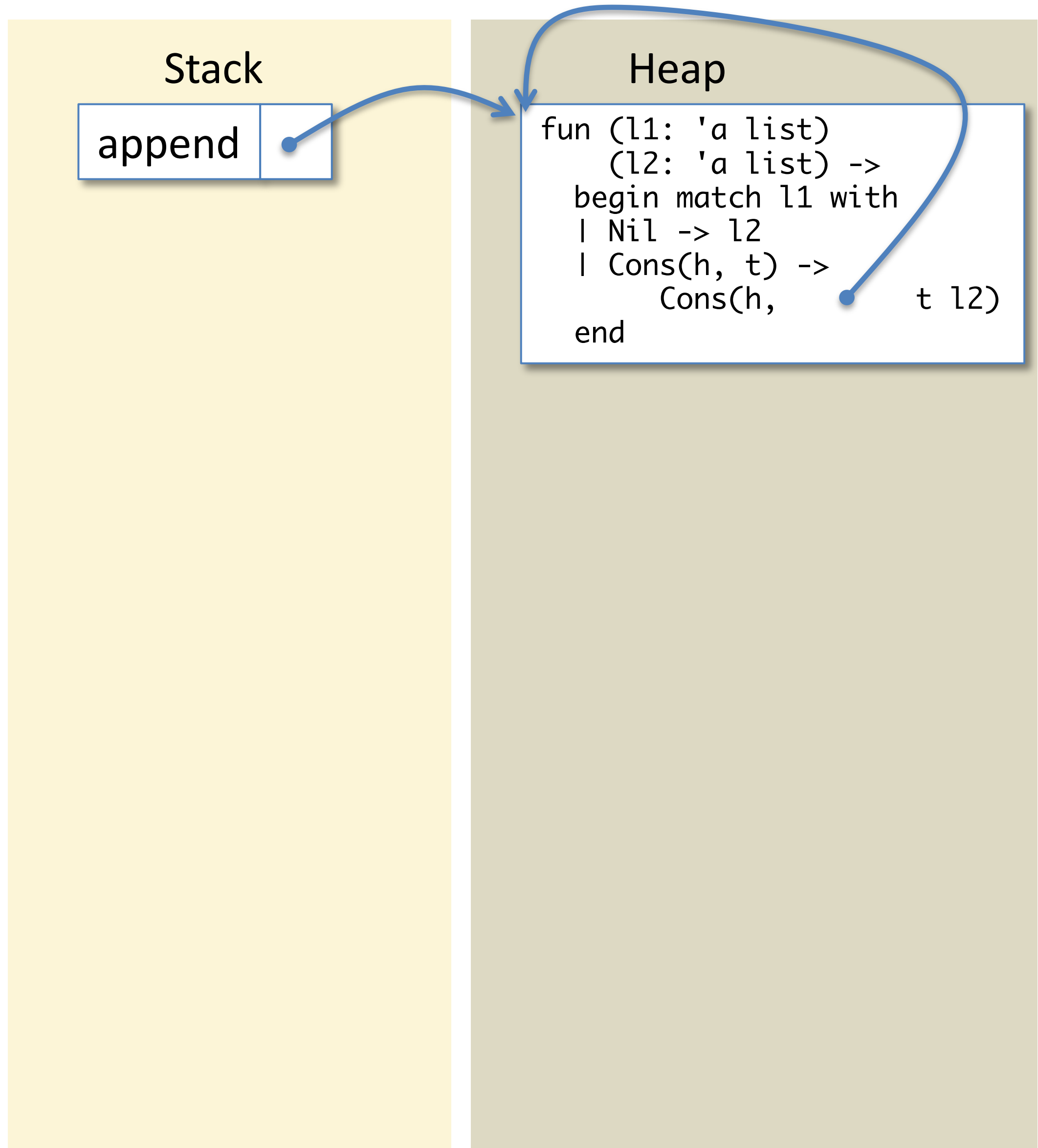
Heap

```
fun (l1: 'a list)
    (l2: 'a list) ->
  begin match l1 with
  | Nil -> l2
  | Cons(h, t) ->
       Cons(h,        t l2)
  end
```

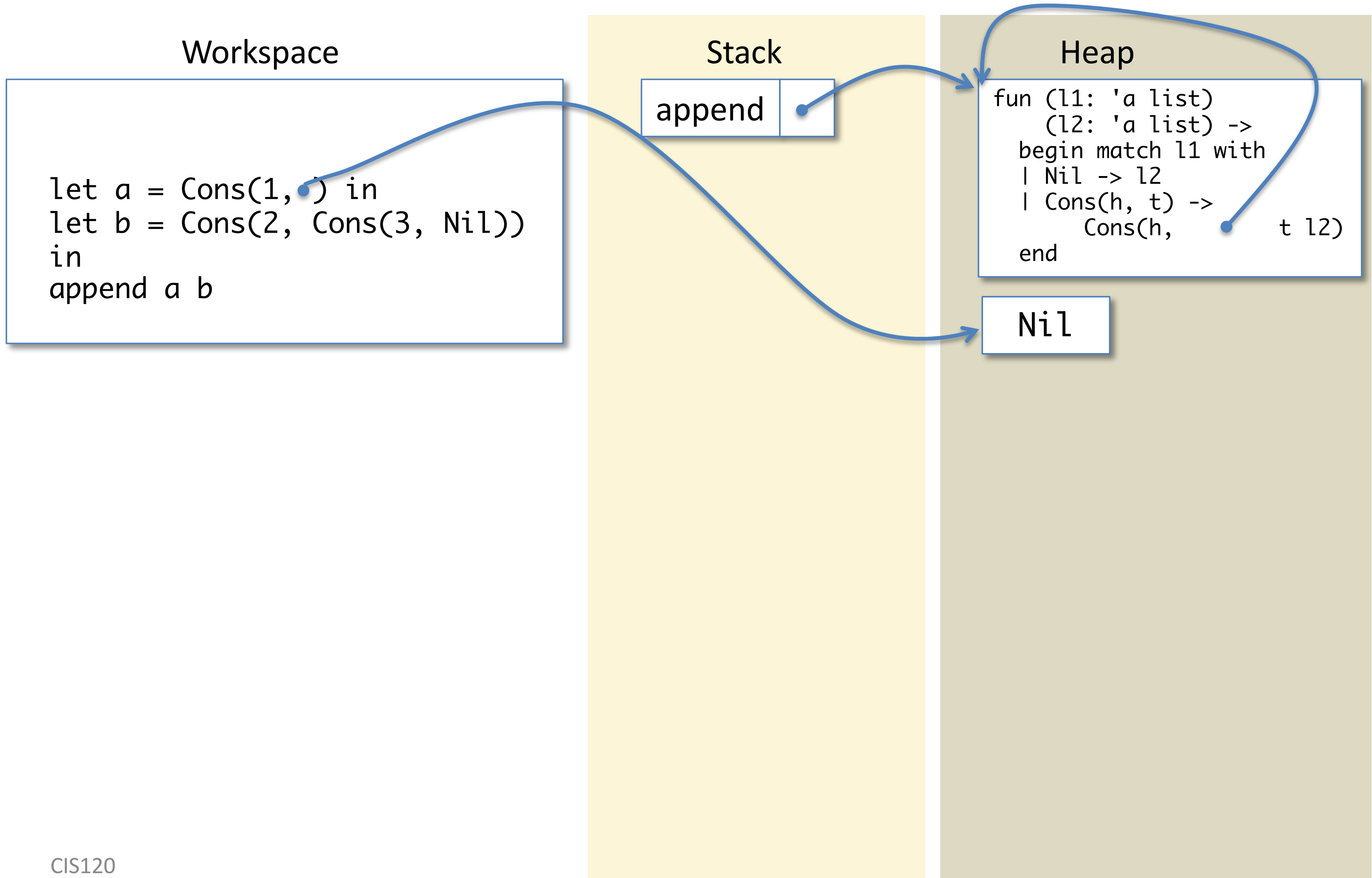
Allocate a Nil cell

Workspace

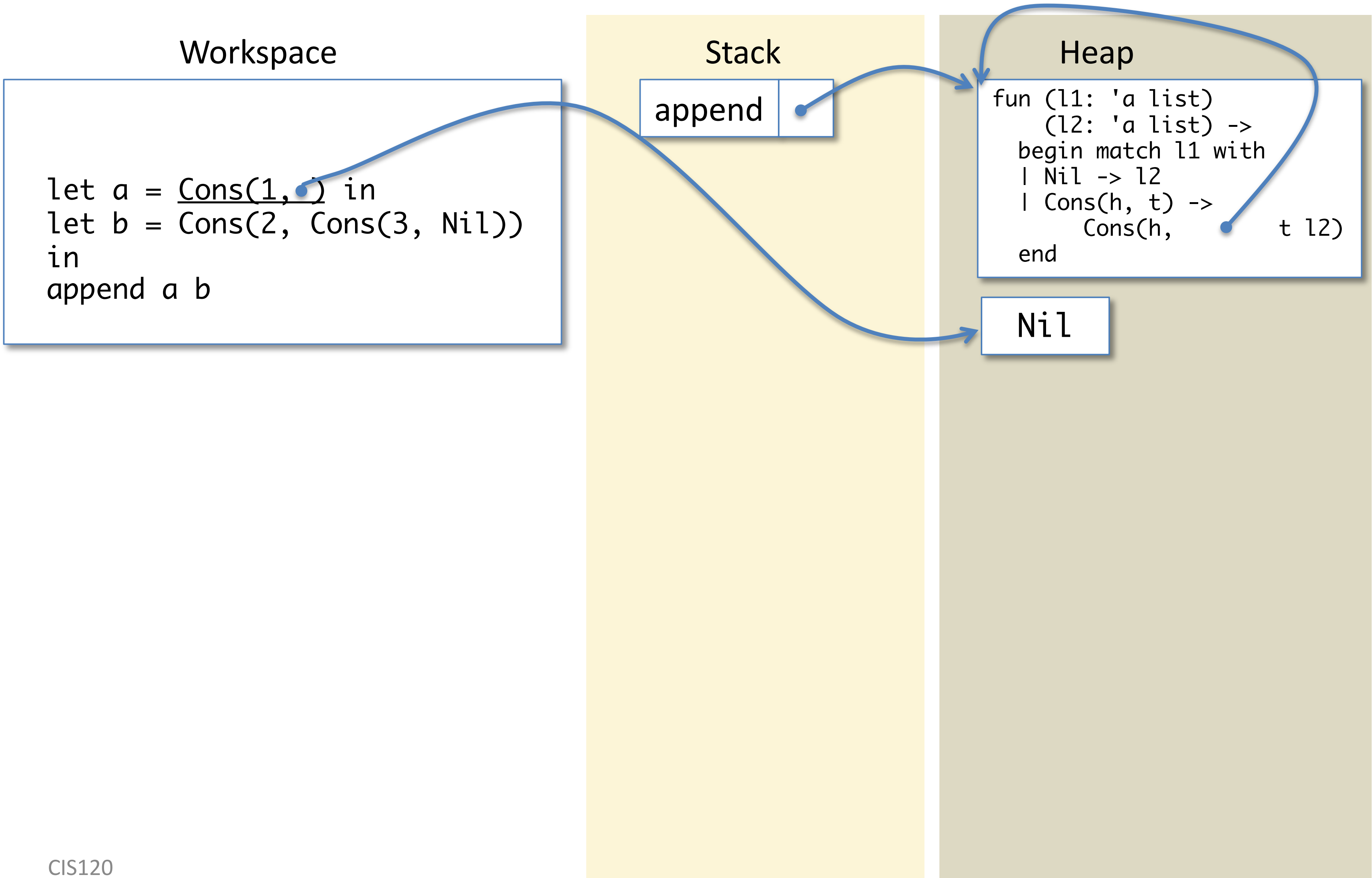
```
let a = Cons(1, Nil) in
let b = Cons(2, Cons(3, Nil))
in
append a b
```



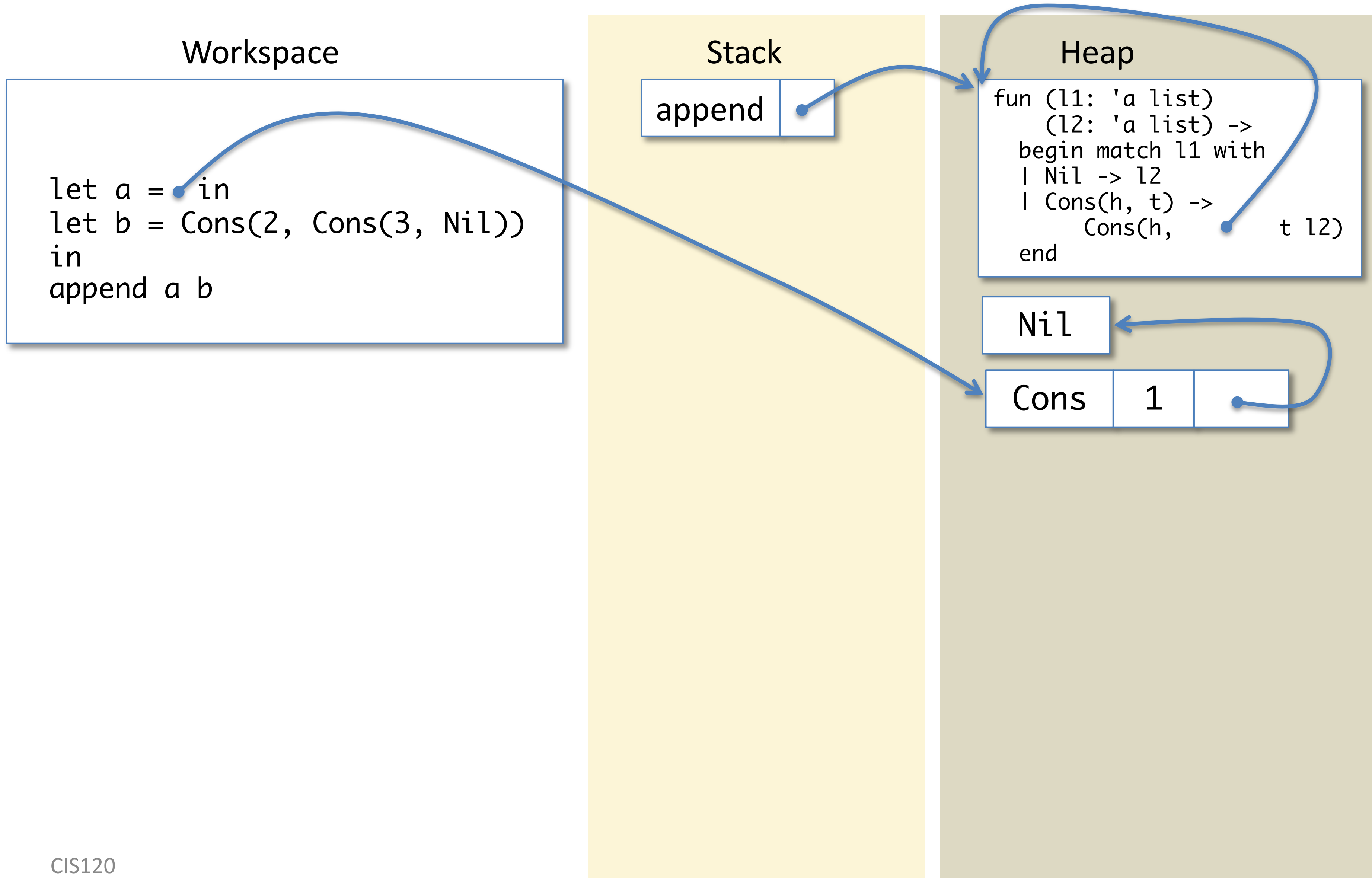
Allocate a Nil cell



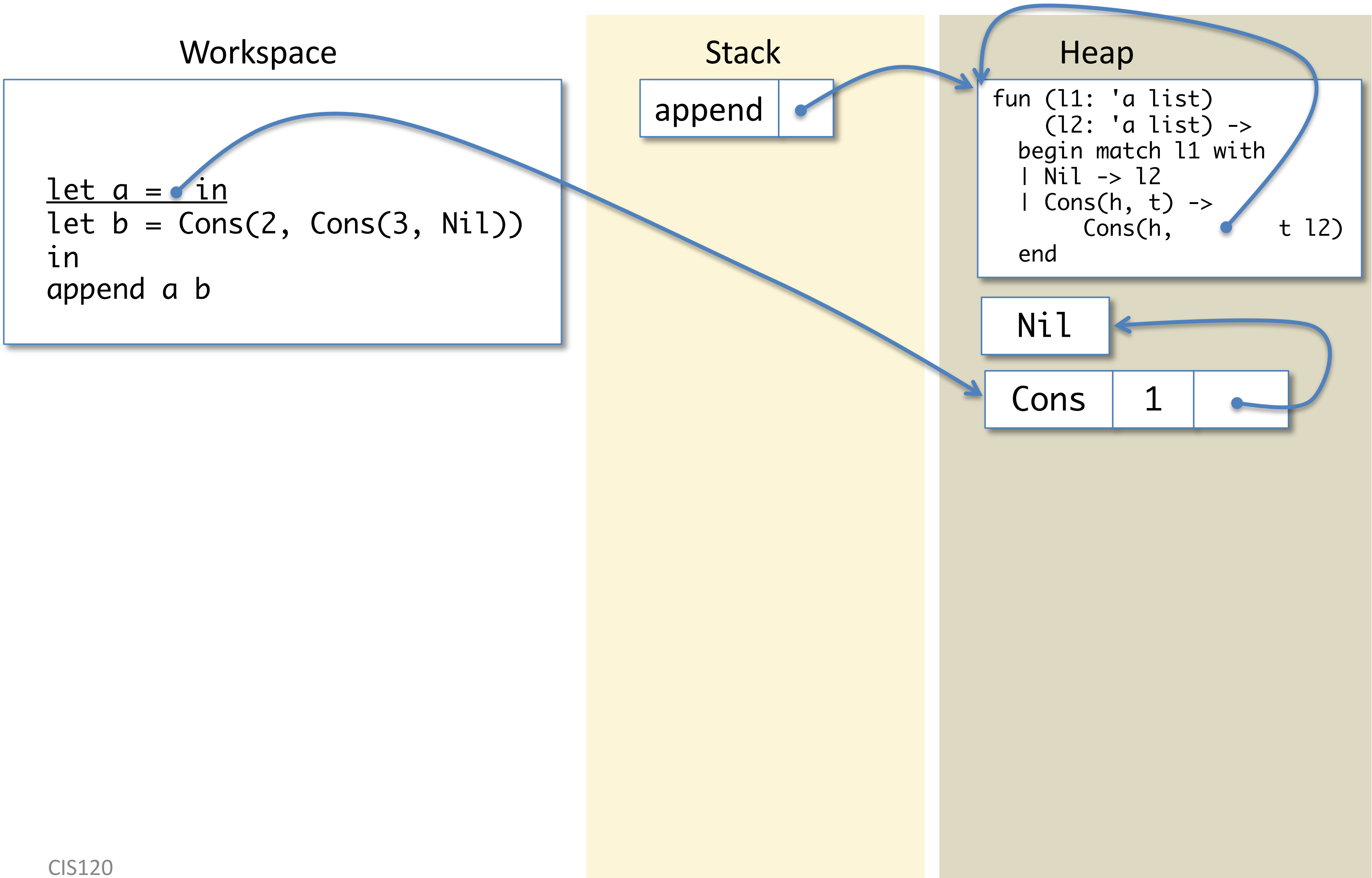
Allocate a Cons cell



Allocate a Cons cell



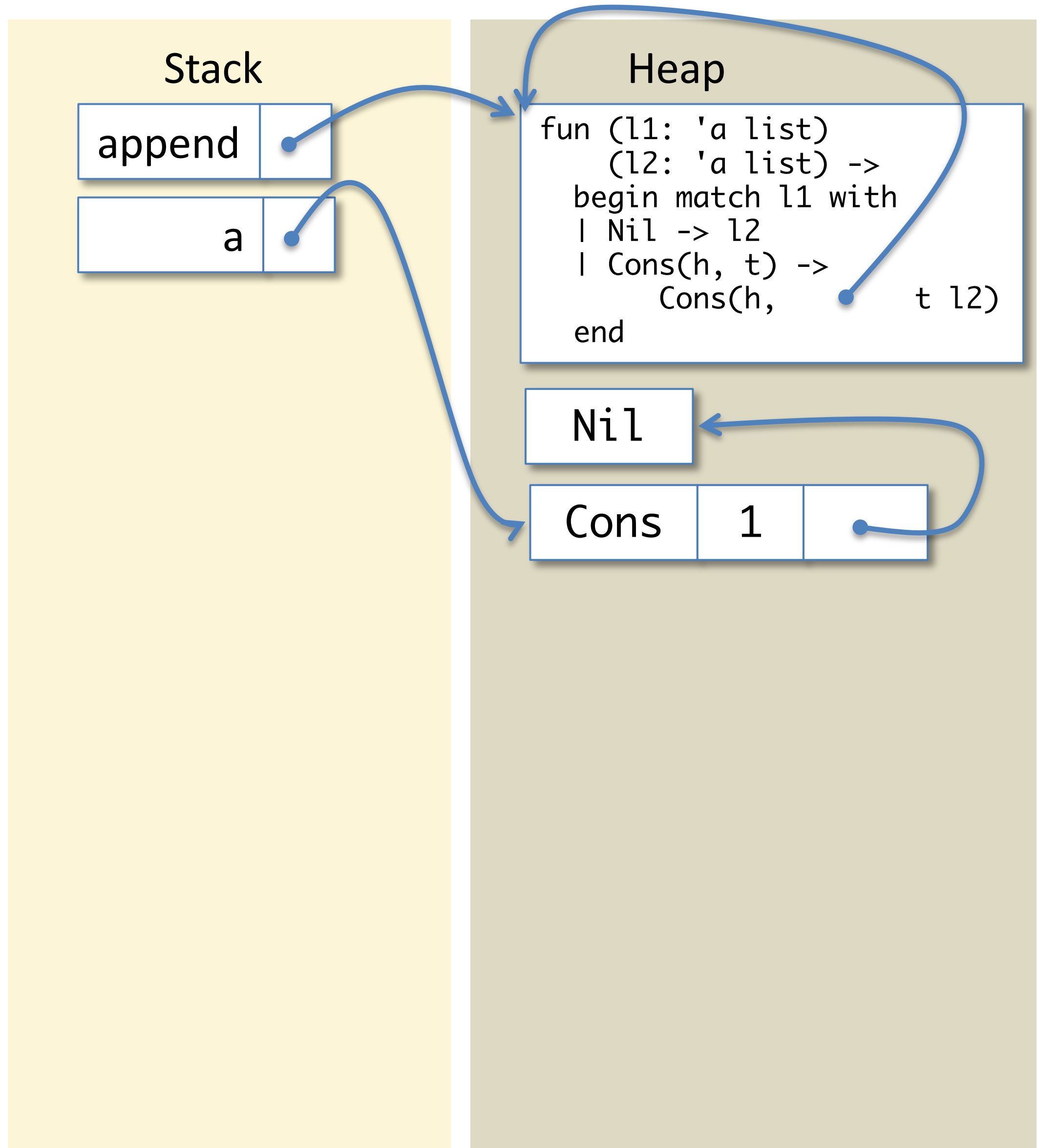
Let Expression



Create a Stack Binding

Workspace

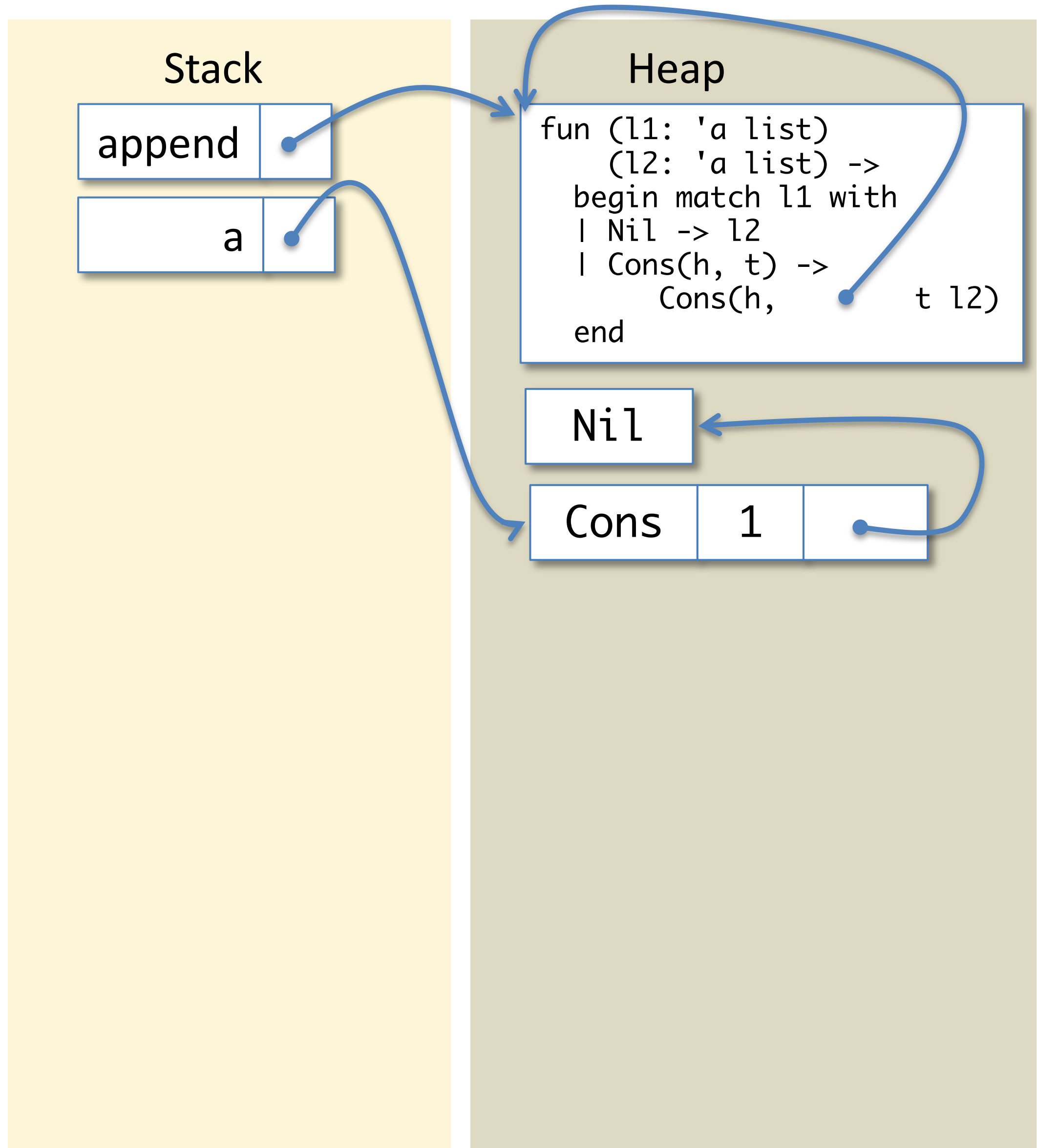
```
let b = Cons(2, Cons(3, Nil))
in
append a b
```



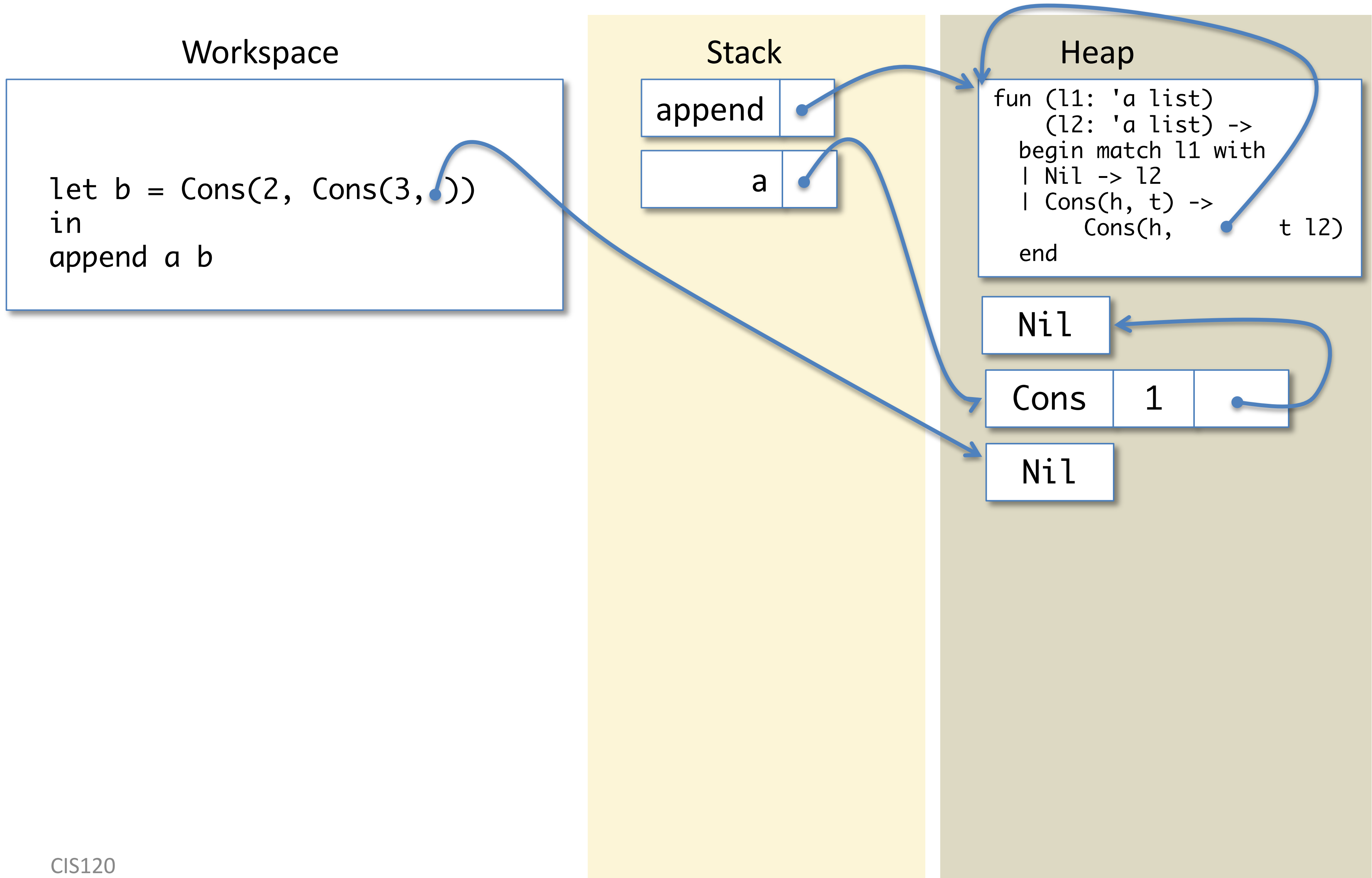
Allocate a Nil cell

Workspace

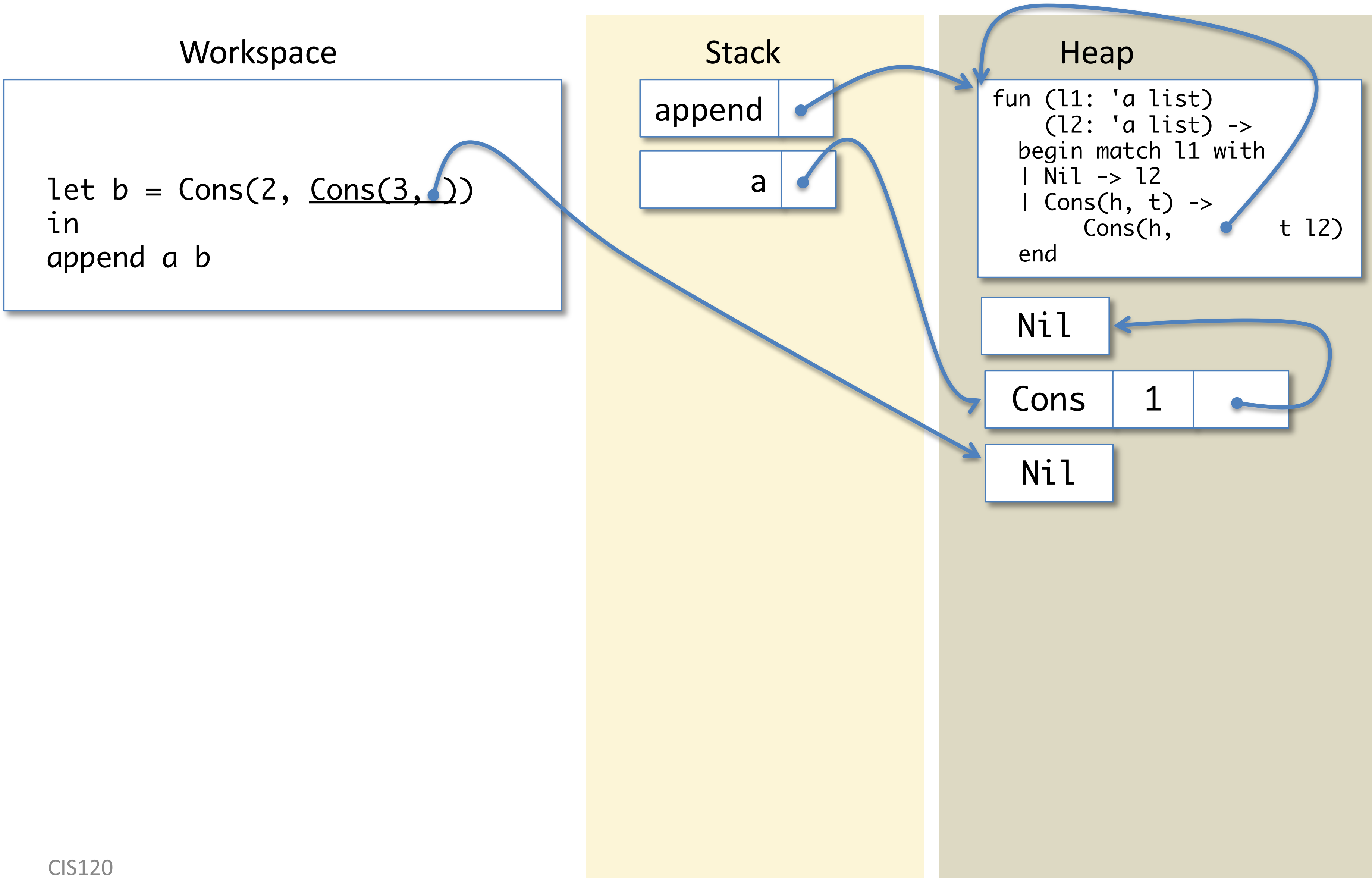
```
let b = Cons(2, Cons(3, Nil))
in
append a b
```



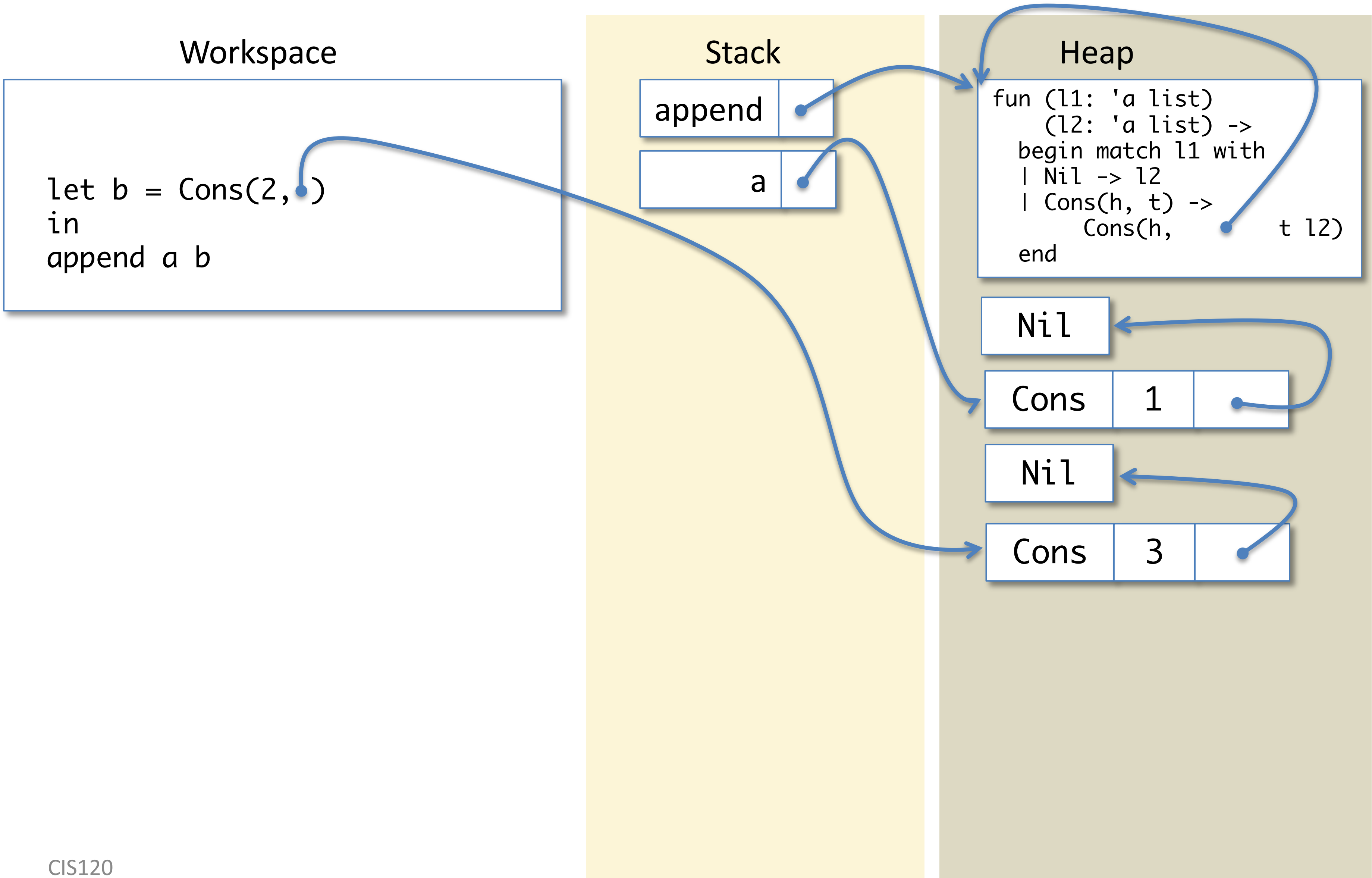
Allocate a Nil cell



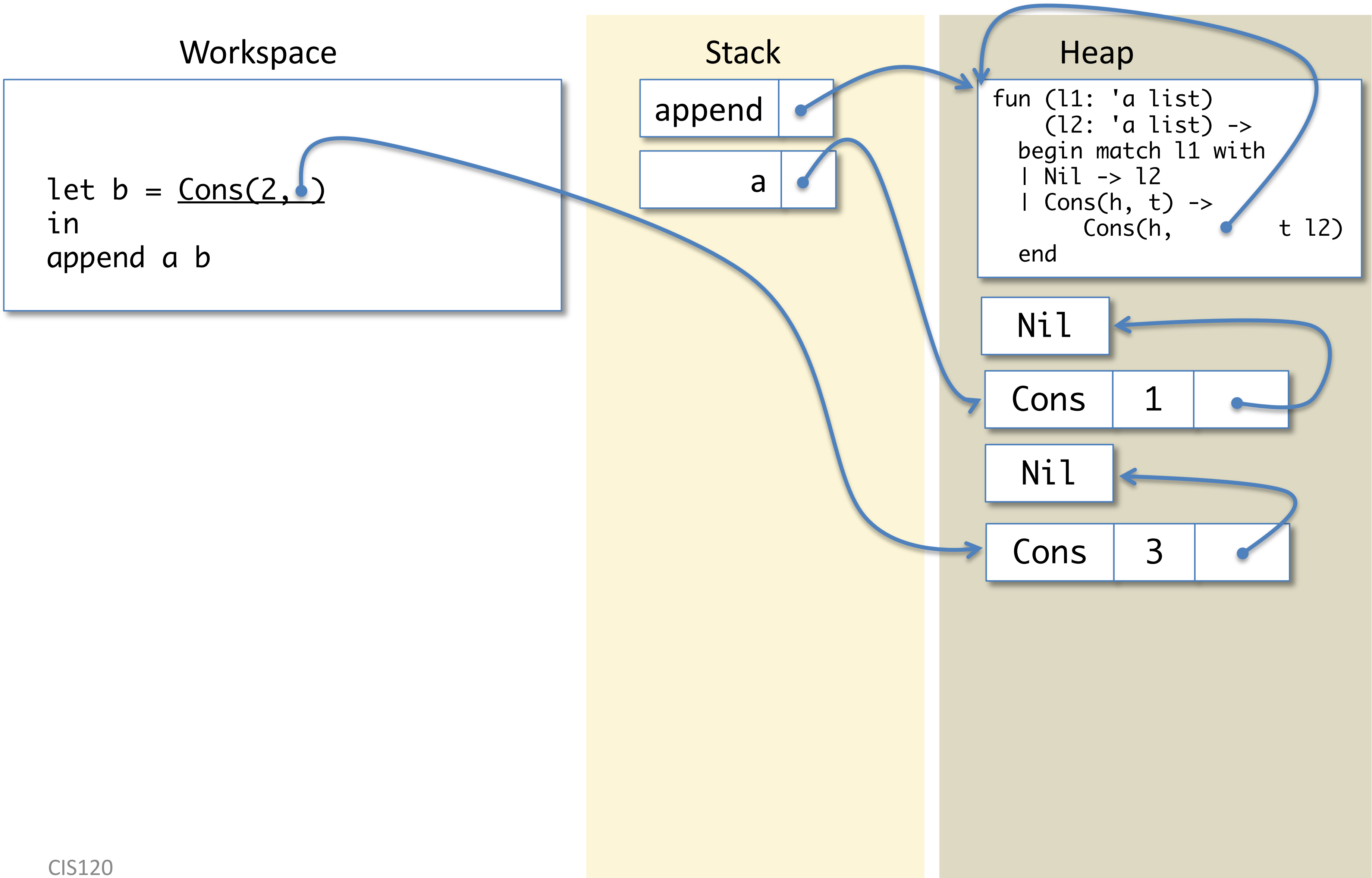
Allocate a Cons cell



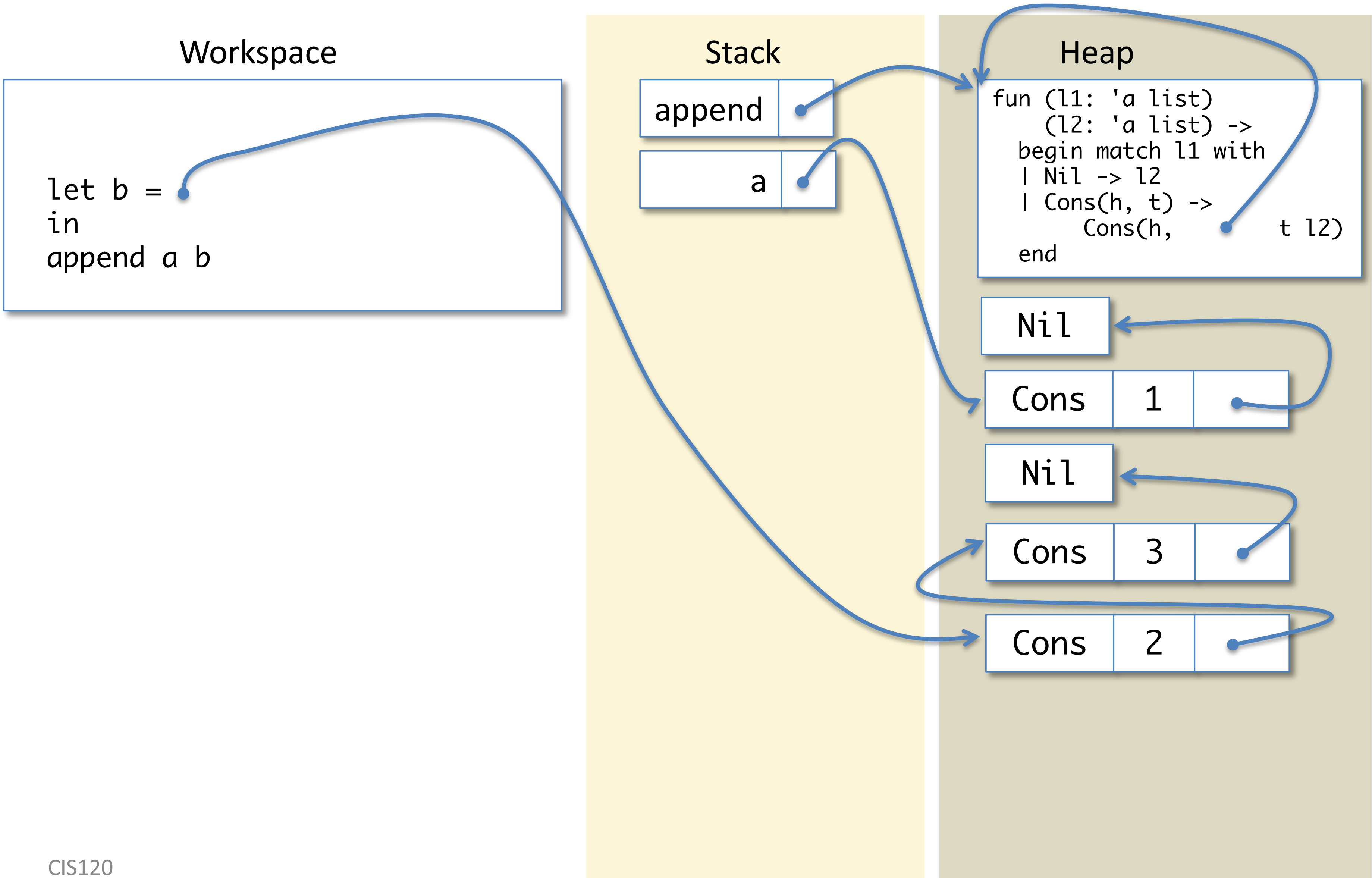
Allocate a Cons cell



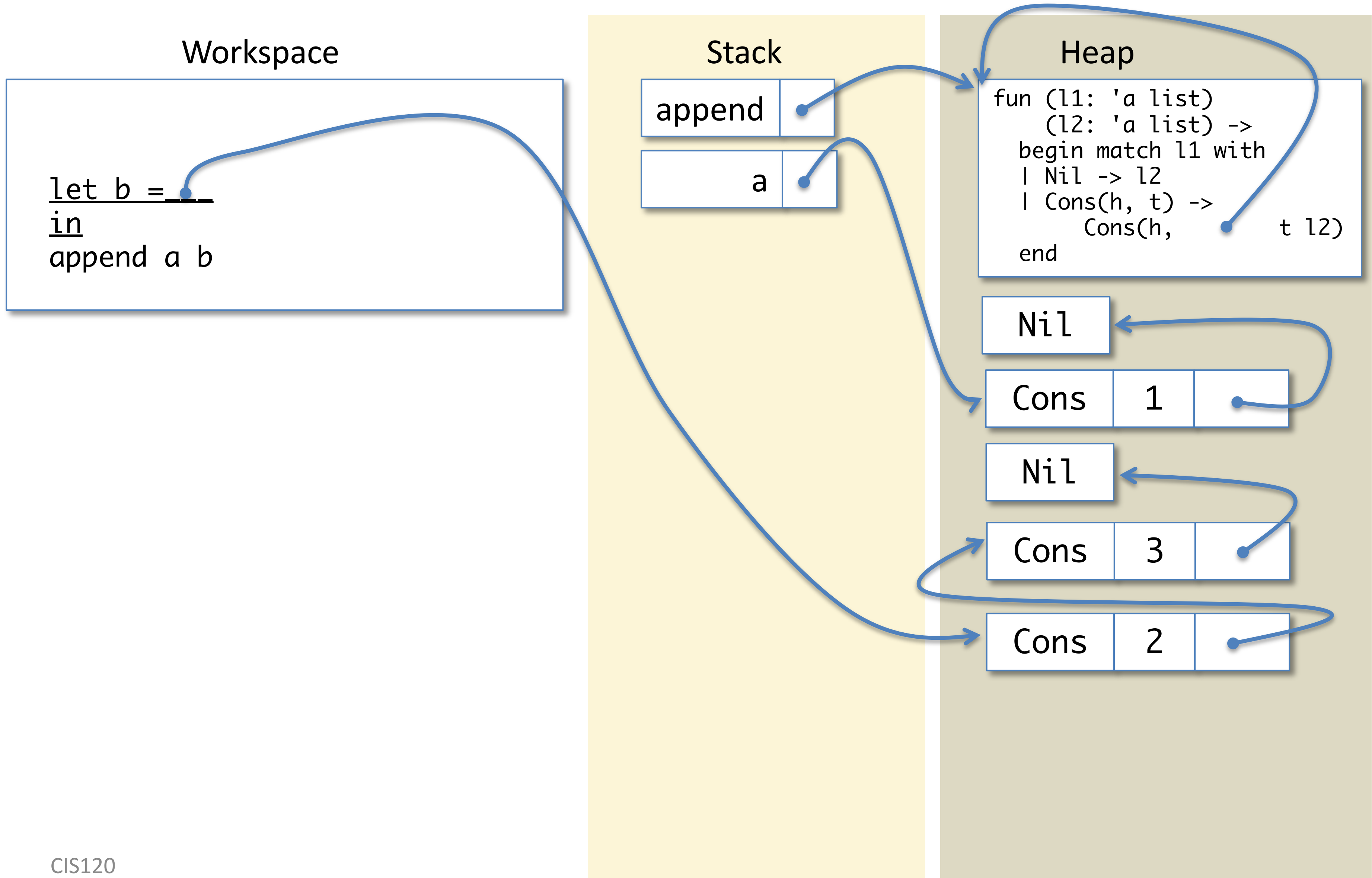
Allocate a Cons cell



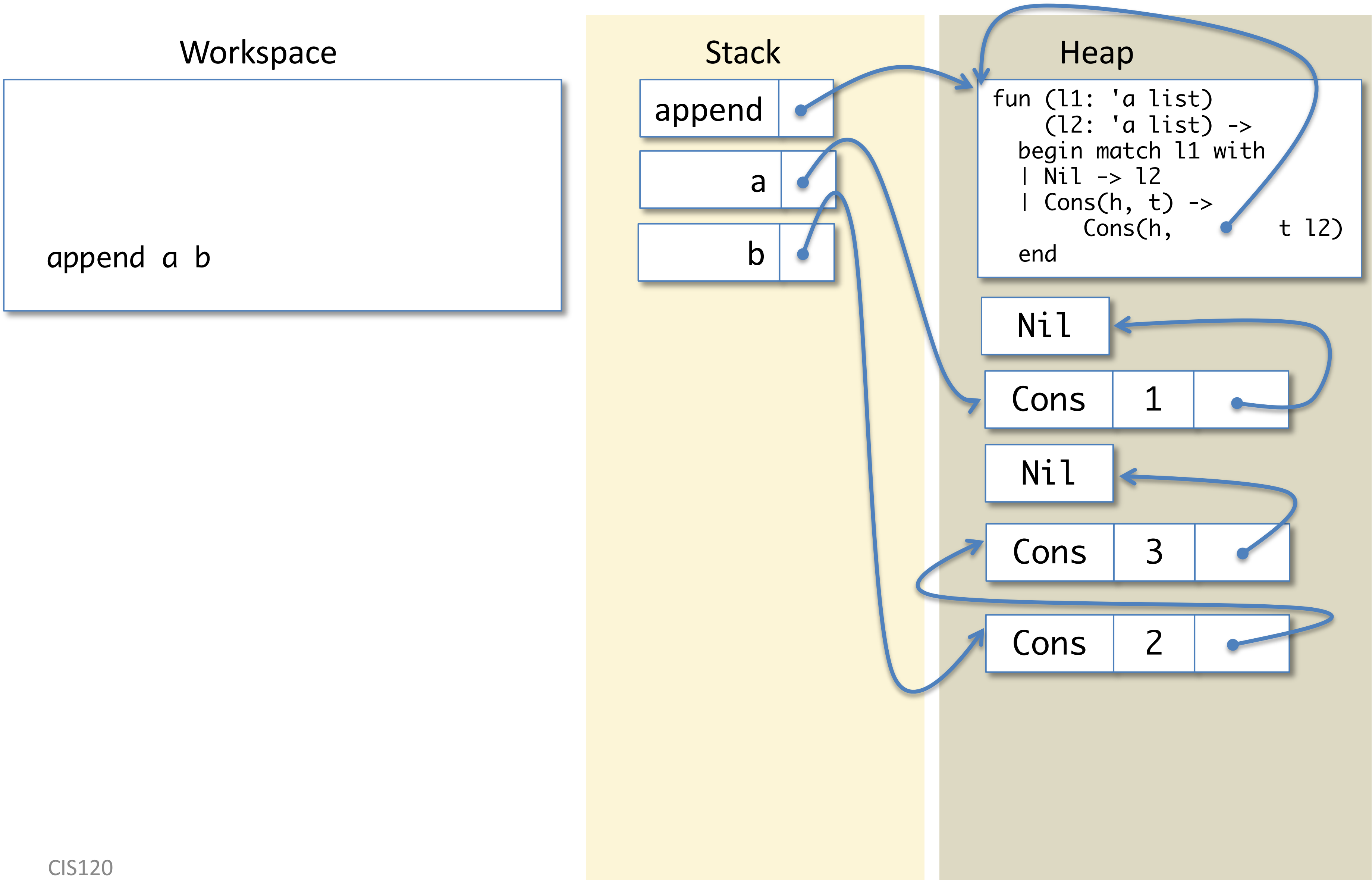
Allocate a Cons cell



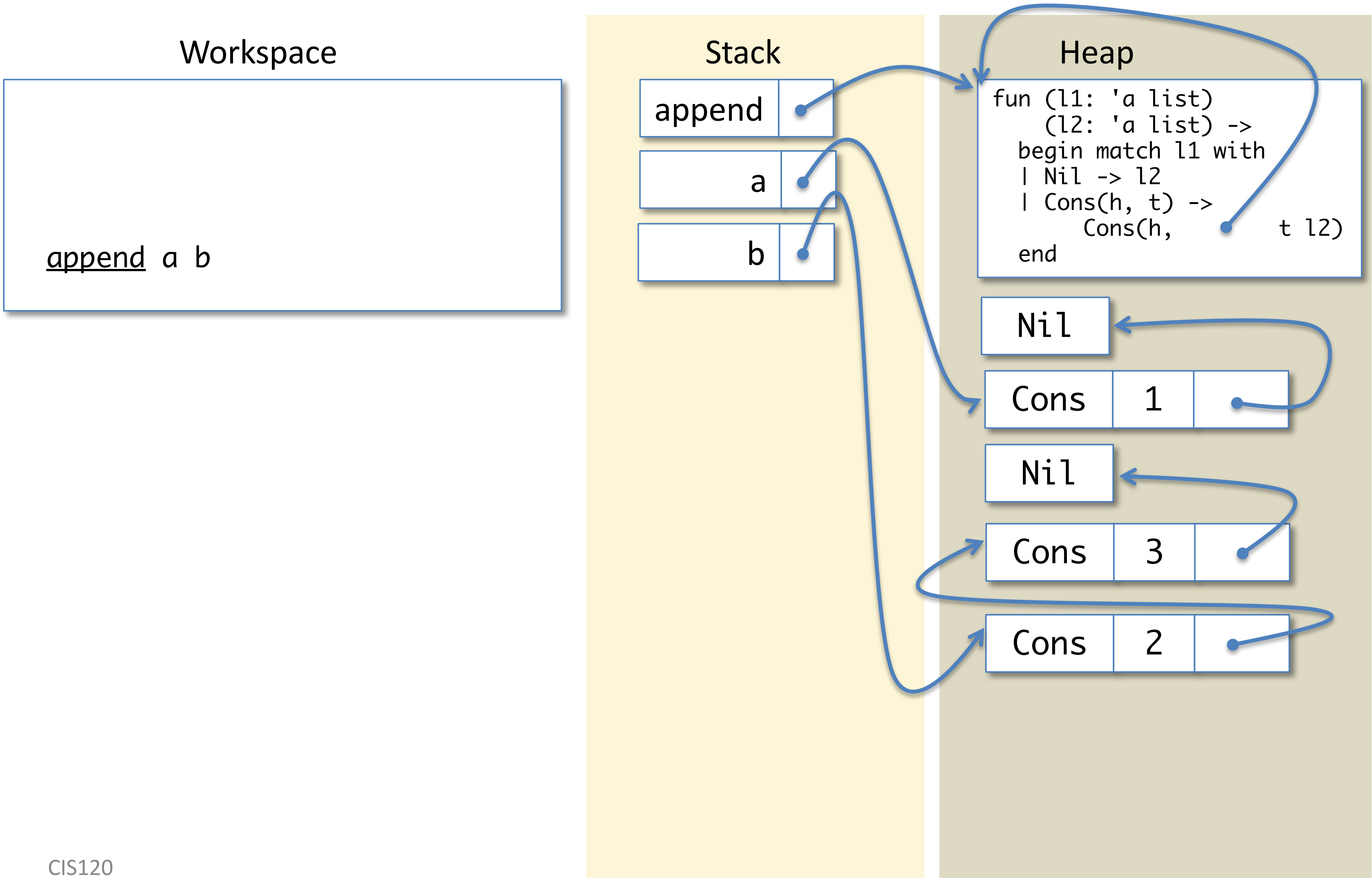
Let Expression



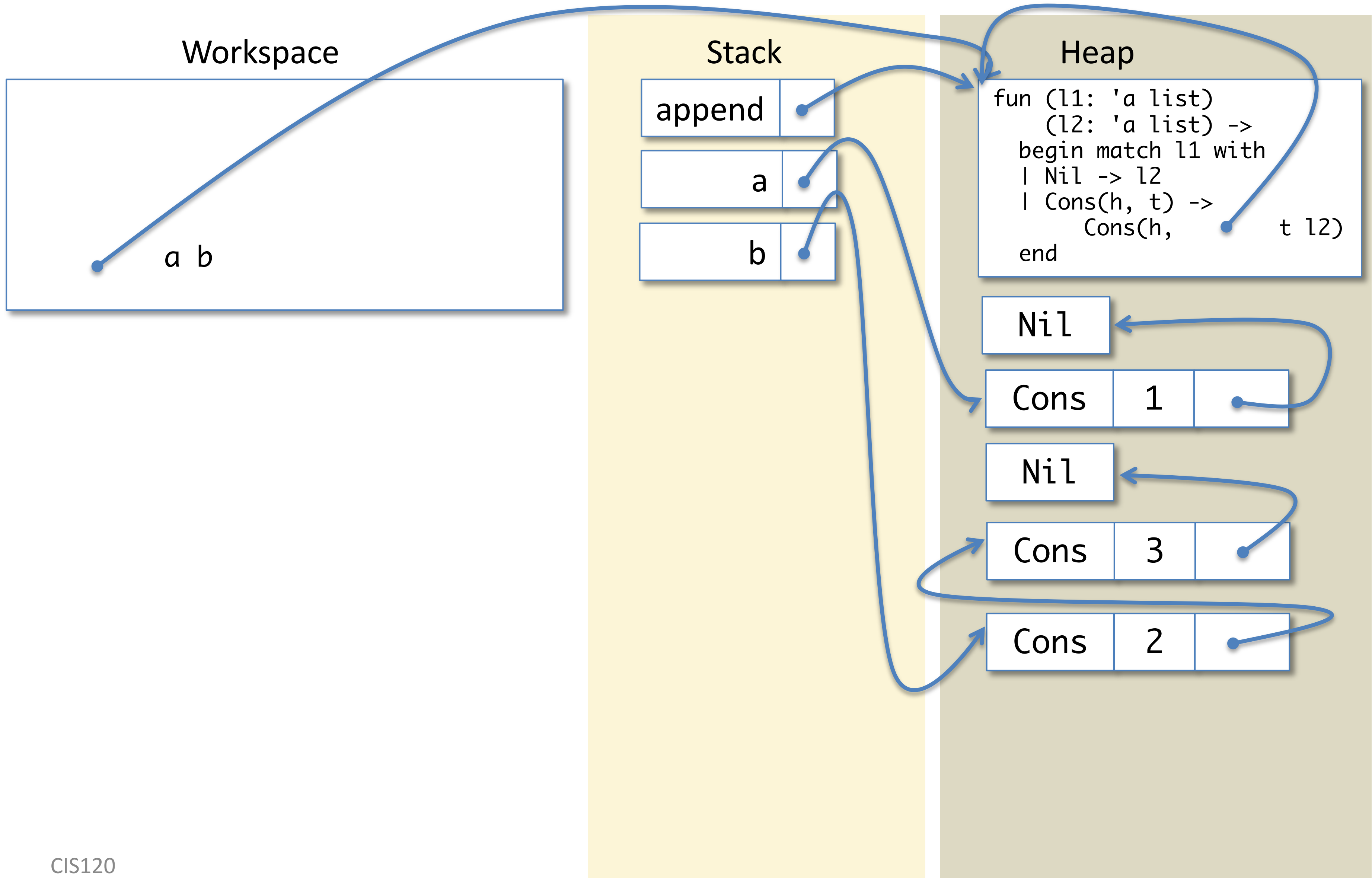
Create a Stack Binding



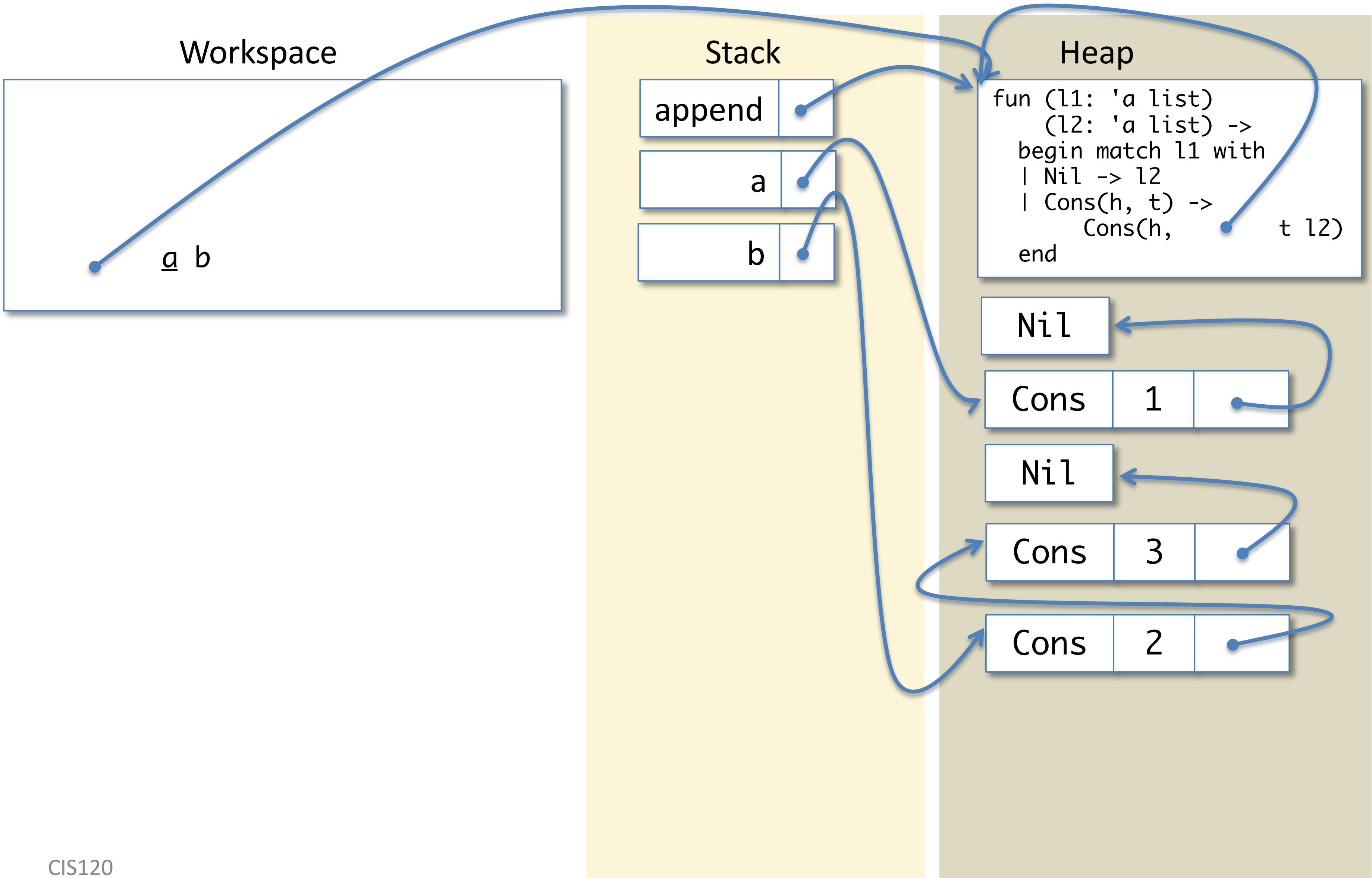
Lookup 'append'



Lookup 'append'

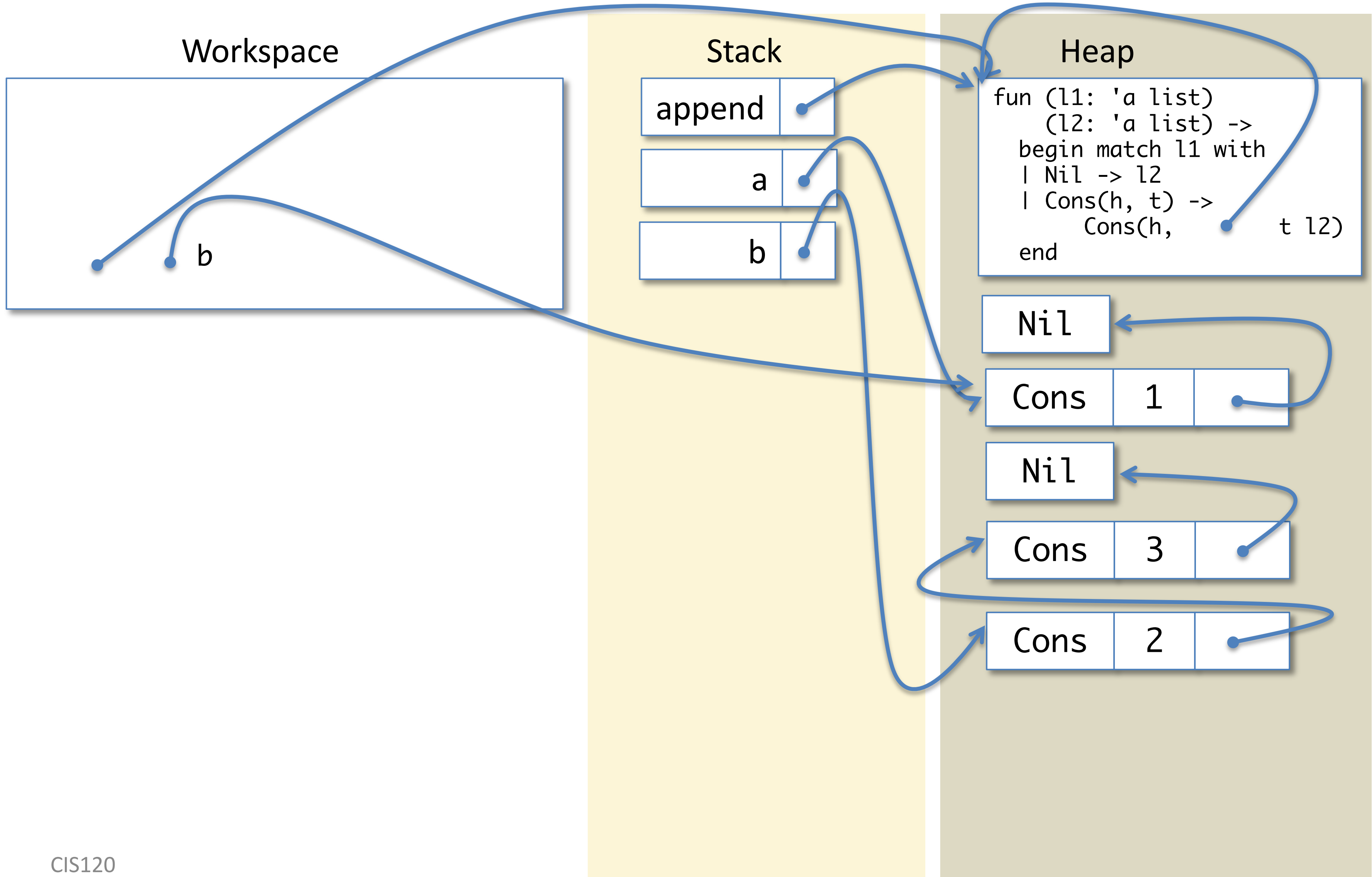


Lookup 'a'

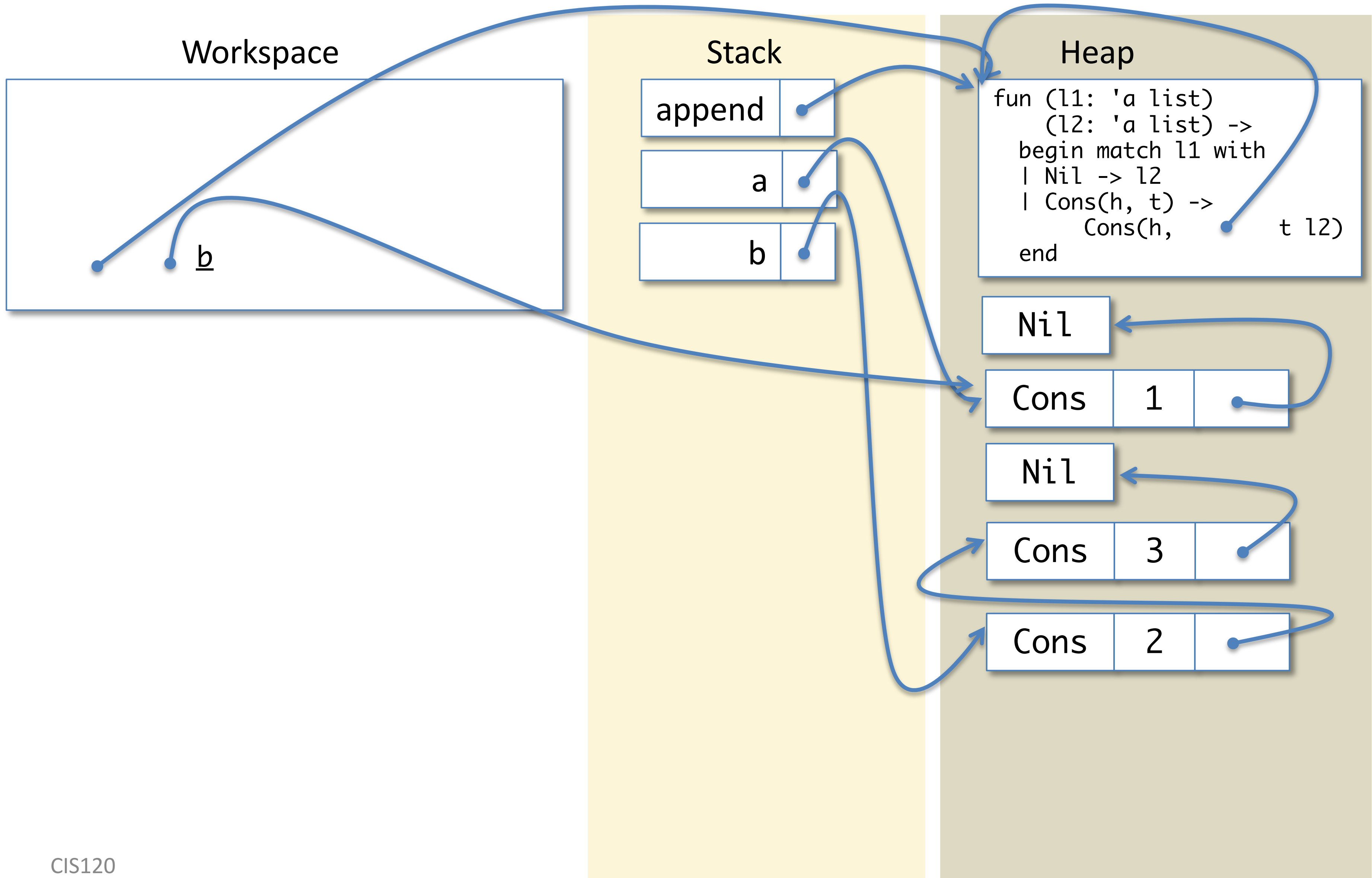


Workspace
a b
Stack
append
a
b
Heap
fun (l1: 'a list)
(l2: 'a list) ->
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Nil
Cons 1
Nil
Cons 3
Cons 2

Lookup 'a'

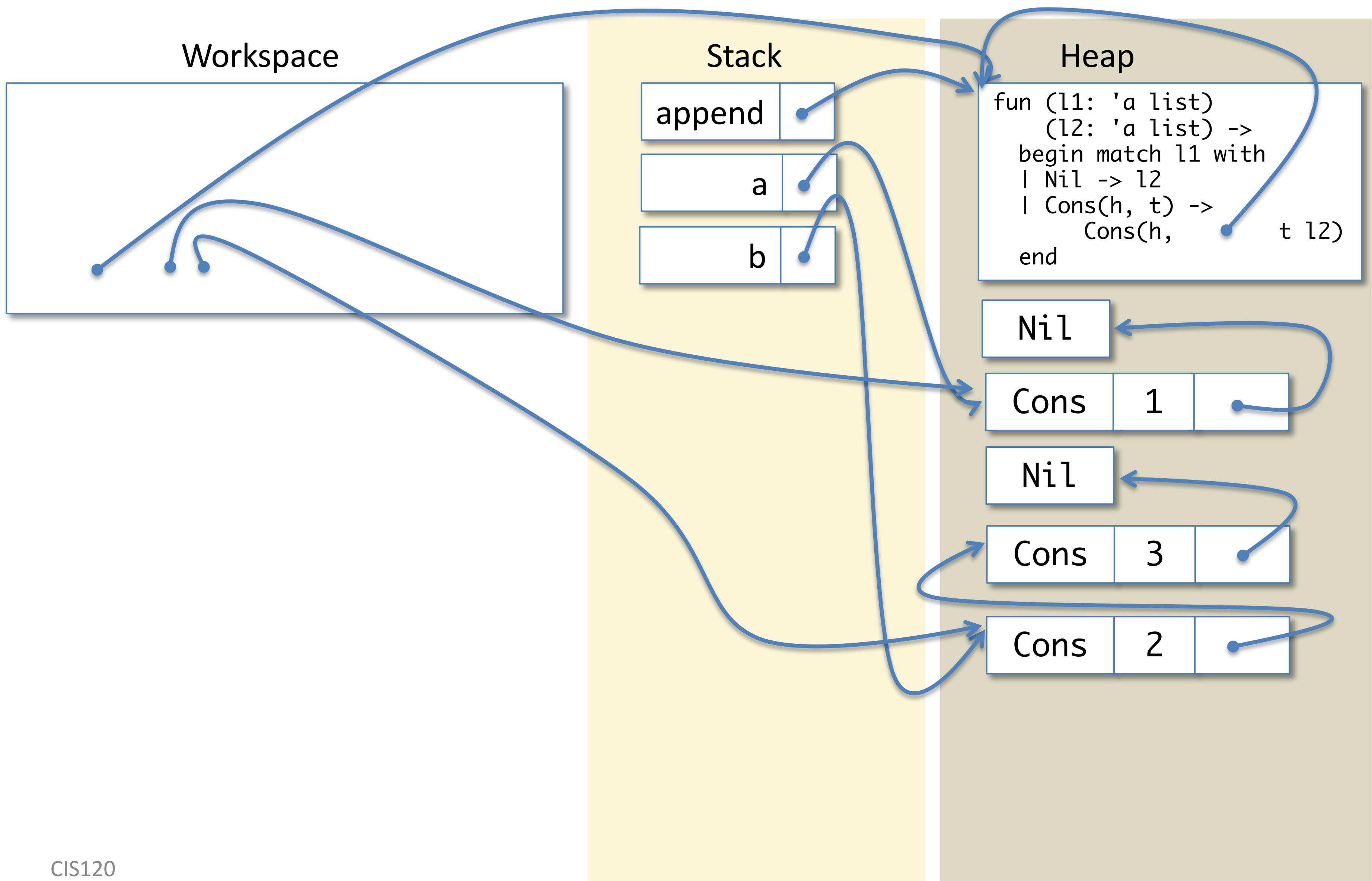

Workspace
b
Stack
append
a
b
Heap
fun (l1: 'a list)
(l2: 'a list) ->
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Nil
Cons 1
Nil
Cons 3
Cons 2

Lookup 'b'



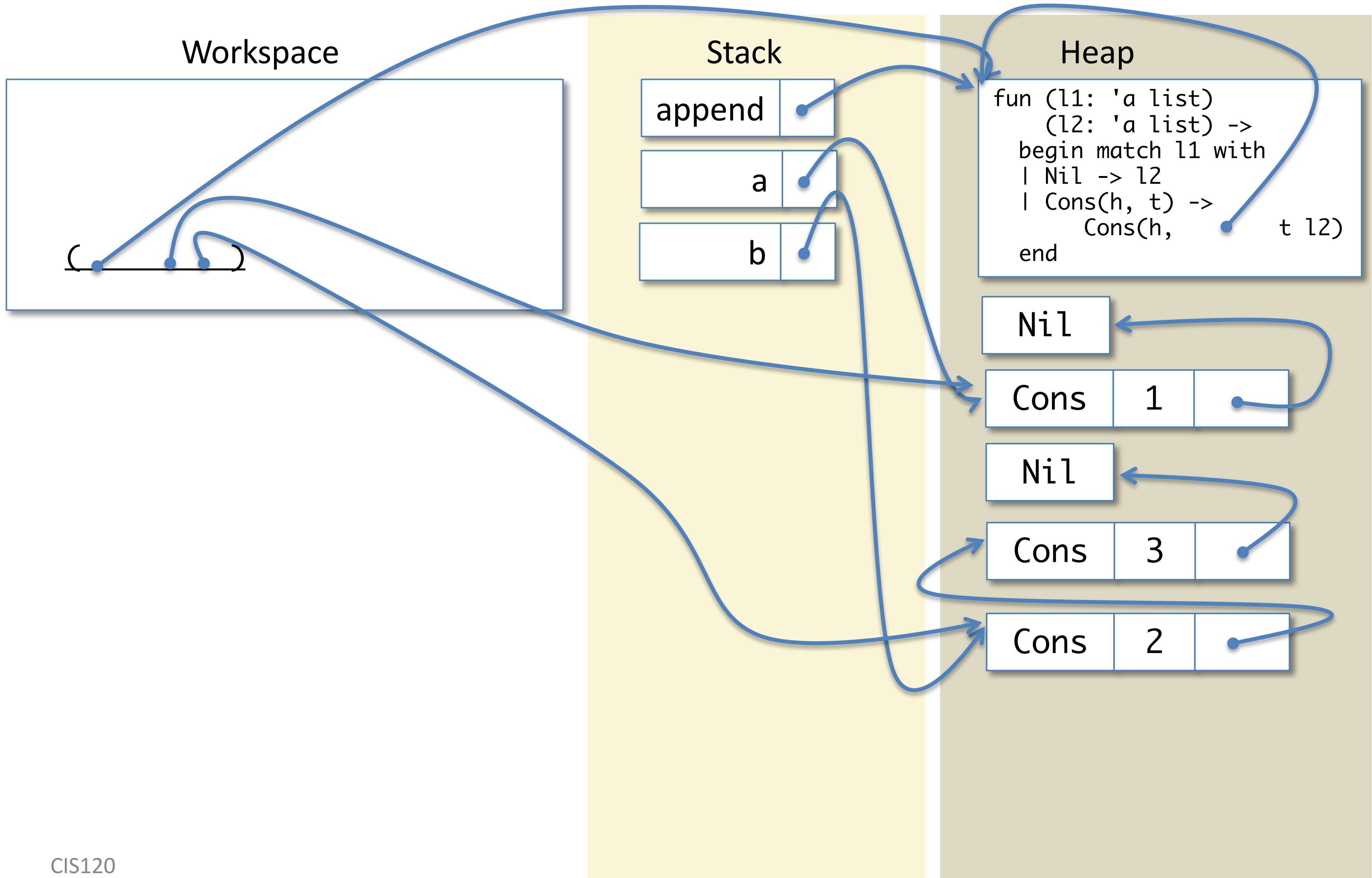
Workspace
b
Stack
append
a
b
Heap
fun (l1: 'a list)
(l2: 'a list) ->
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Nil
Cons
1
Nil
Cons
3
Cons
2

Lookup 'b'

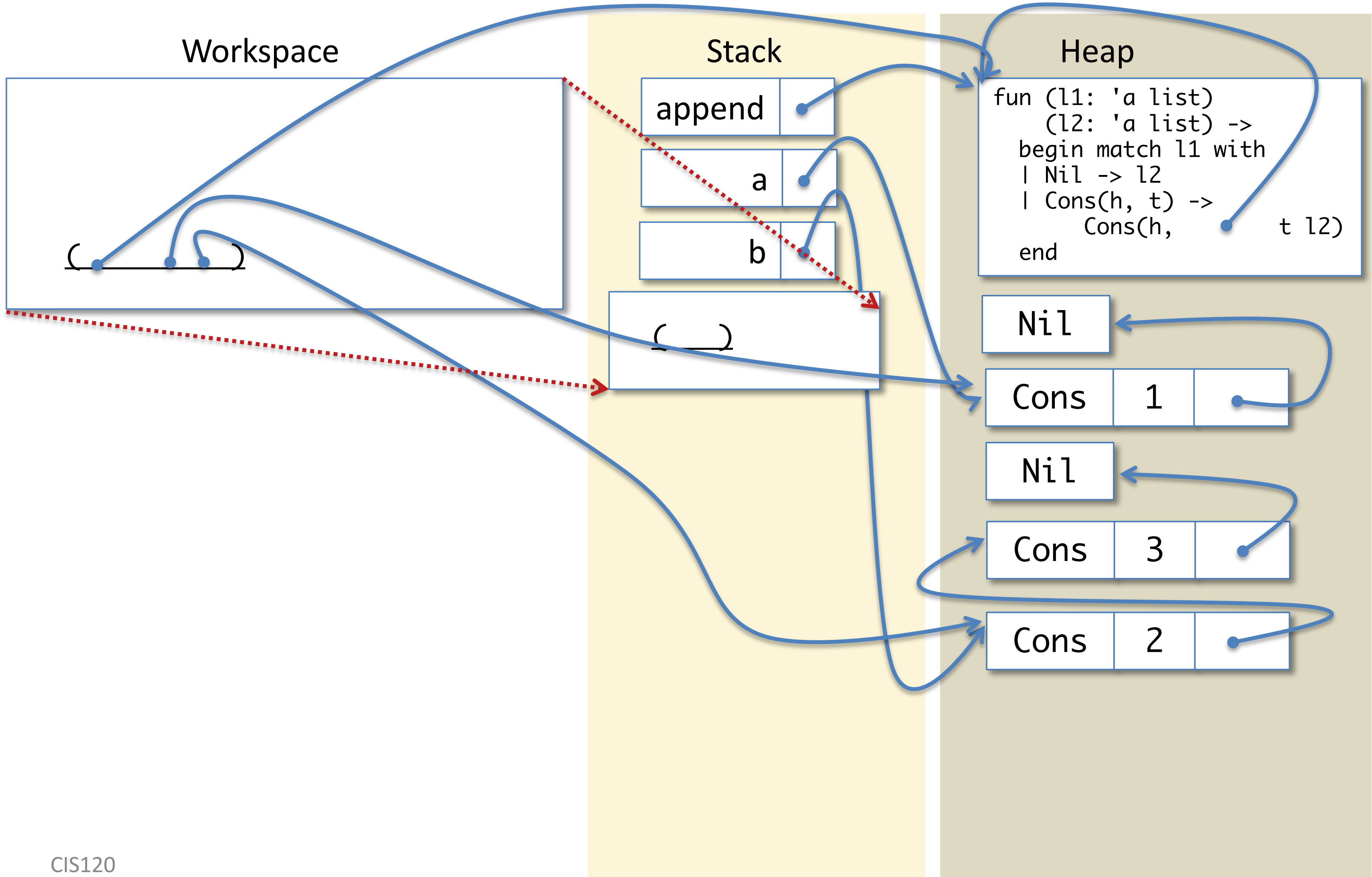


Workspace
Stack
Heap
append
a
b
fun (l1: 'a list)
(l2: 'a list) ->
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Nil
Cons 1
Nil
Cons 3
Cons 2

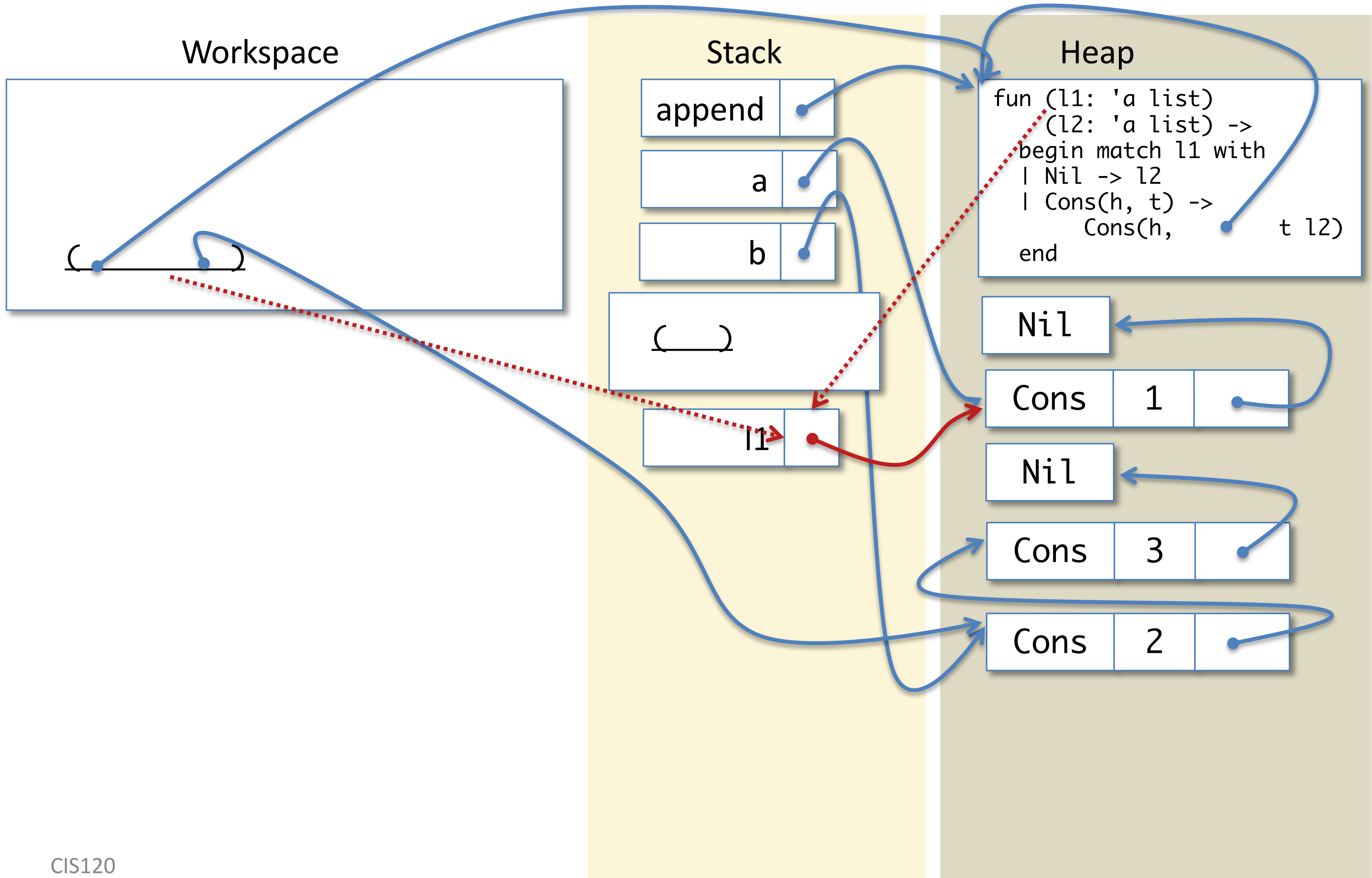
Do the Function call



Call (1): Save Workspace

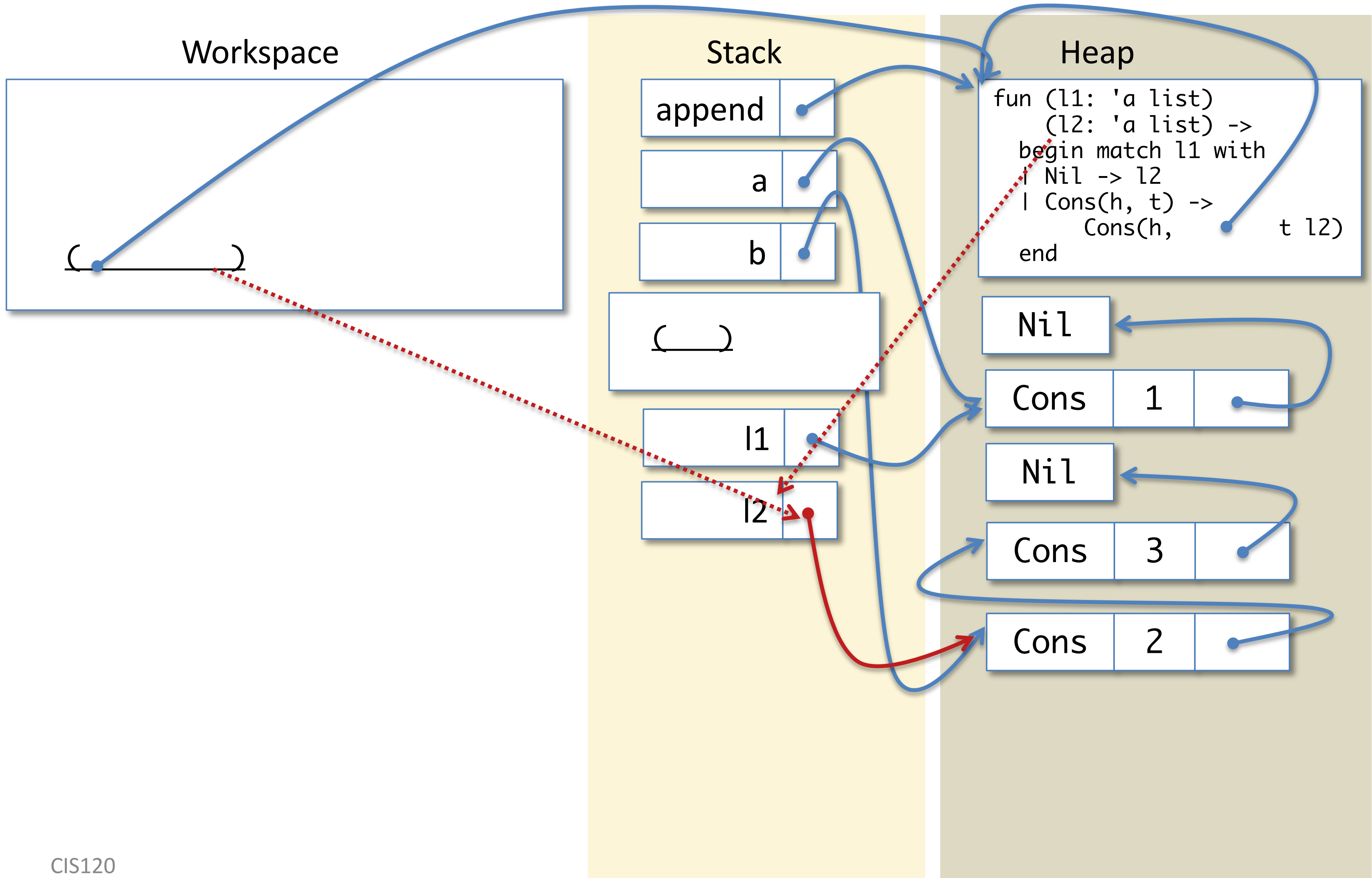


push l1

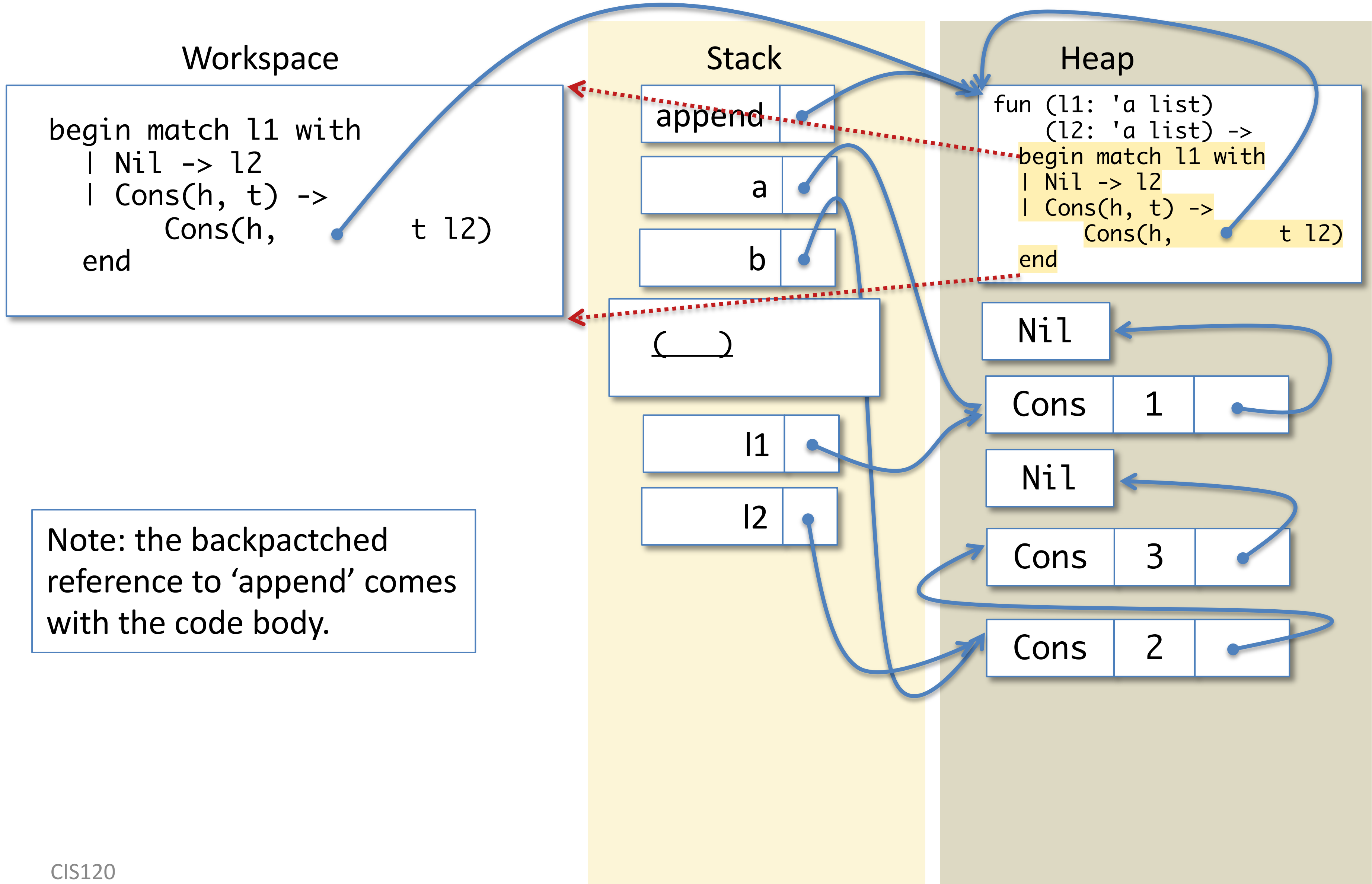


Workspace
Stack
Heap
append
a
b
l1
fun (l1: 'a list)
(l2: 'a list) ->
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Nil
Cons
1
Nil
Cons
3
Cons
2

push l2

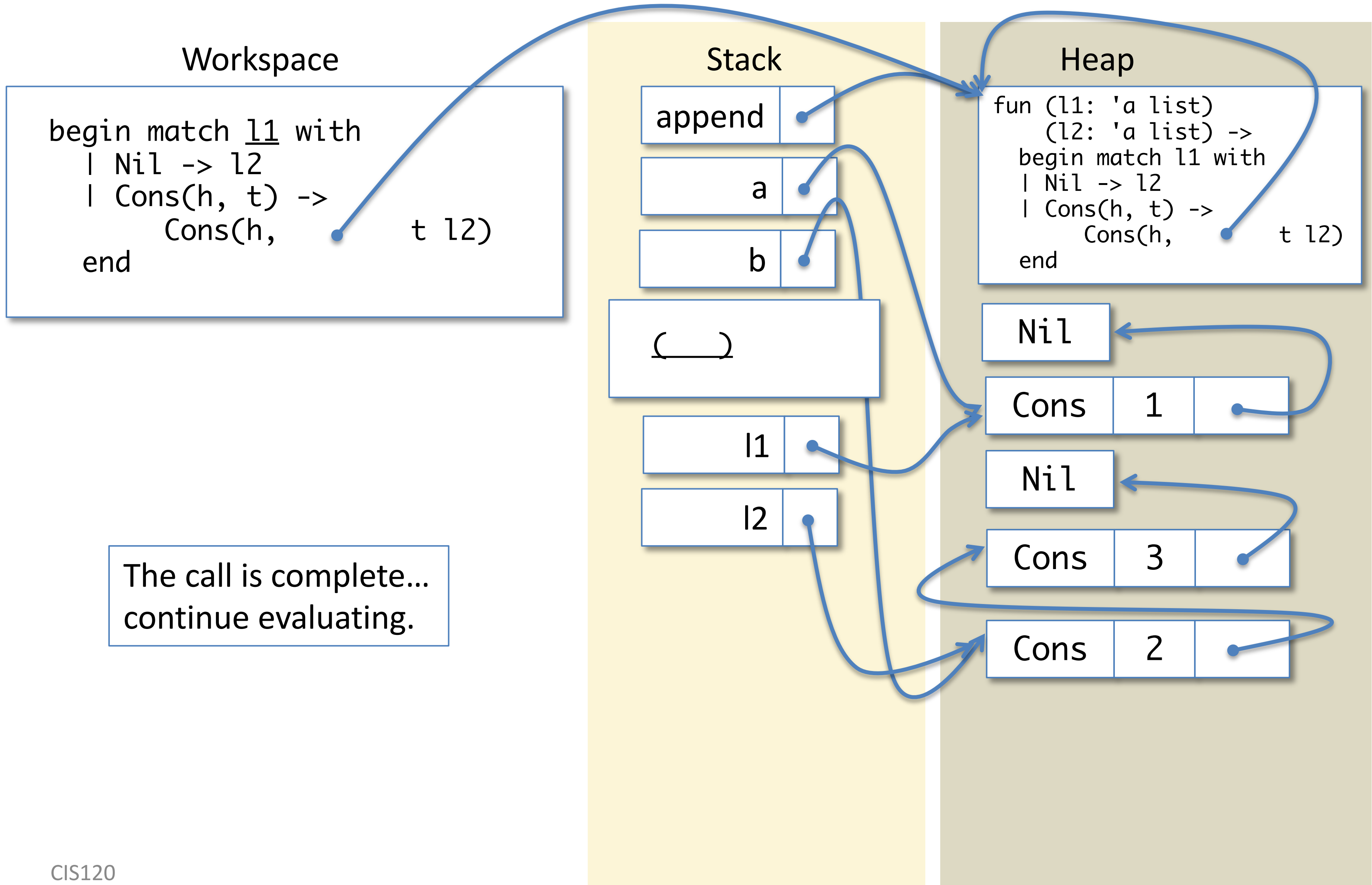

Workspace
Stack
Heap
append
a
b
l1
l2
fun (l1: 'a list)
(l2: 'a list) ->
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Nil
Cons 1
Nil
Cons 3
Cons 2

Install Function Body in Workspace



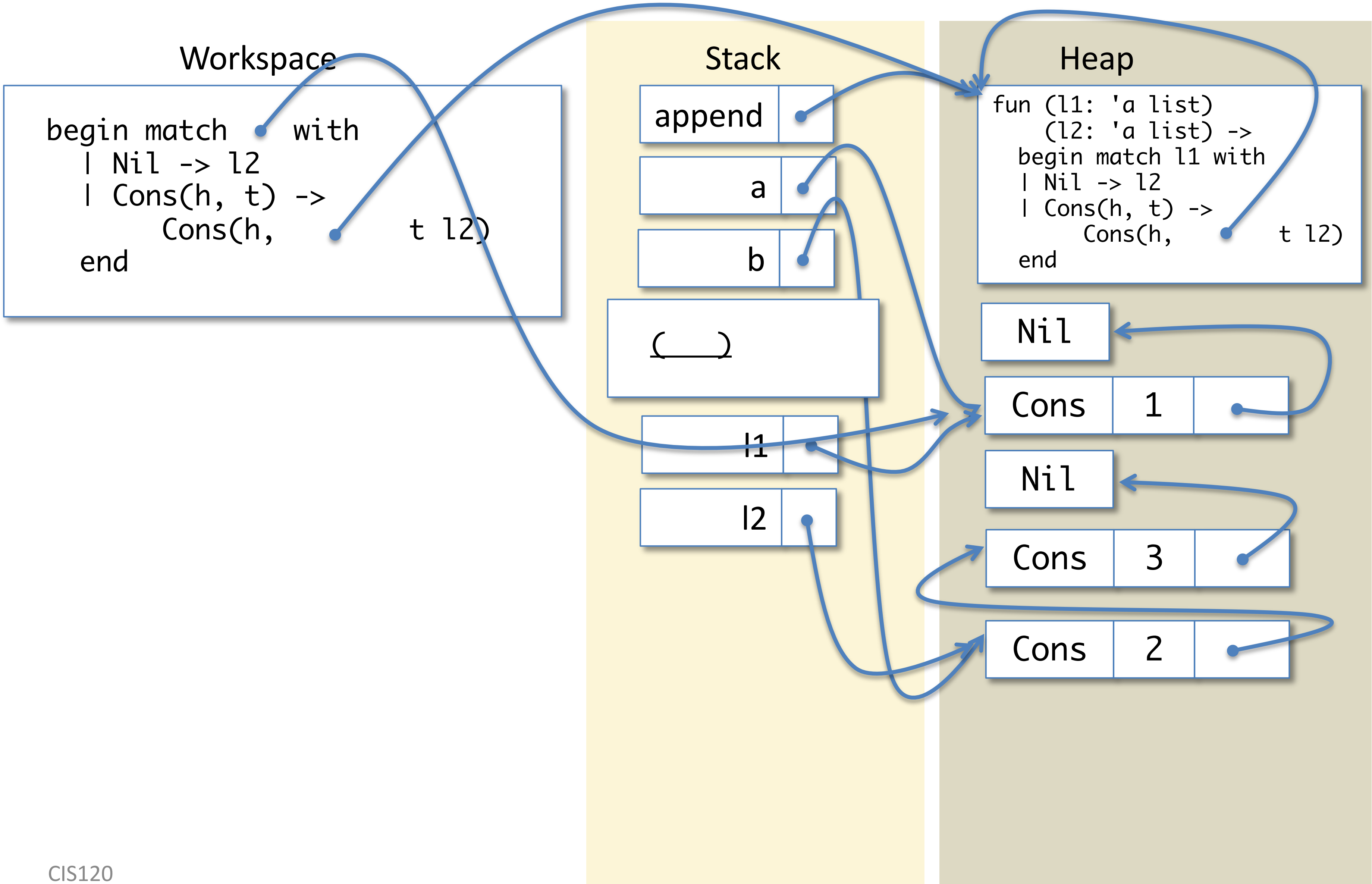
Note: the backpactched reference to 'append' comes with the code body.

Lookup l1



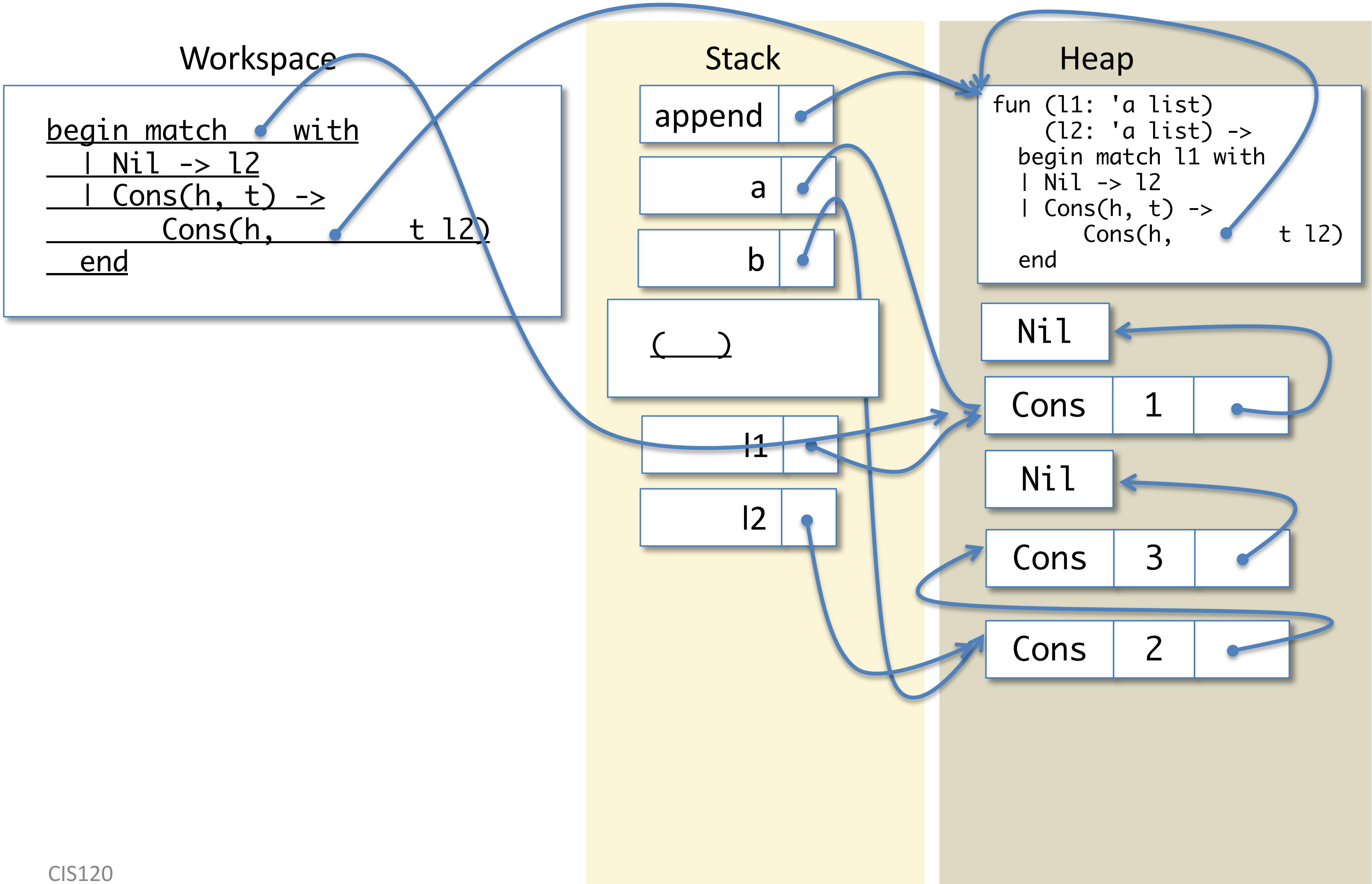
The call is complete… continue evaluating.

Lookup l1

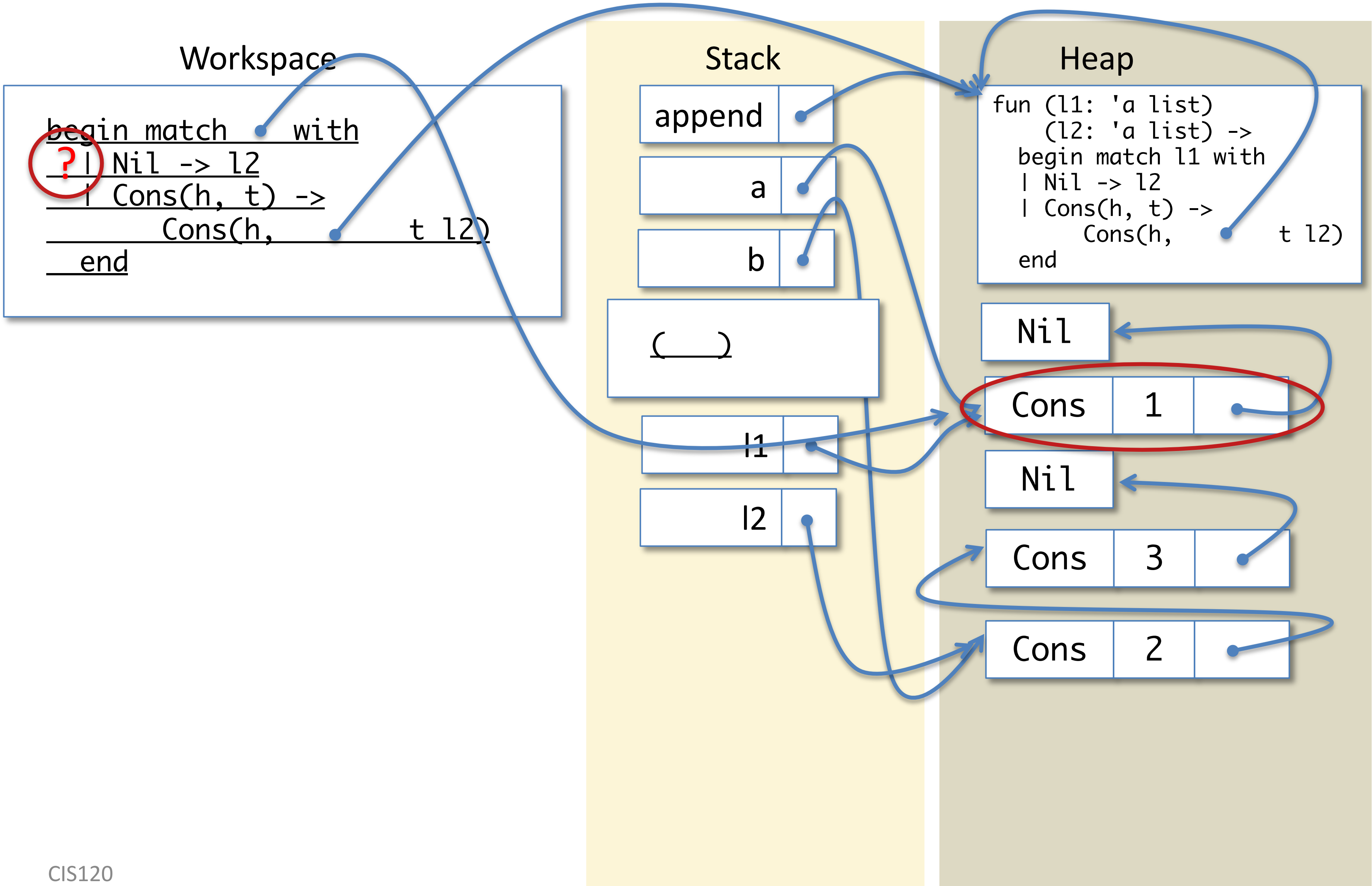


Workspace
begin match with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Stack
append
a
b
( )
l1
l2
Heap
fun (l1: 'a list)
(l2: 'a list) ->
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Nil
Cons
1
Nil
Cons
3
Cons
2

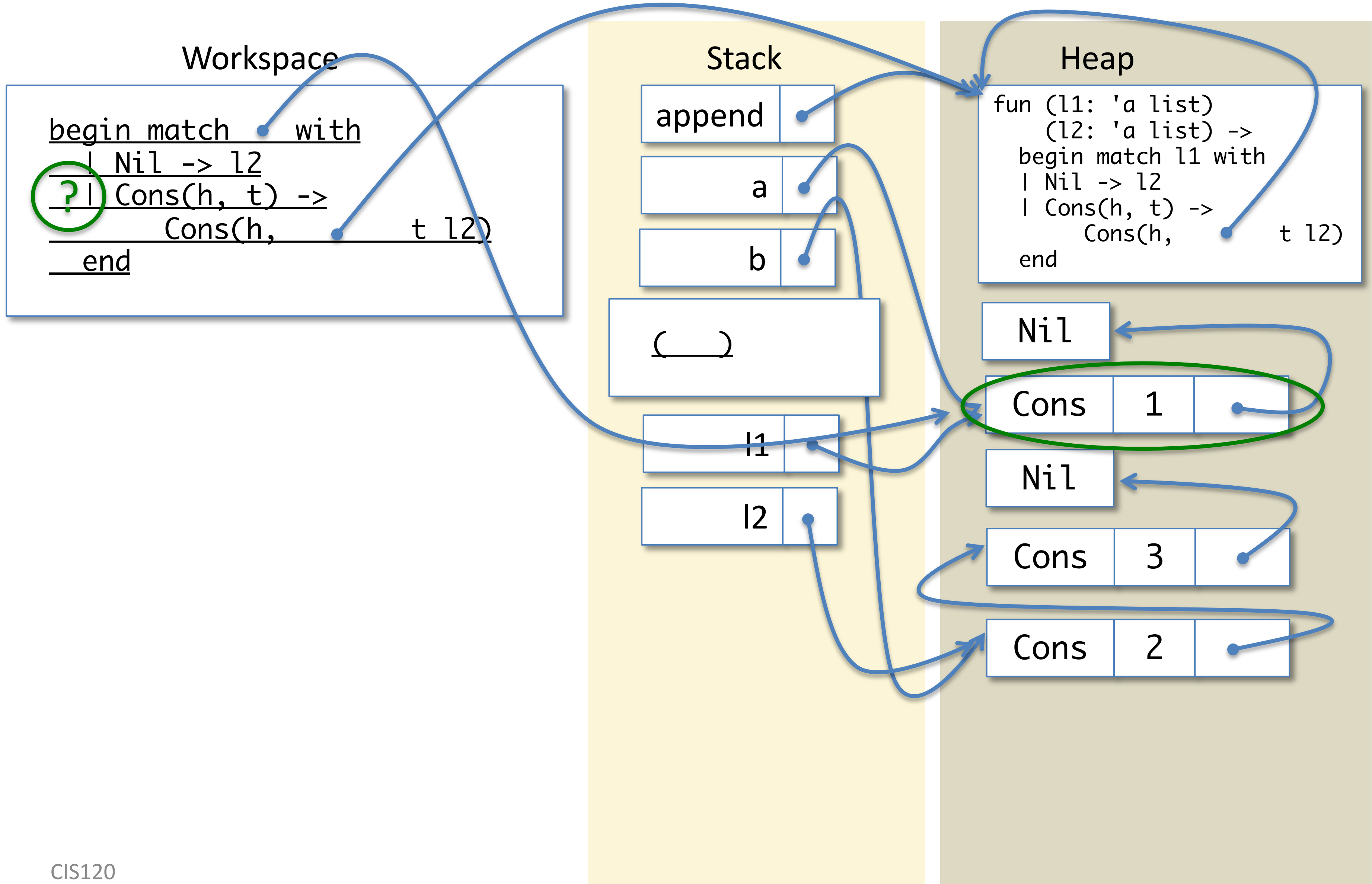
Match Expression



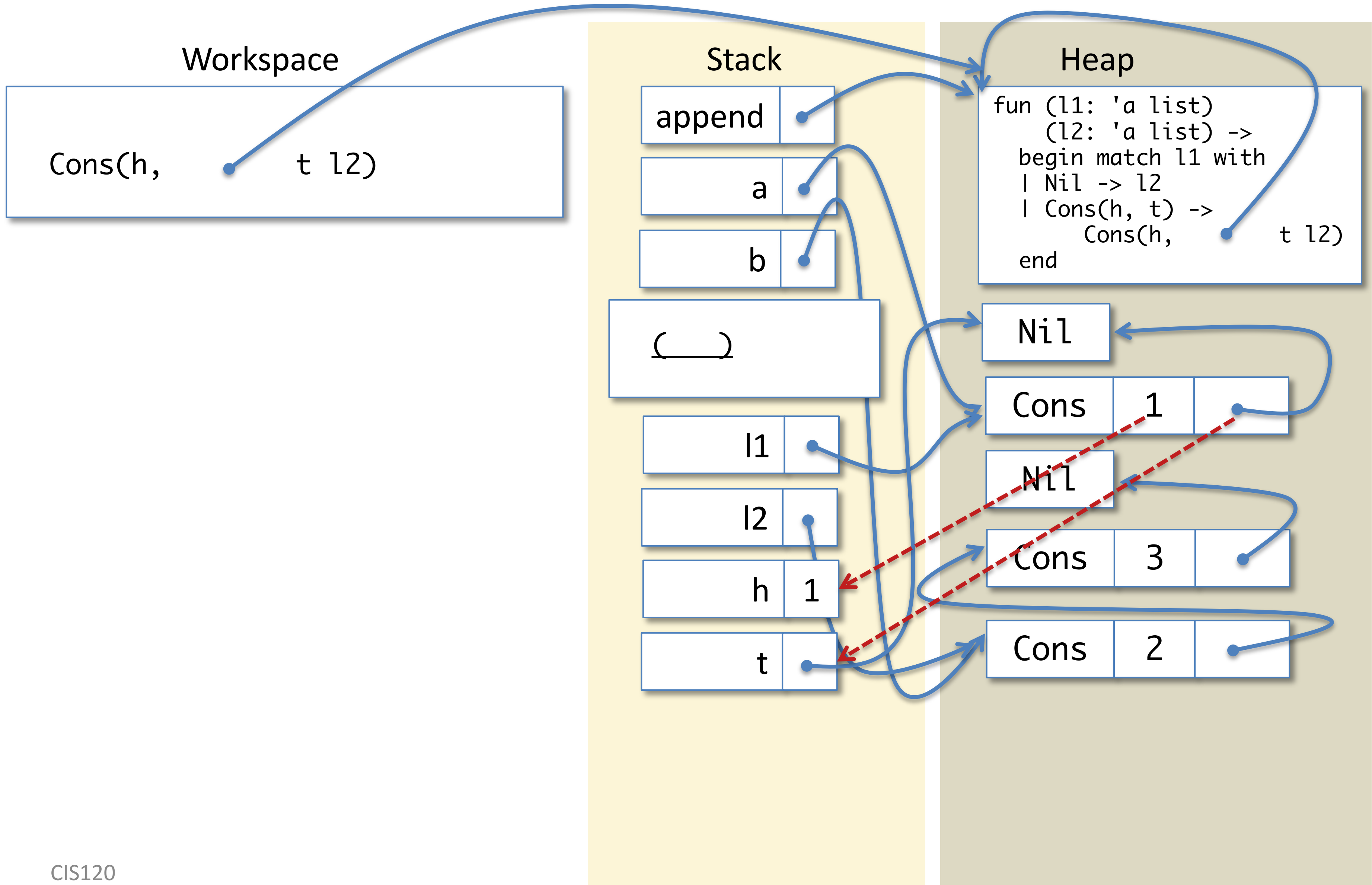
Nil case Doesn't Match



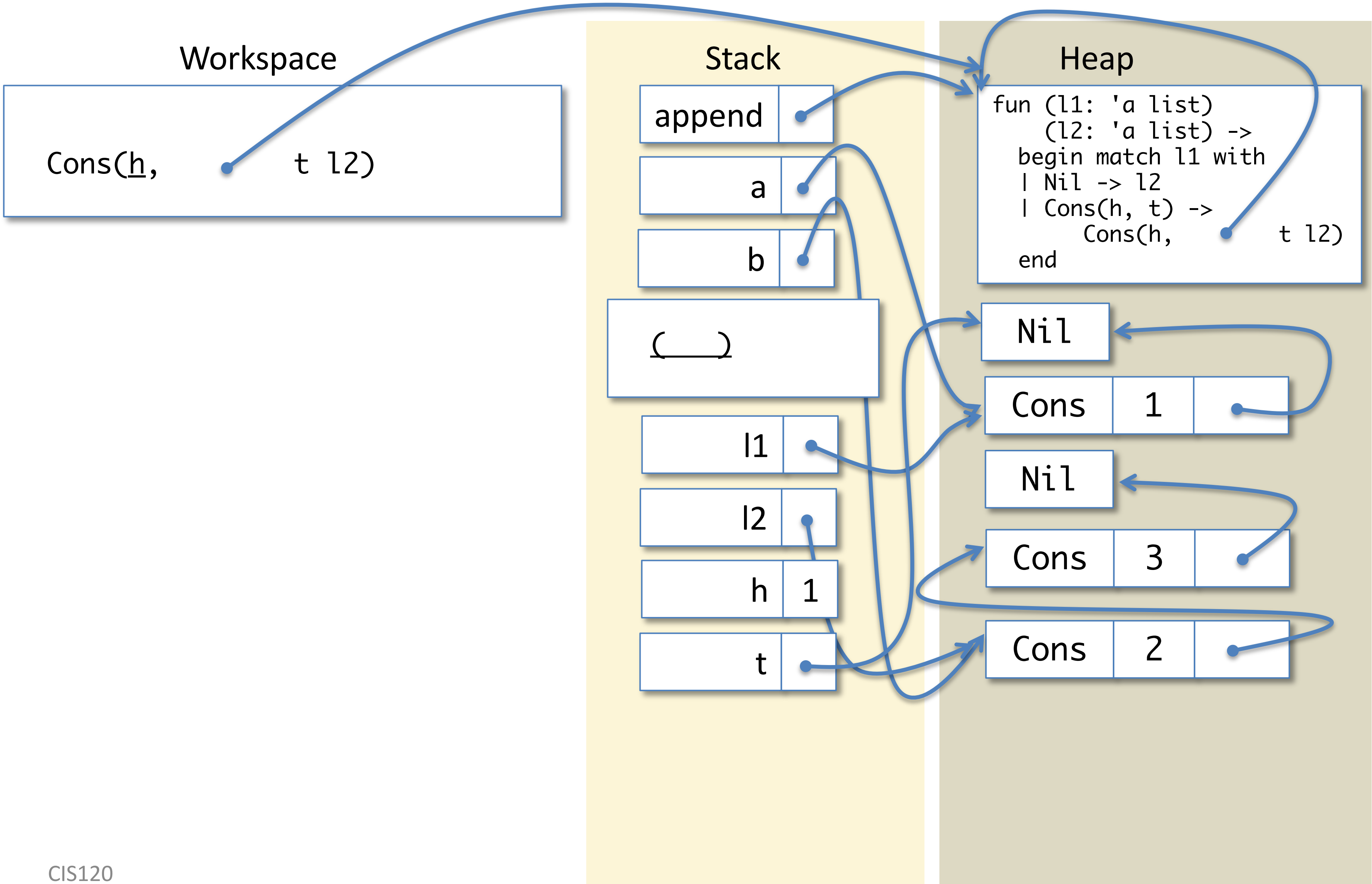
Cons case *Does* Match



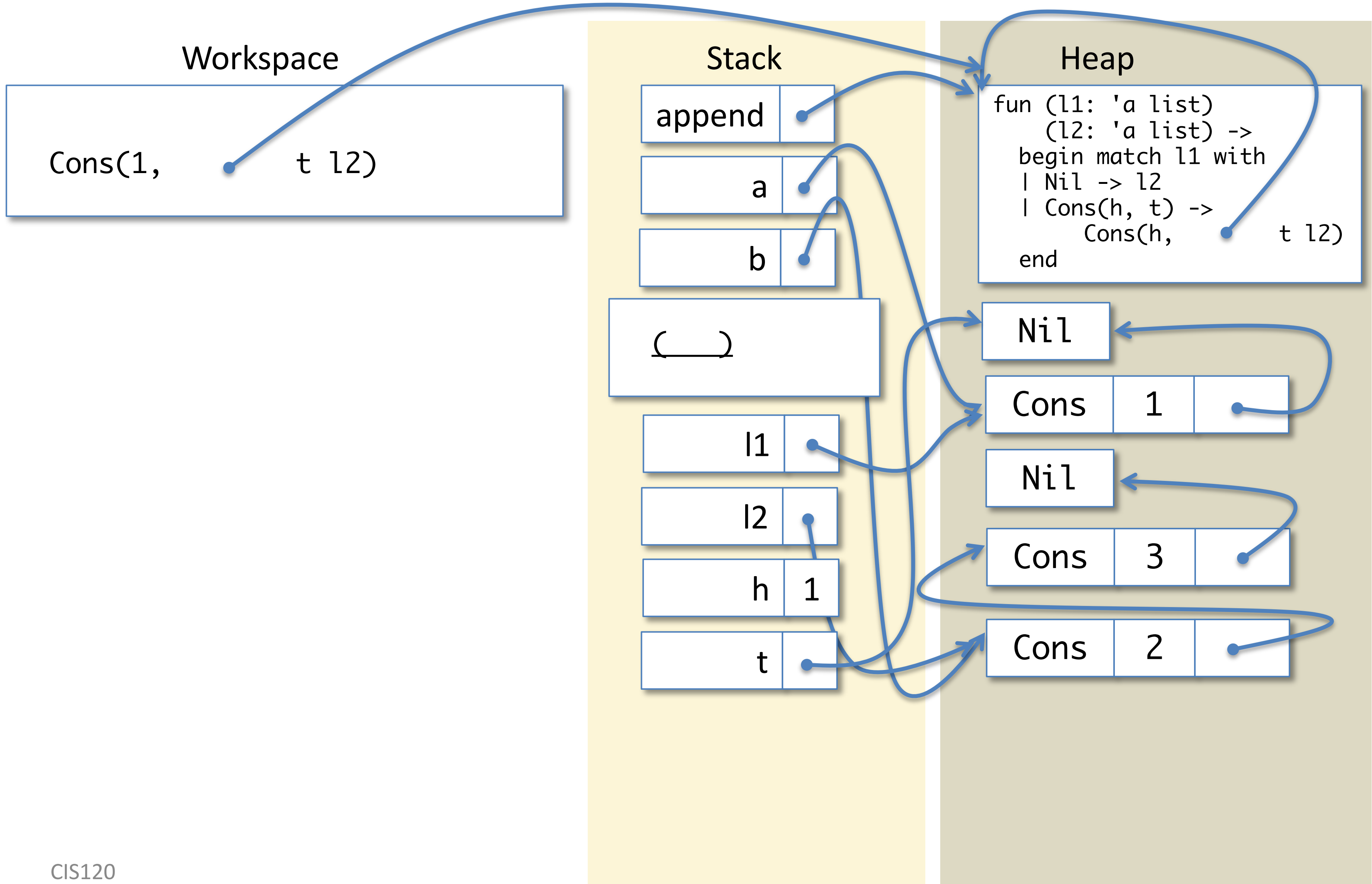
Simplify the Branch: push h, t



Lookup 'h'

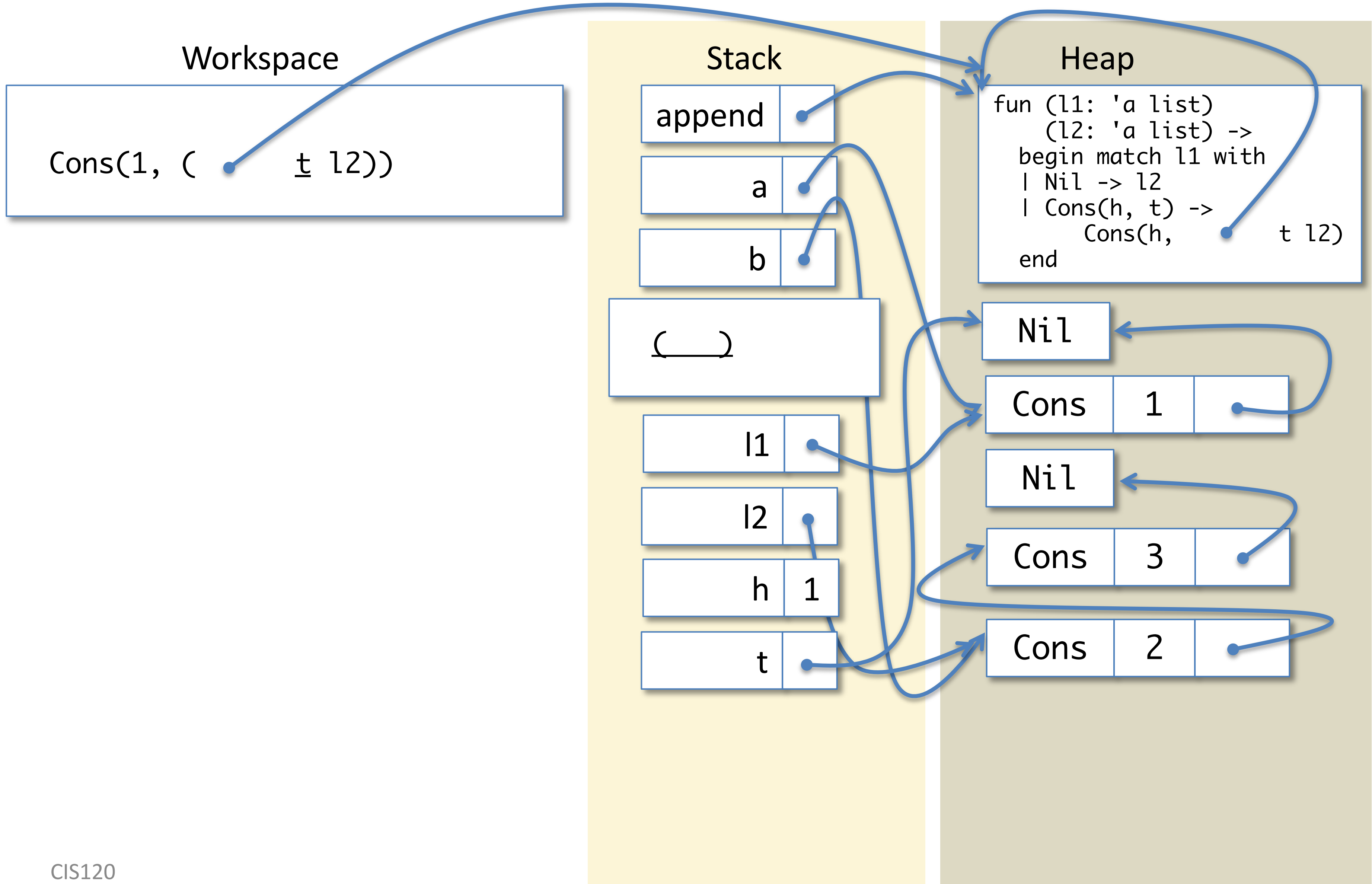

Workspace
Cons(h, t l2)
Stack
append
a
b
( )
l1
l2
h 1
t
Heap
fun (l1: 'a list)
(l2: 'a list) ->
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Nil
Cons 1
Nil
Cons 3
Cons 2

Lookup 'h'



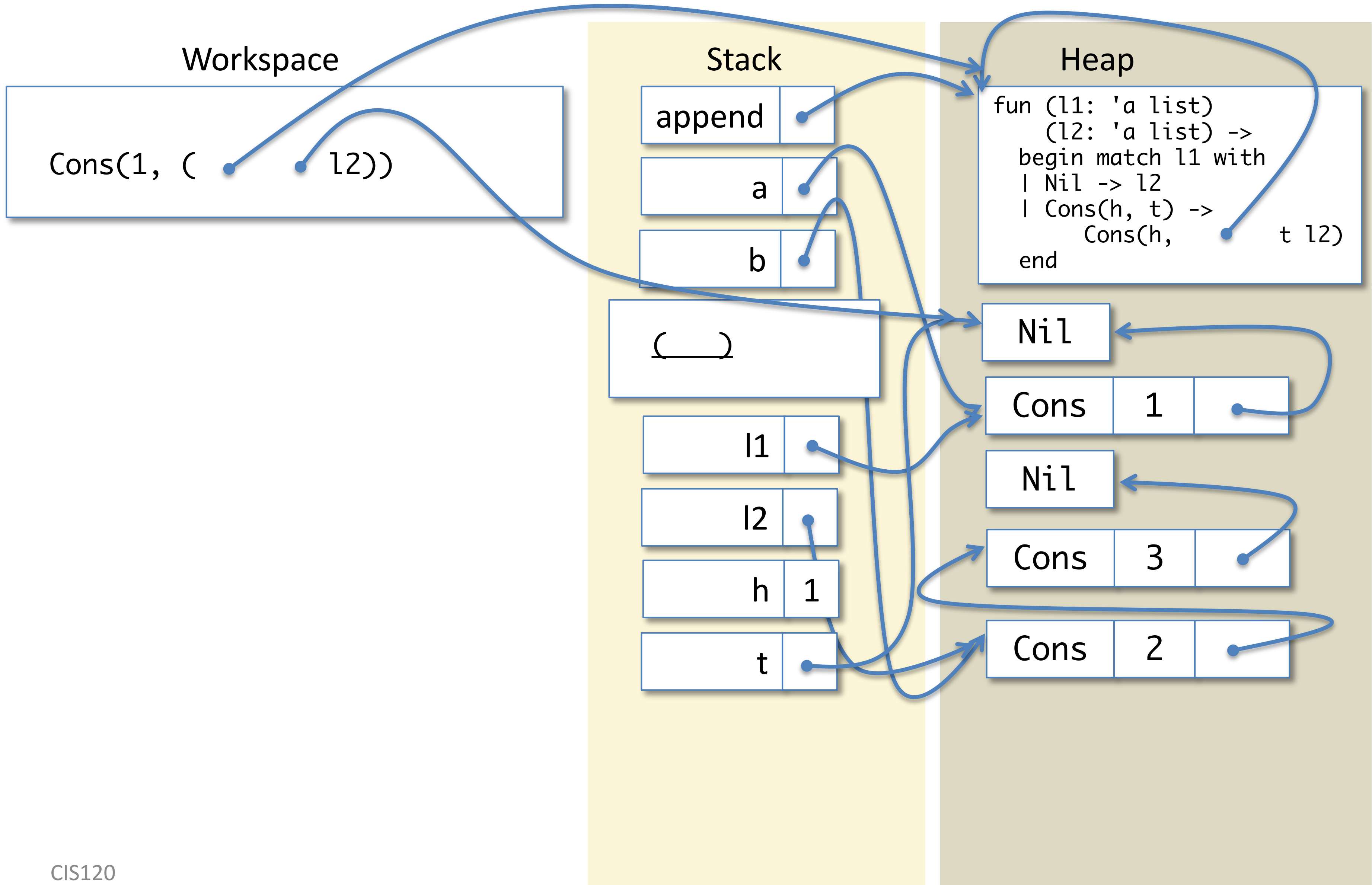
Workspace
Cons(1, t l2)
Stack
append
a
b
( )
l1
l2
h 1
t
Heap
fun (l1: 'a list)
(l2: 'a list) ->
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Nil
Cons 1
Nil
Cons 3
Cons 2

Lookup ‘t’

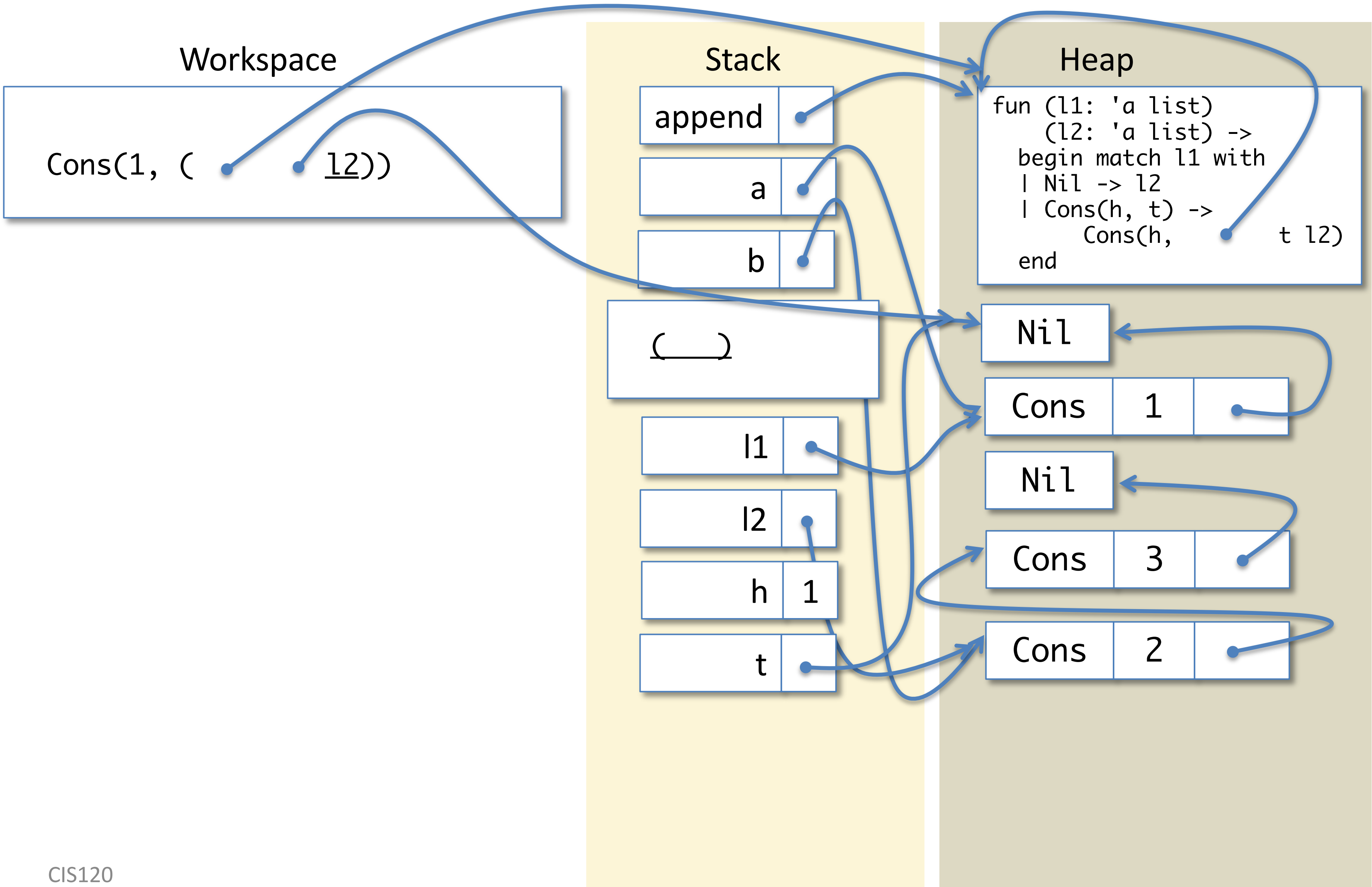


Workspace
Cons(1, (t l2))
Stack
append
a
b
( )
l1
l2
h 1
t
Heap
fun (l1: 'a list)
(l2: 'a list) ->
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Nil
Cons 1
Nil
Cons 3
Cons 2

Lookup ‘t’

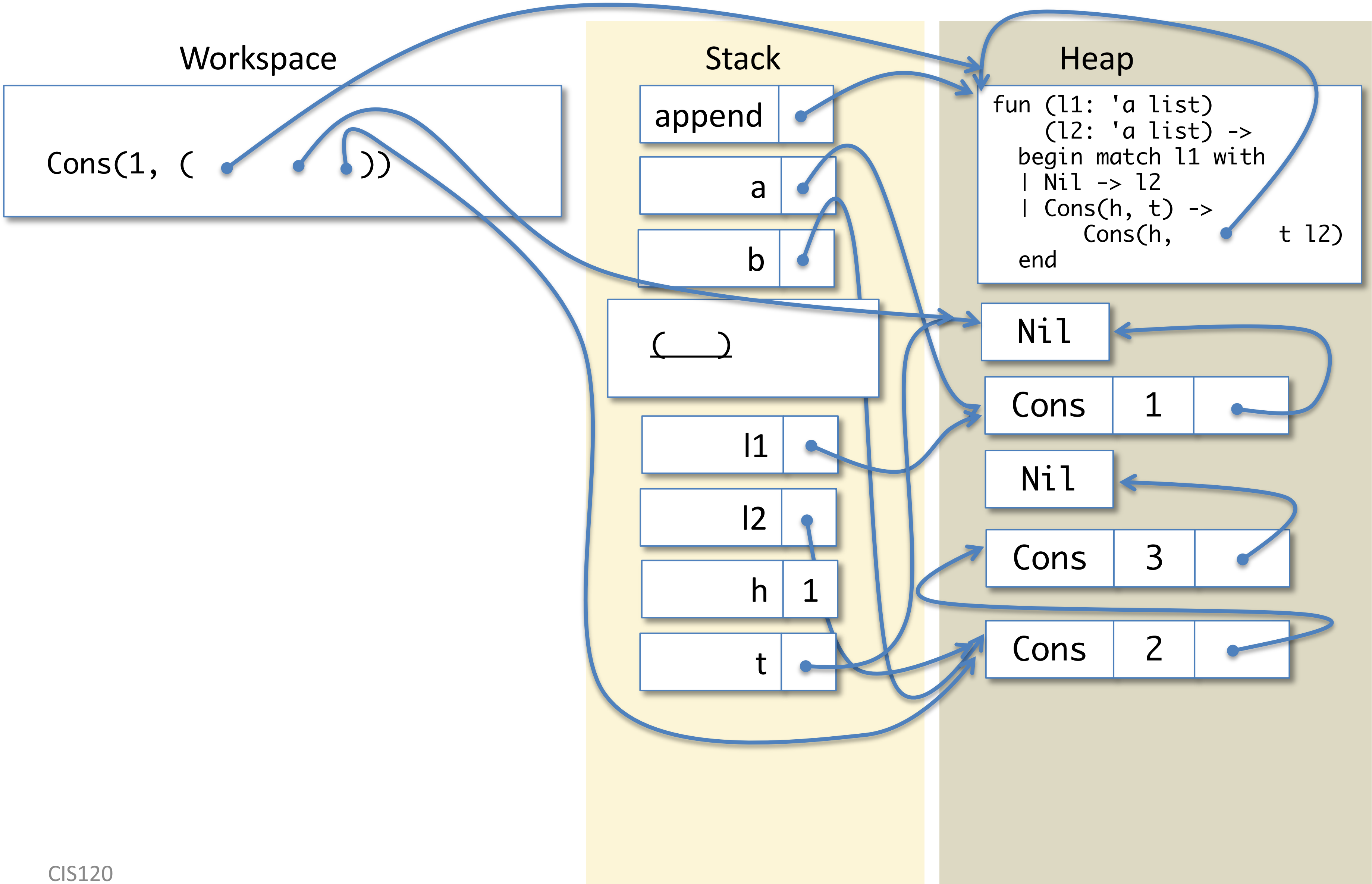

Workspace
Cons(1, (l2))
Stack
append
a
b
( )
l1
l2
h 1
t
Heap
fun (l1: 'a list)
(l2: 'a list) ->
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Nil
Cons 1
Nil
Cons 3
Cons 2

Lookup 'l2'



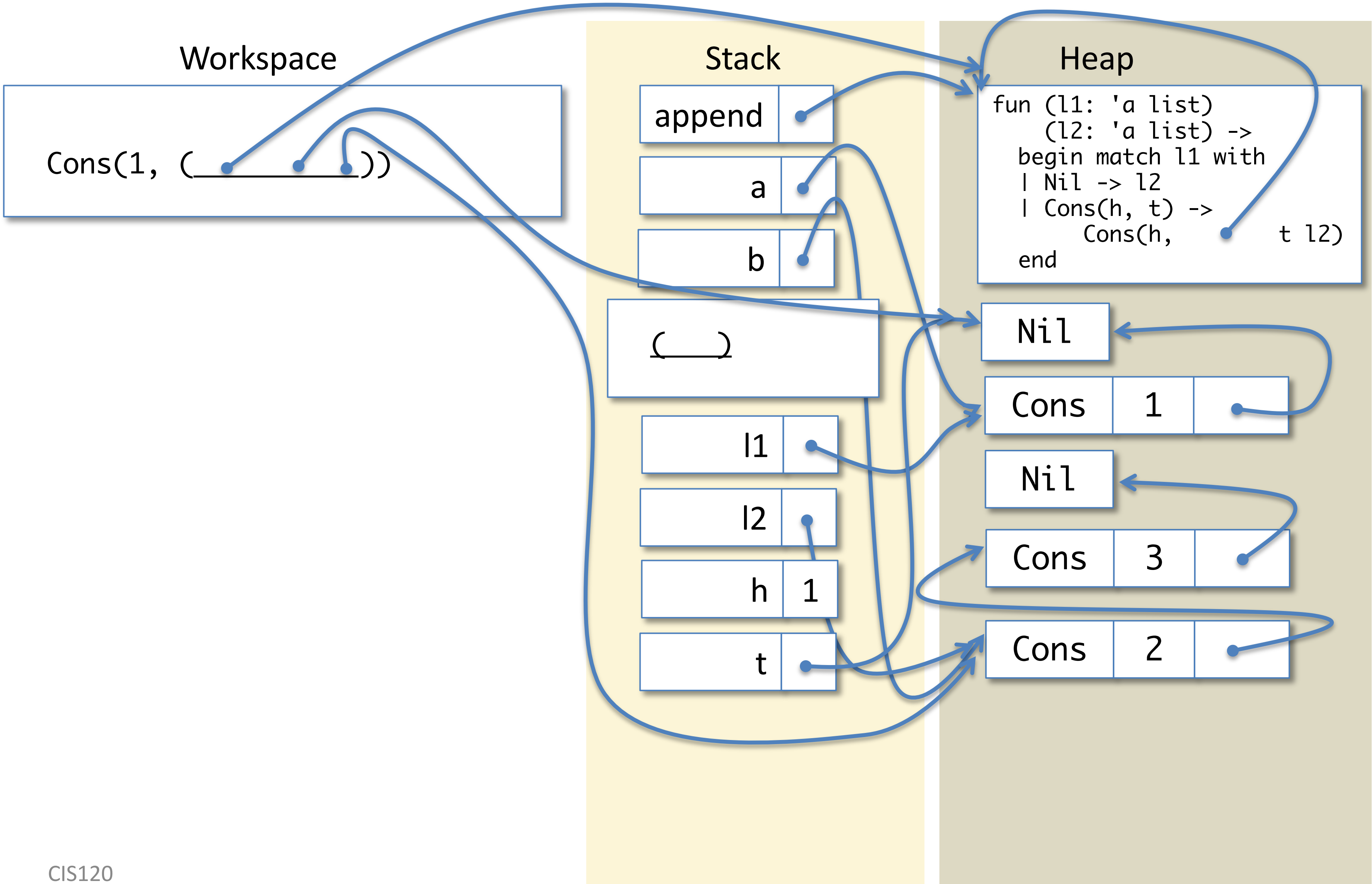
Workspace
Cons(1, (l2))
Stack
append
a
b
( )
l1
l2
h 1
t
Heap
fun (l1: 'a list)
(l2: 'a list) ->
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Nil
Cons 1
Nil
Cons 3
Cons 2

Lookup 'l2'

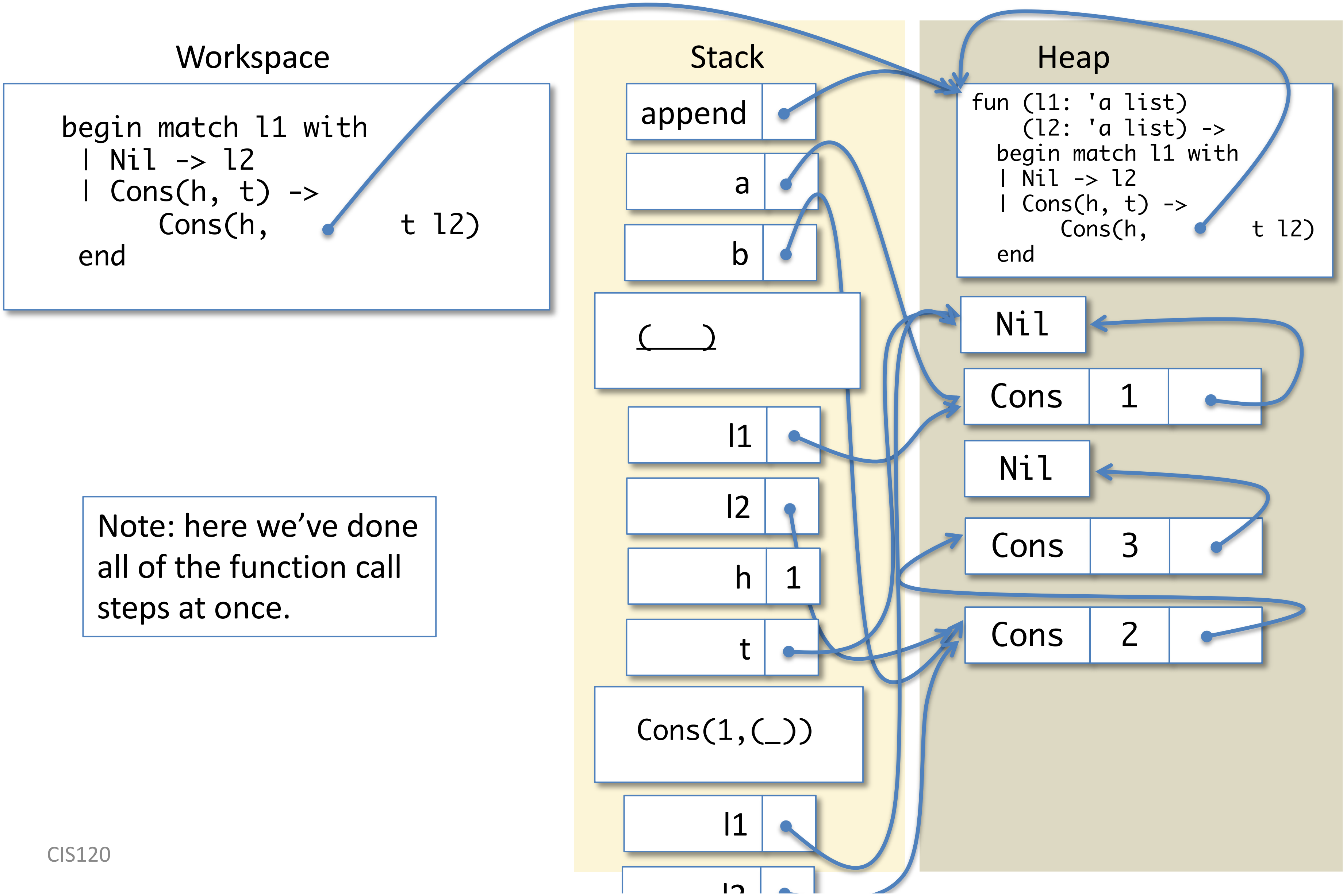


Workspace
Cons(1, ( ))
Stack
append
a
b
( )
l1
l2
h
1
t
Heap
fun (l1: 'a list)
(l2: 'a list) ->
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Nil
Cons
1
Nil
Cons
3
Cons
2

Do the Function Call

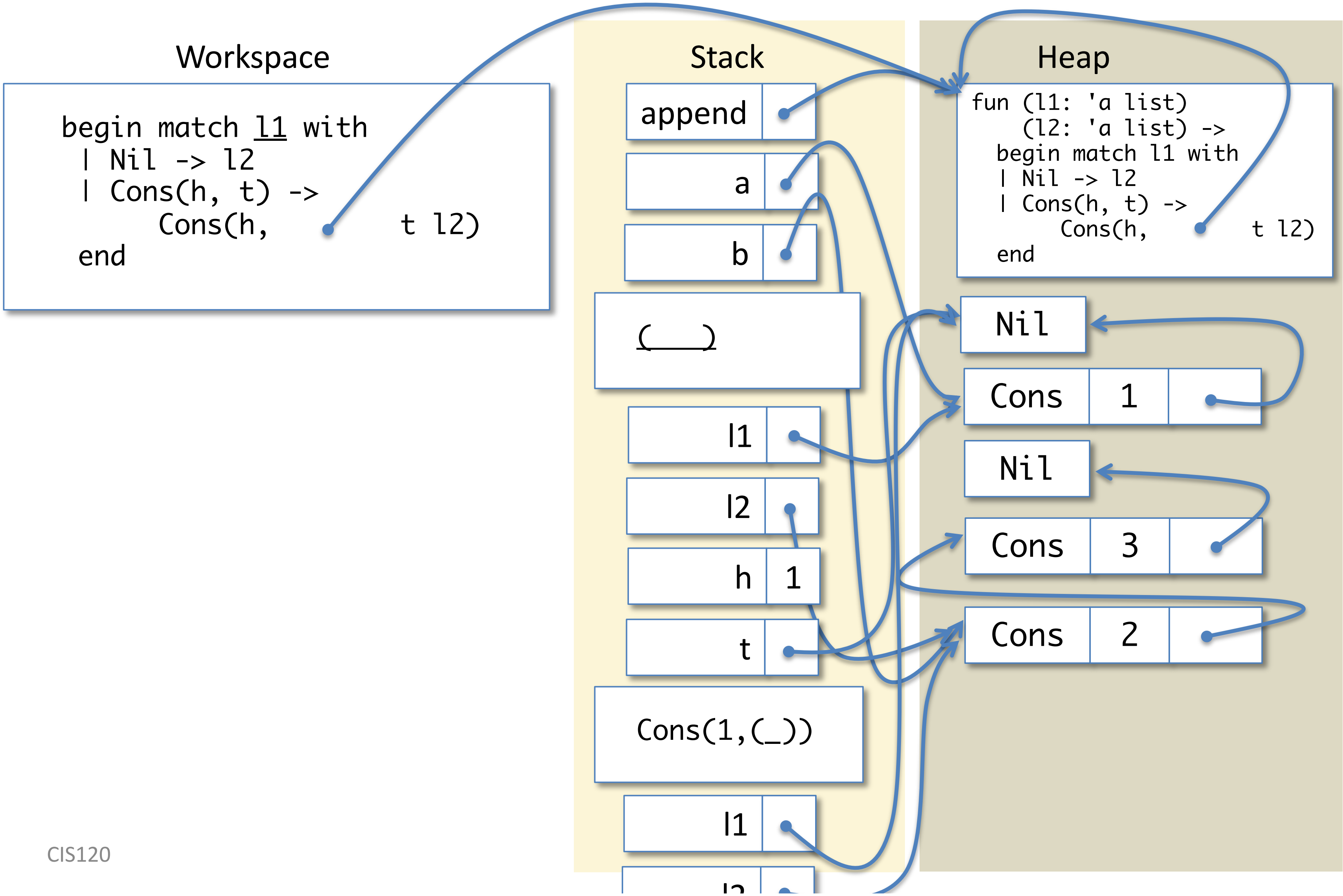


Save the Workspace; push l1, l2

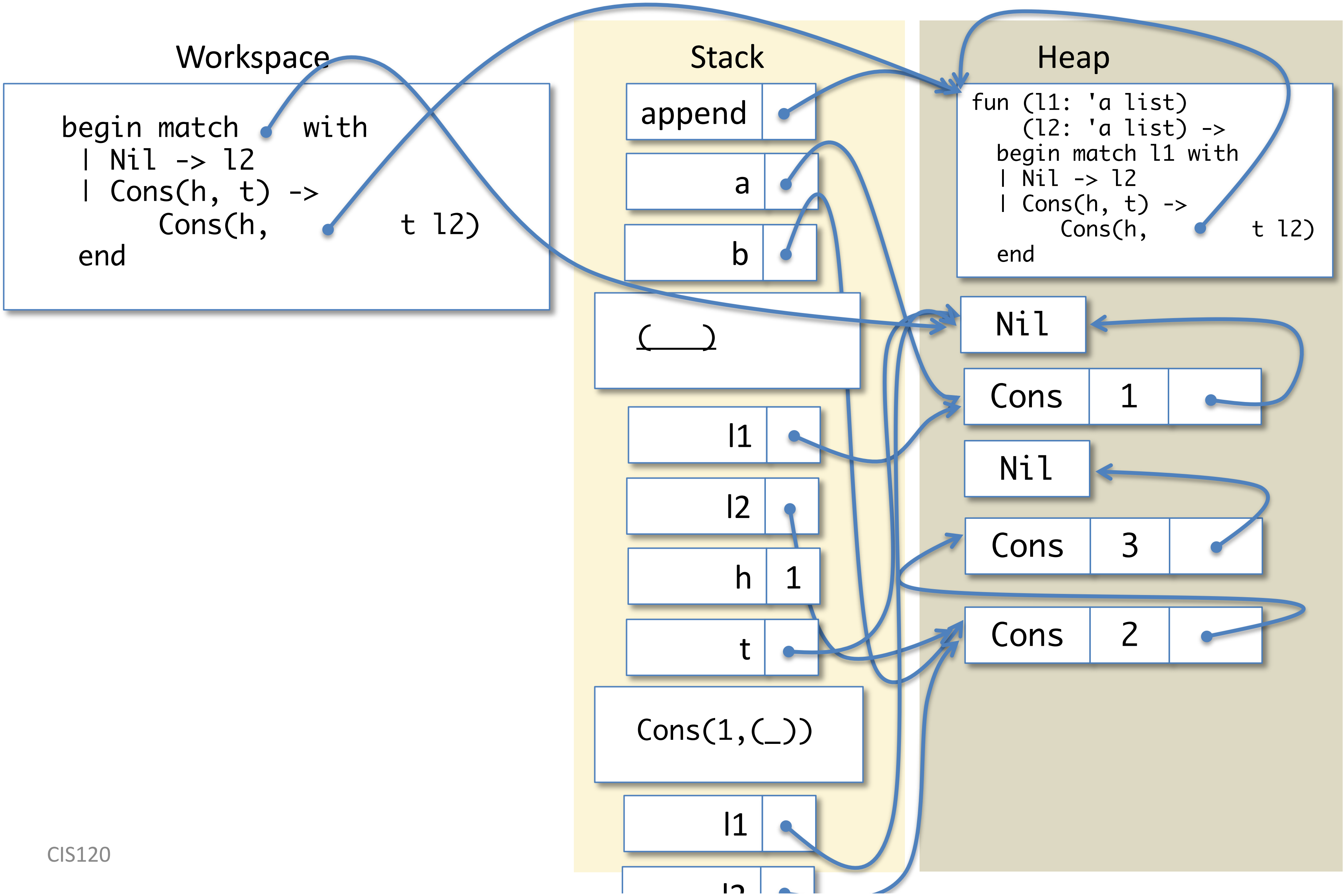


Note: here we've done all of the function call steps at once.

Lookup ‘l1’

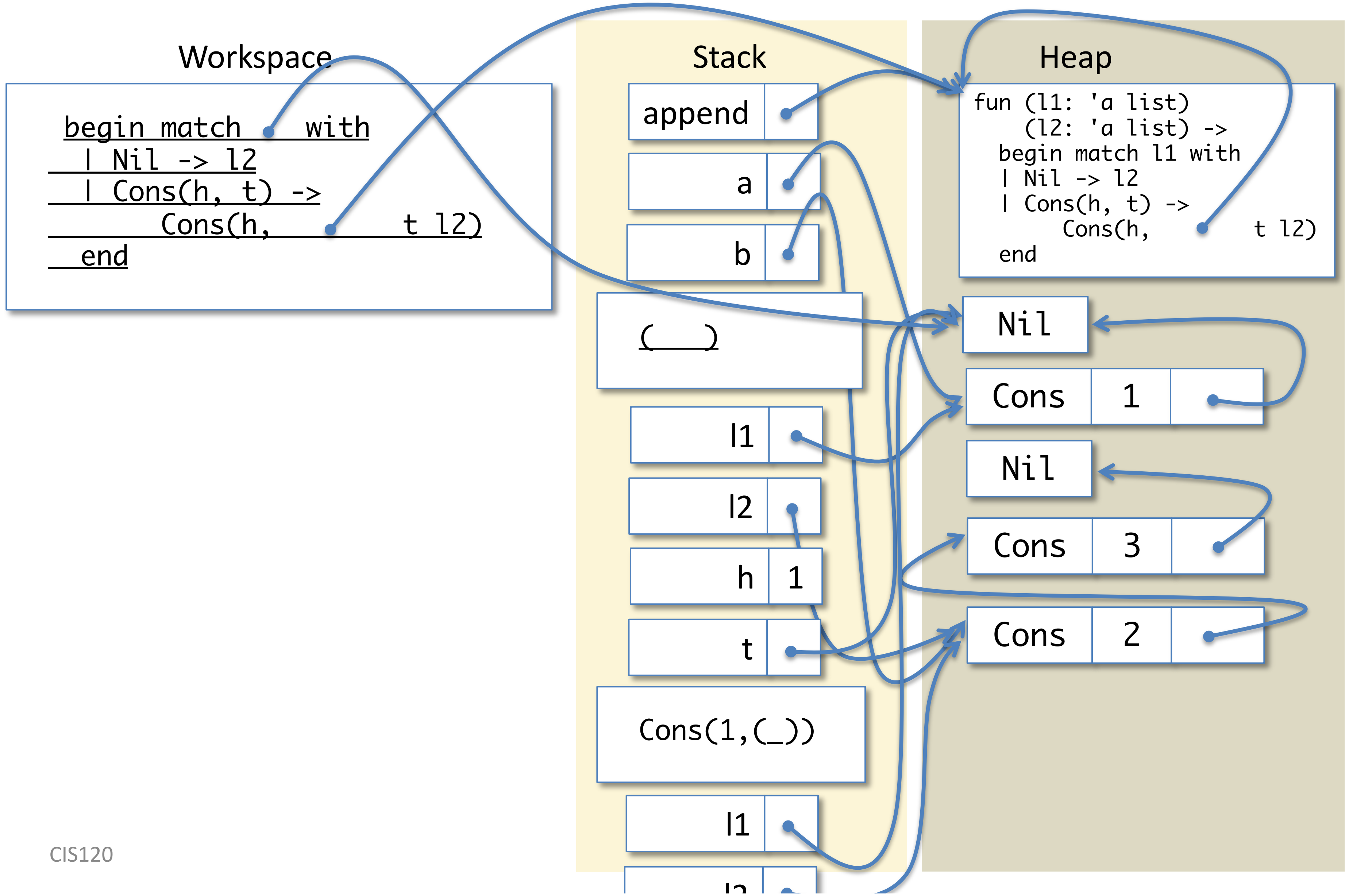

Workspace
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Stack
append
a
b
( )
l1
l2
h 1
t
Cons(1,(_))
l1
l2
Heap
fun (l1: 'a list)
(l2: 'a list) ->
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Nil
Cons 1
Nil
Cons 3
Cons 2

Lookup ‘l1’

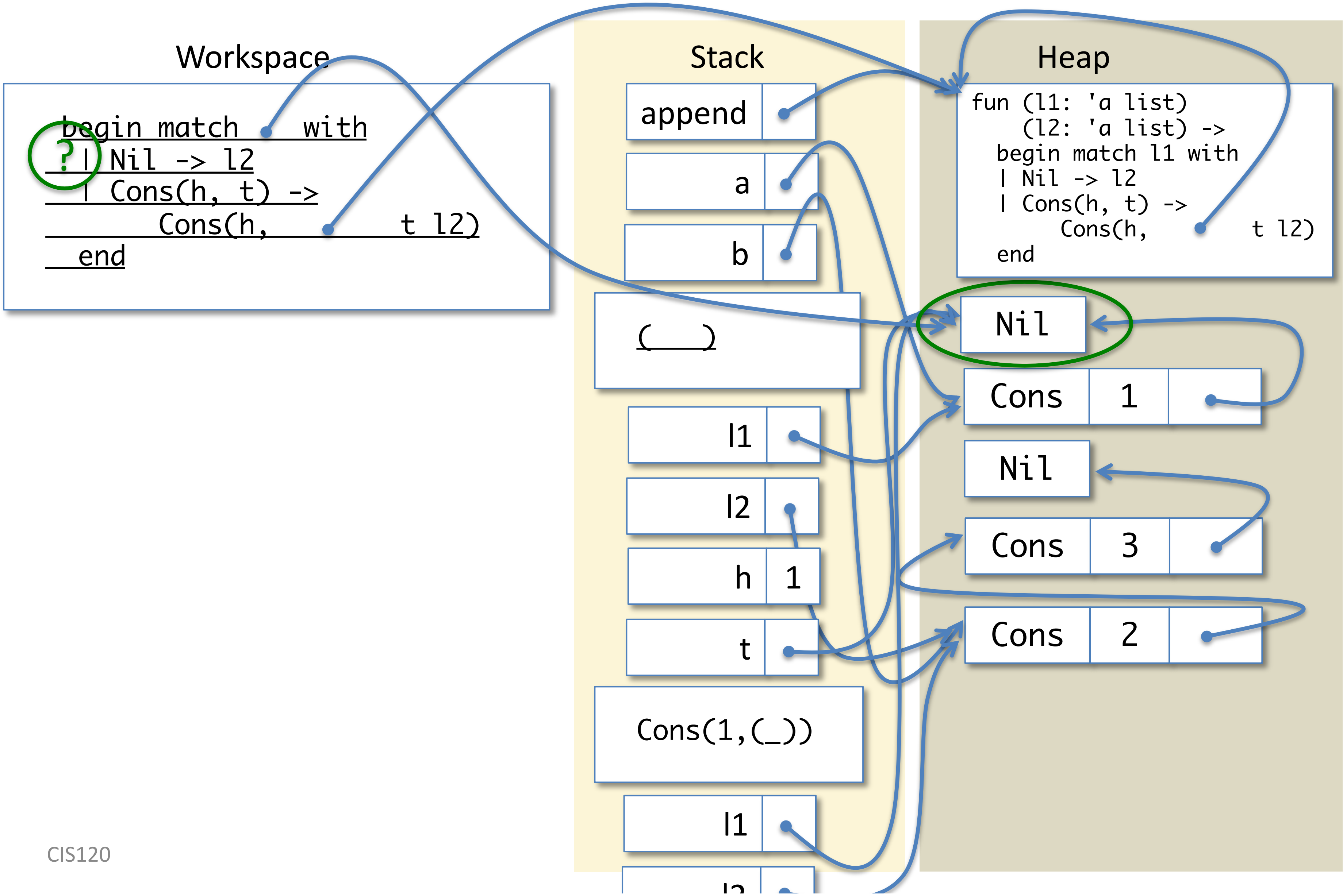


Workspace
begin match with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Stack
append
a
b
( )
l1
l2
h
1
t
Cons(1,(_))
l1
Heap
fun (l1: 'a list)
(l2: 'a list) ->
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Nil
Cons
1
Nil
Cons
3
Cons
2

Match Expression



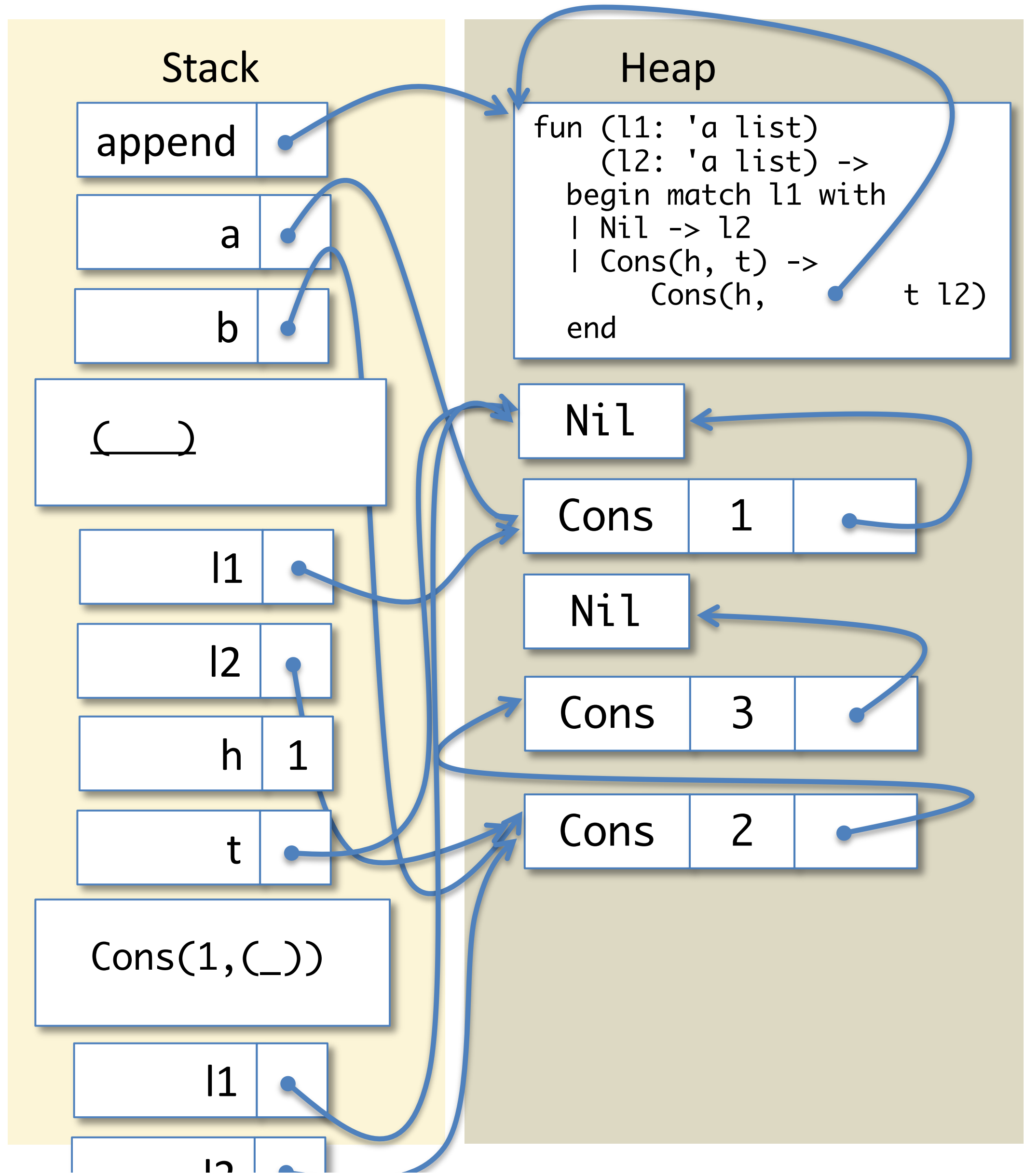
The Nil case Matches



Simplify the Branch (nothing to push)

Workspace

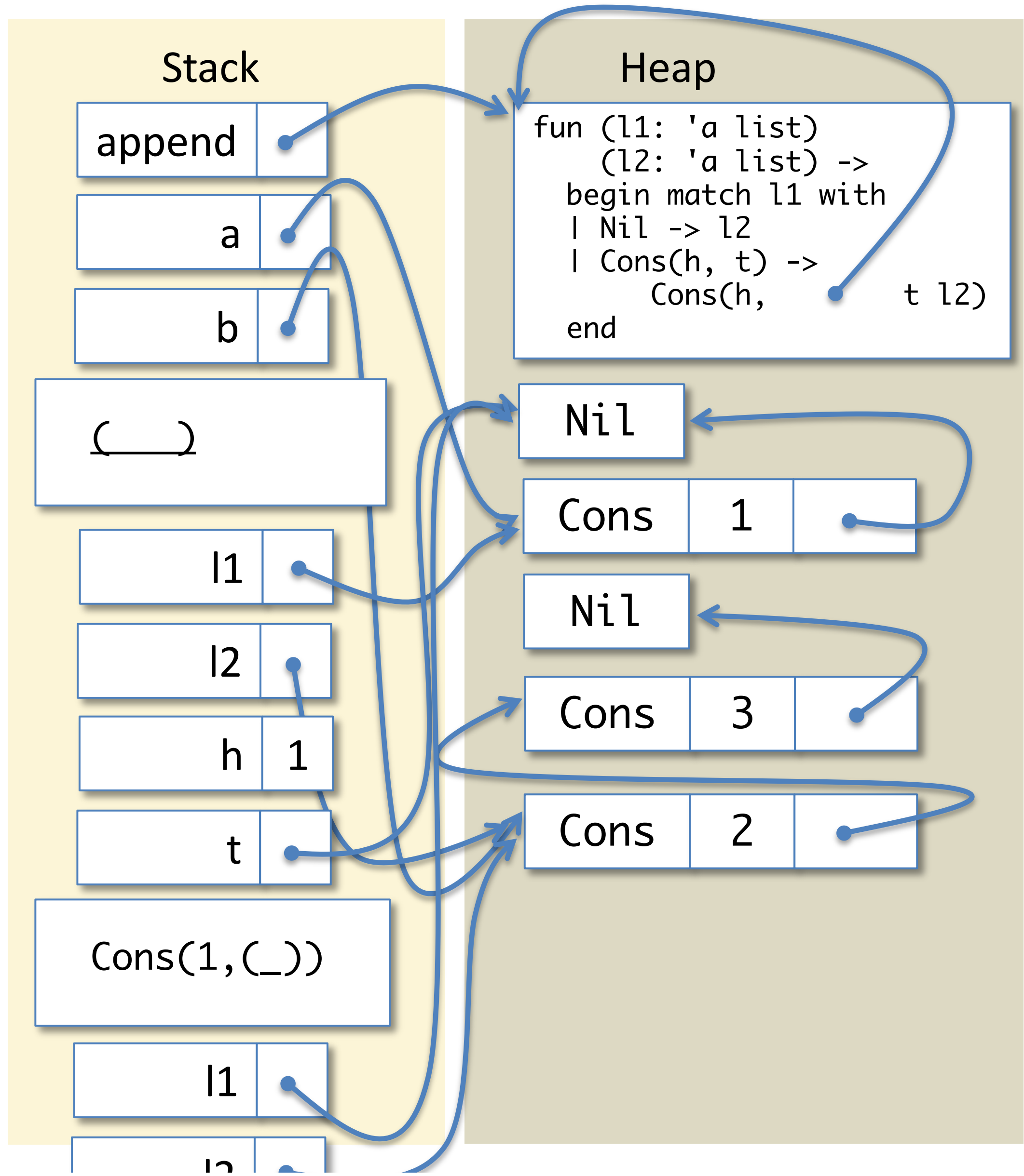
```
l2
```



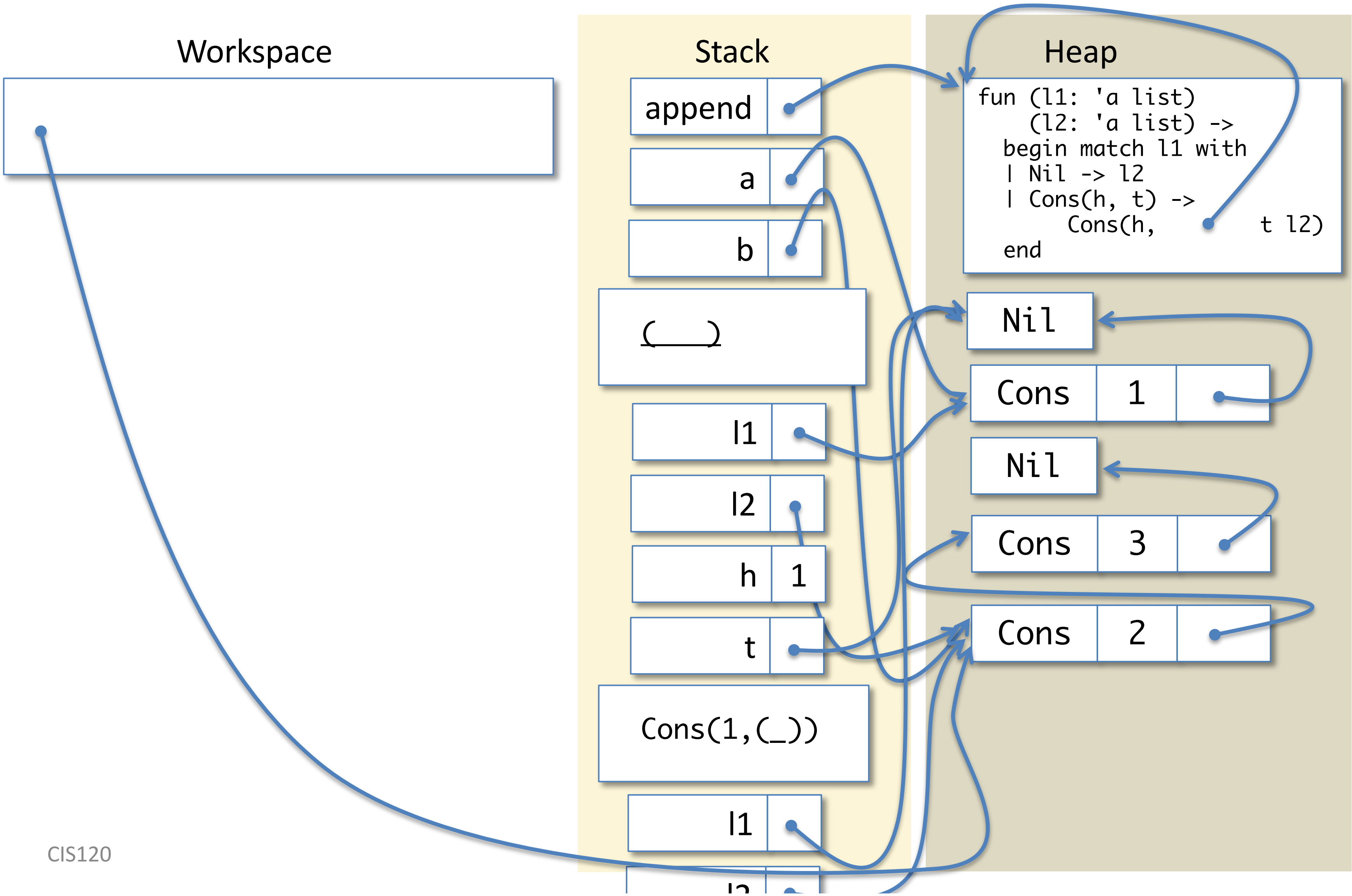
Lookup ‘l2’

Workspace

`l2`

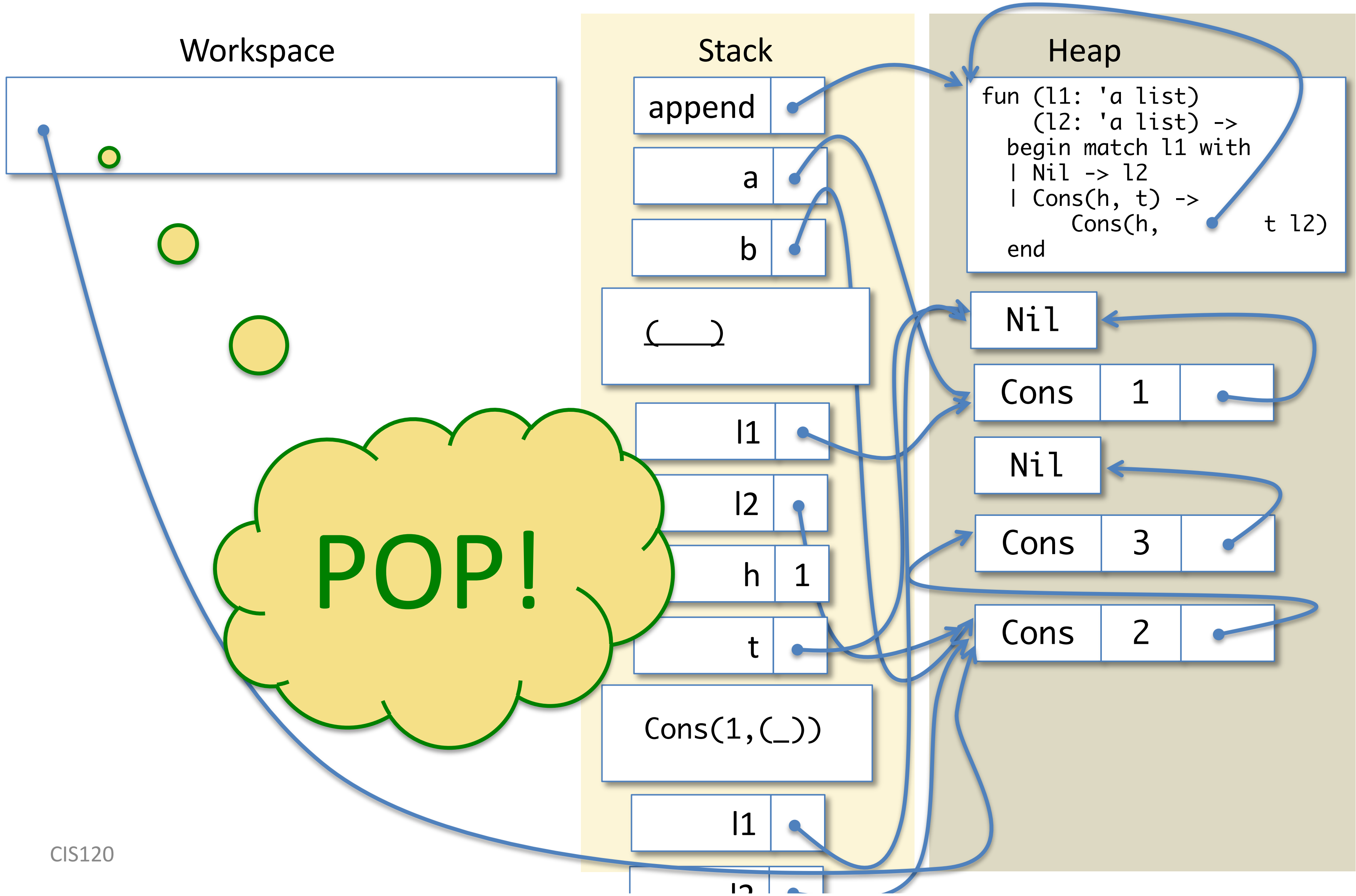


Lookup 'l2'

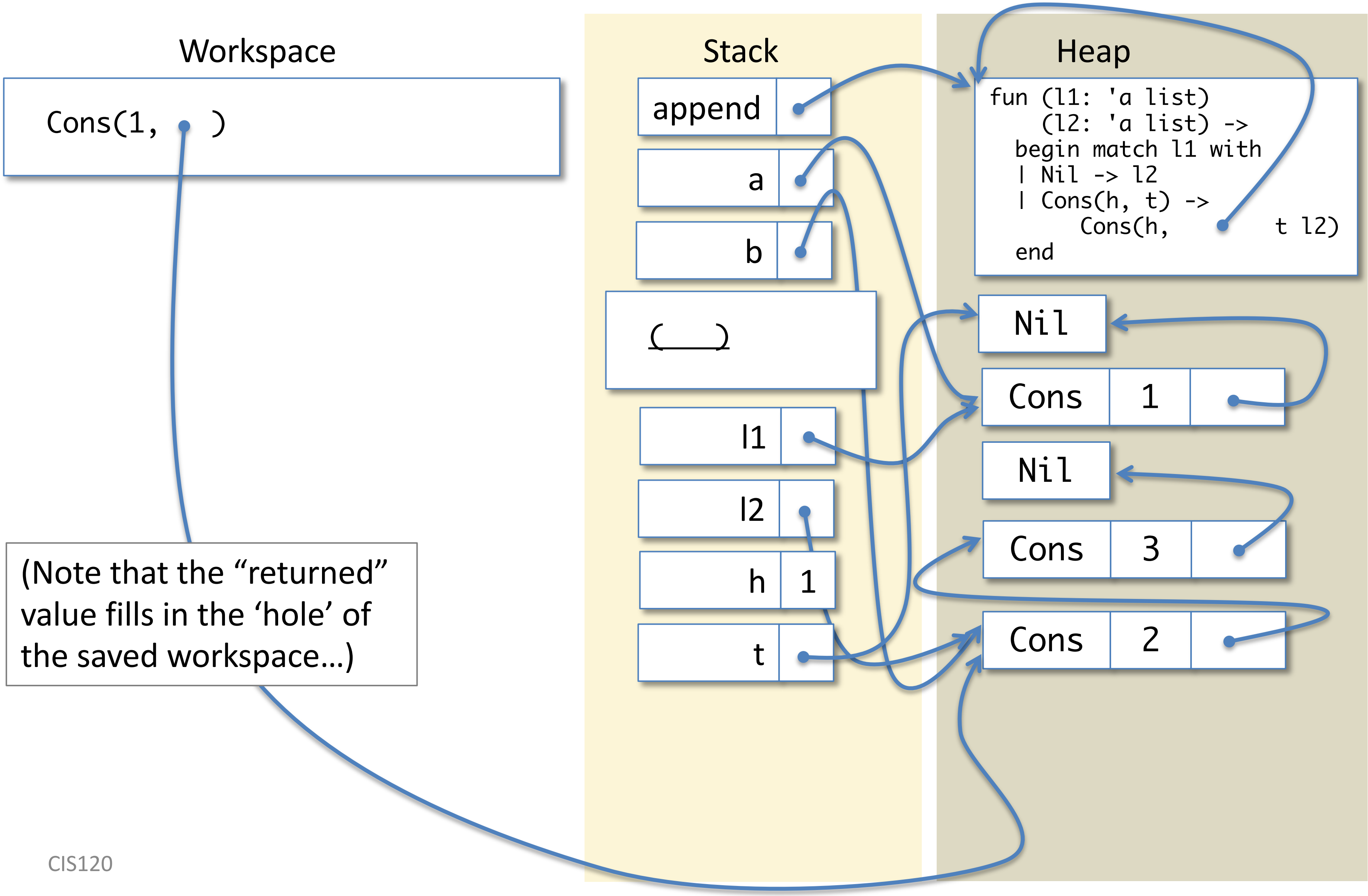


Workspace
Stack
Heap
append
a
b
(___)
l1
l2
h
1
t
Cons(1,(_))
l1
fun (l1: 'a list)
(l2: 'a list) ->
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Nil
Cons 1
Nil
Cons 3
Cons 2

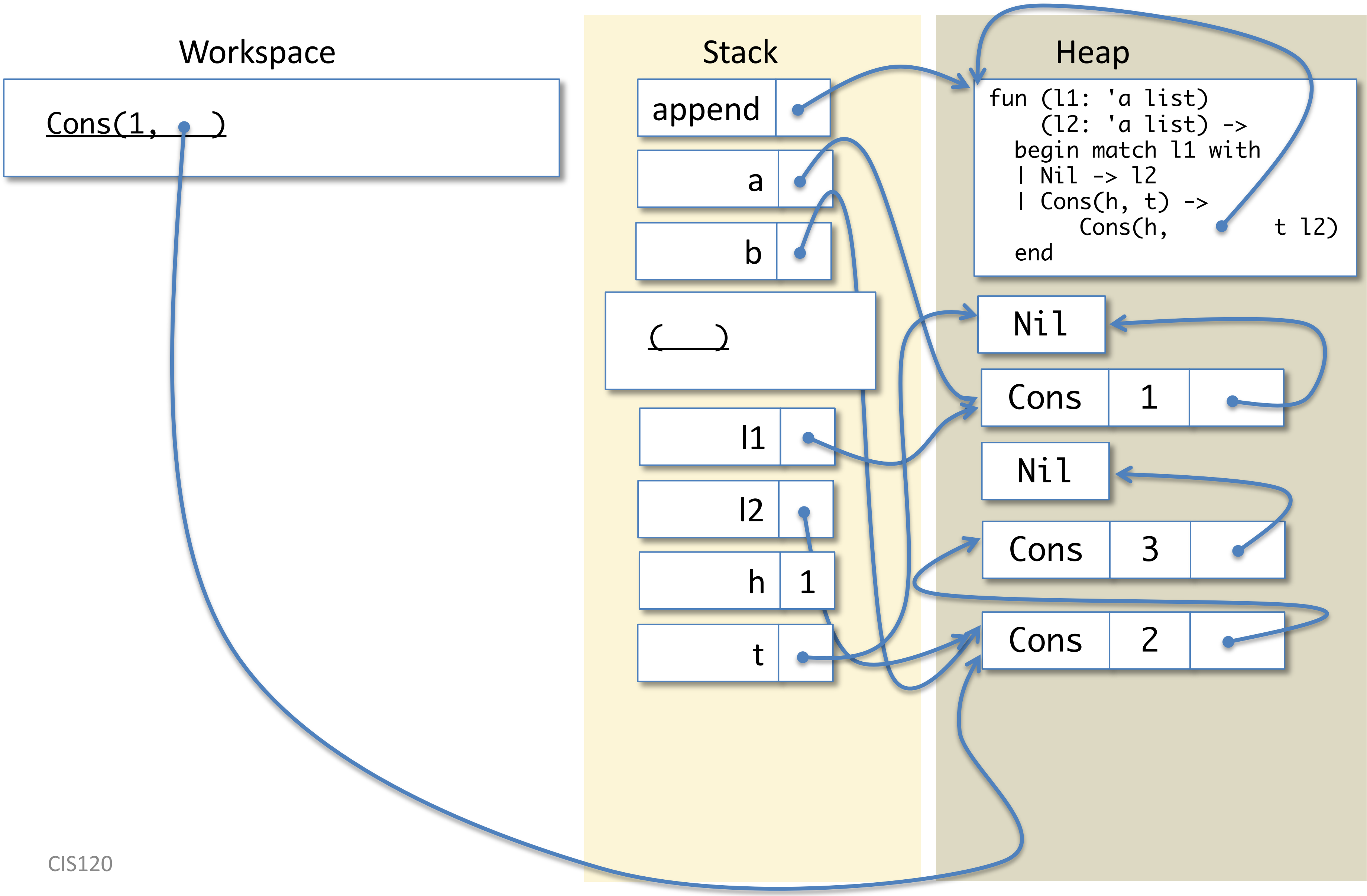
Done! Pop stack to last Workspace



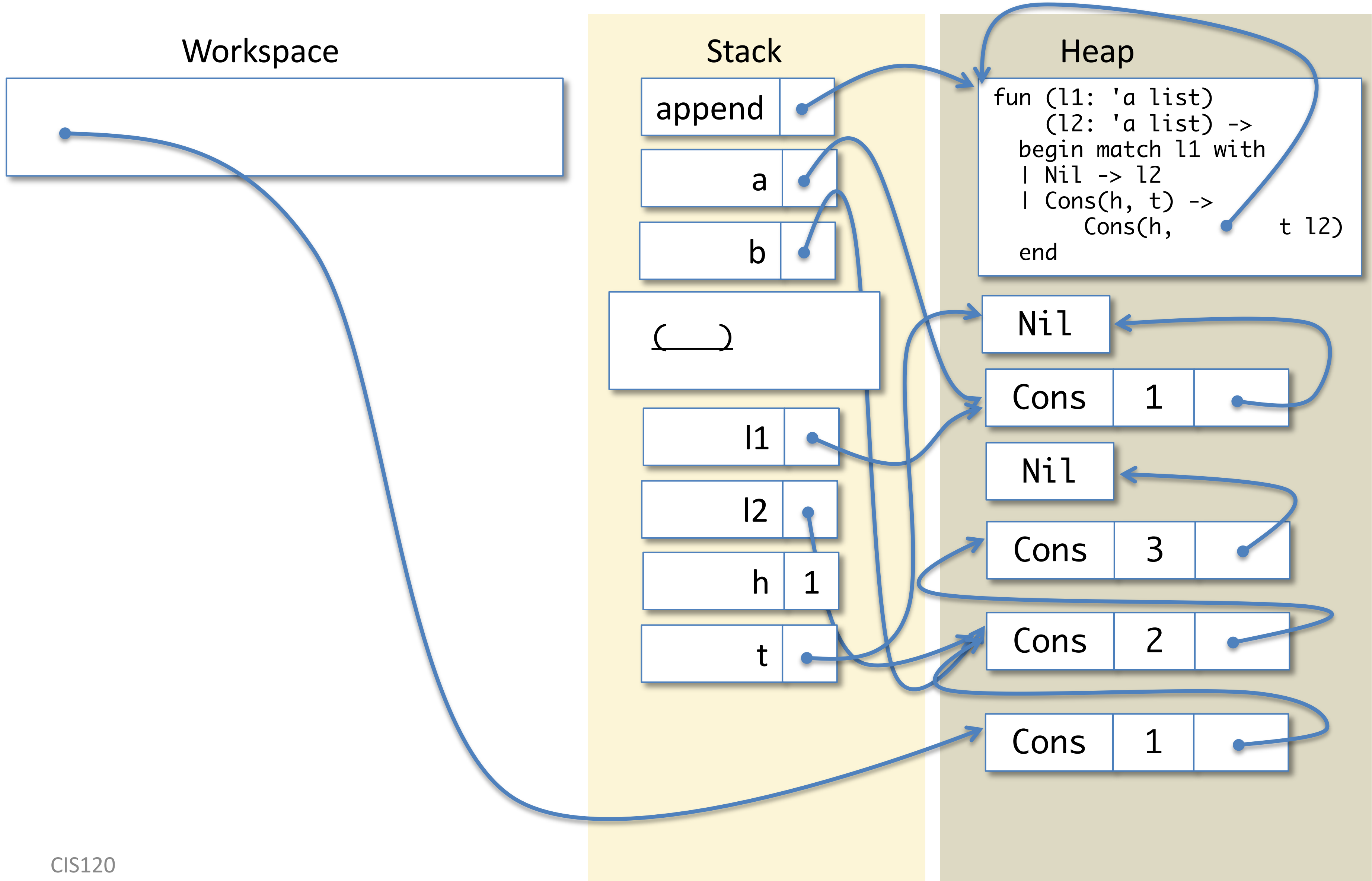
Done! Pop stack to last Workspace



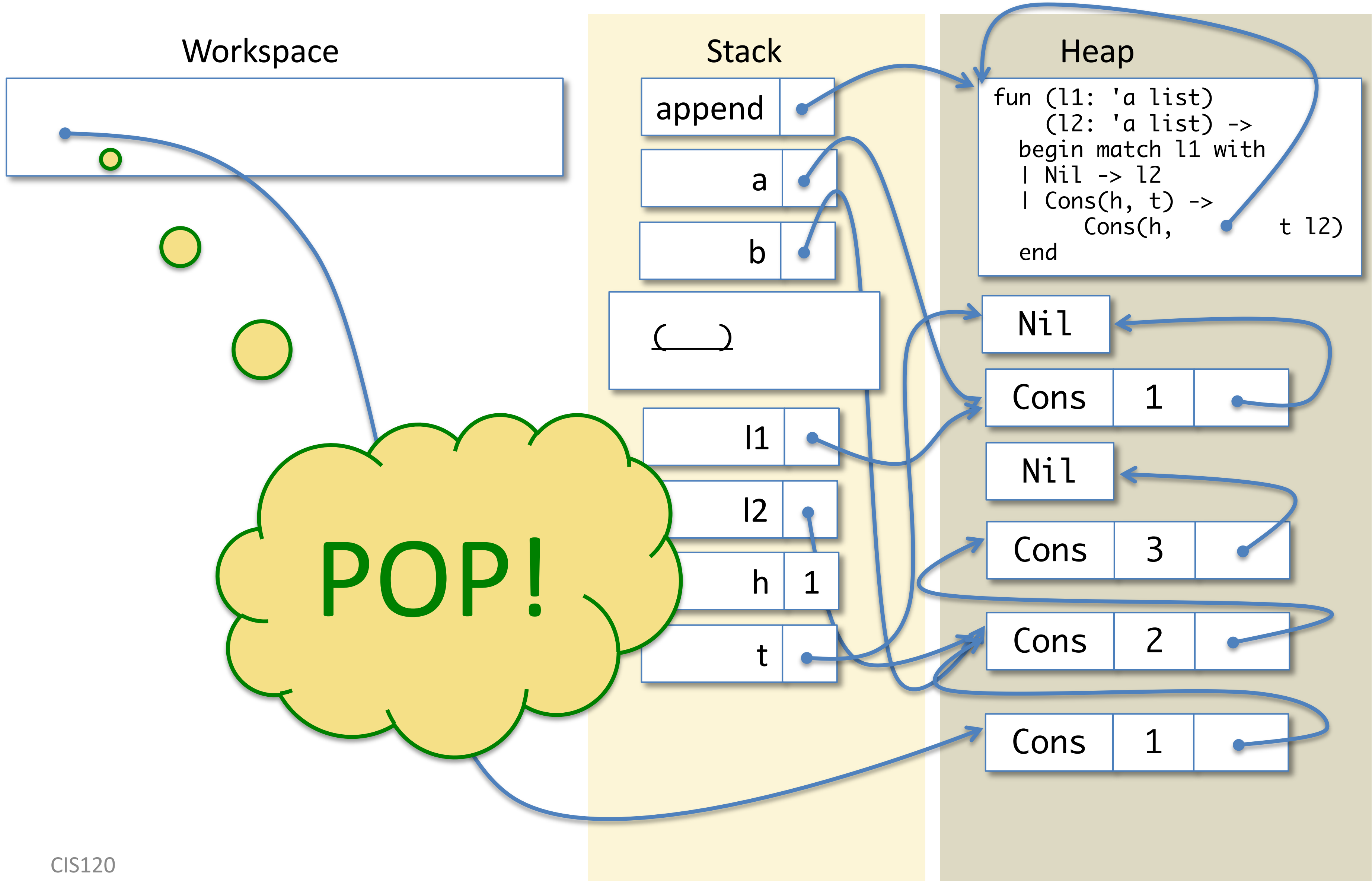
Allocate a Cons cell



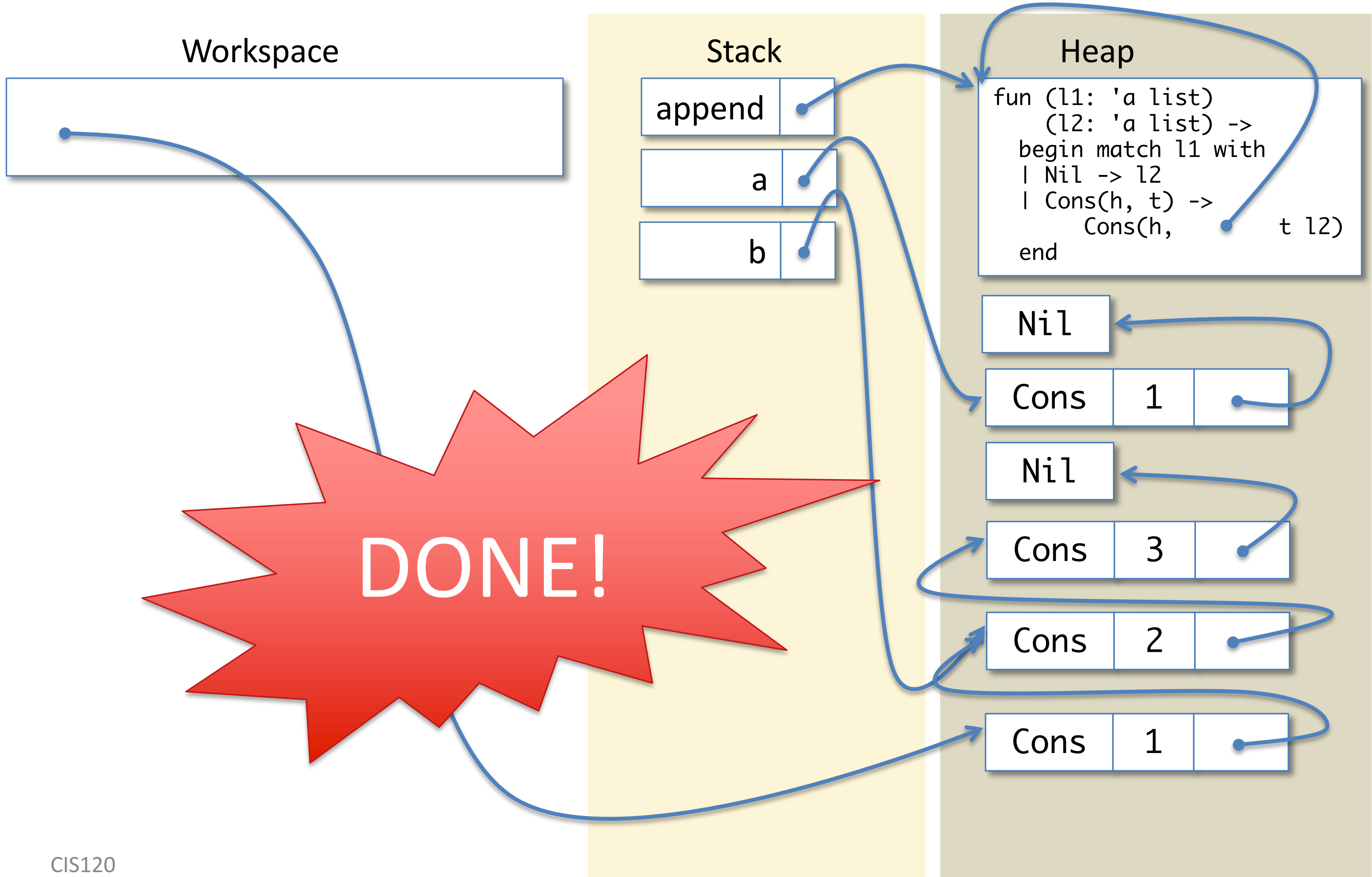
Allocate a Cons cell



Done! Pop stack to last Workspace

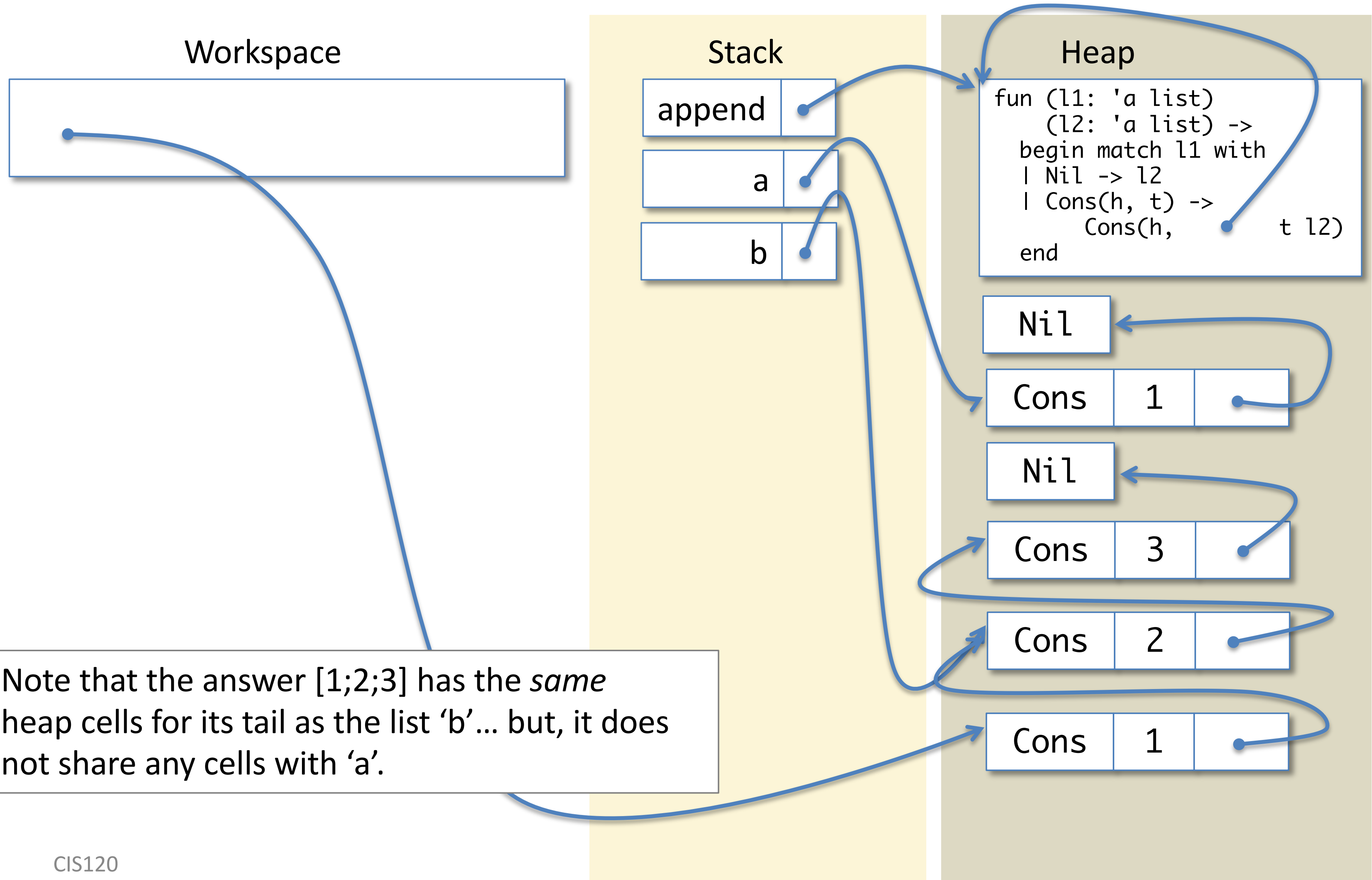


Done! (PHEW!)



Workspace
Stack
Heap
append
a
b
fun (l1: 'a list)
(l2: 'a list) ->
begin match l1 with
| Nil -> l2
| Cons(h, t) ->
Cons(h, t l2)
end
Nil
Cons 1
Nil
Cons 3
Cons 2
Cons 1
DONE!

Done! (PHEW!)



Note that the answer [1;2;3] has the *same* heap cells for its tail as the list 'b'… but, it does not share any cells with 'a'.

Simplifying Match

- A match expression

```
begin match e with
 | pat_1 -> branch_1
 | …
 | pat_n -> branch_n
end
```

  is ready if e is a value
  - Note that e will always be a pointer to a constructor cell in the heap
  - This expression is simplified by finding the first pattern $\texttt{pat}_i$ that matches the cell and adding new bindings for the pattern variables (to the parts of `e` that line up) to the end of the stack
  - replacing the whole match expression in the workspace with the corresponding $\texttt{branch}_i$

Putting State to Work: Mutable Queues

A design problem

Suppose you are implementing a website for constituents to submit questions to their political representatives. To be fair, you would like to deal with questions in first-come, first-served order. How would you do it?

- Understand the problem
  - Need to keep track of pending questions, in the order in which they were submitted
- Define the interface
  - Need a data structure to store questions
  - Need to add questions to the *end* of the queue
  - Need to allow responders to retrieve questions from the *beginning* of the queue
  - Both kinds of access must be efficient to handle large volume

(Mutable) Queue Interface

```
module type QUEUE =
sig
  (* abstract type *)
  type 'a queue

  (* Make a new, empty queue *)
  val create : unit -> 'a queue

  (* Determine if a queue is empty *)
  val is_empty : 'a queue -> bool

  (* Add a value to the end of a queue *)
  val enq : 'a -> 'a queue -> unit

  (* Remove the first value (if any) and return it *)
  val deq : 'a queue -> 'a

end
```

Q: We can tell, just looking at this interface, that it is for a MUTABLE data structure. How?

Since queues are mutable, we must allocate a new one every time we need one.

A: Adding an element to a queue returns `unit` because it *modifies* the given queue.

Specify the behavior via test cases

```
let test () : bool =
  let q : int queue = create () in
  enq 1 q;
  enq 2 q;
  1 = deq q
;; run_test "queue test 1" test

let test () : bool =
  let q : int queue = create () in
  enq 1 q;
  enq 2 q;
  let _ = deq q in
  2 = deq q
;; run_test "queue test 2" test
```

Implementing Linked Queues

Representing links

Data Structure for Mutable Queues

```
type 'a qnode = {
     v: 'a;
     mutable next : 'a qnode option
}

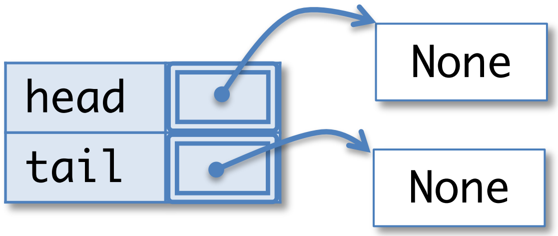
type 'a queue = { mutable head : 'a qnode option;
                  mutable tail : 'a qnode option }
```

There are two parts to a mutable queue:

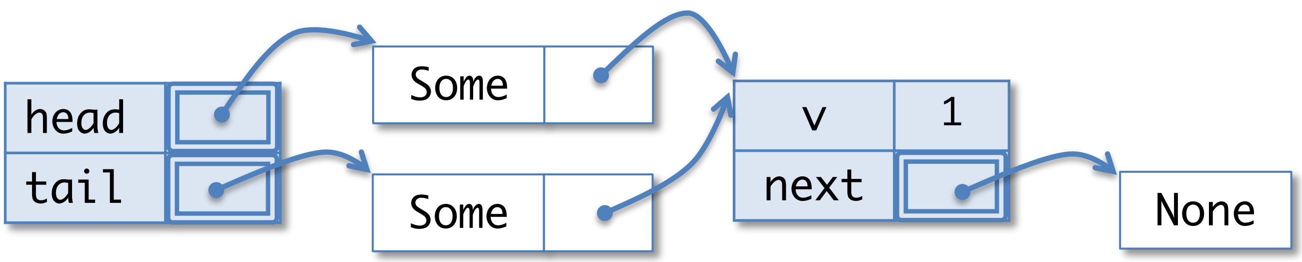
1. the "internal nodes" of the queue, with links from one to the next
2. a record with links to the head and tail nodes

All of the links are *optional* so that the queue can be empty.

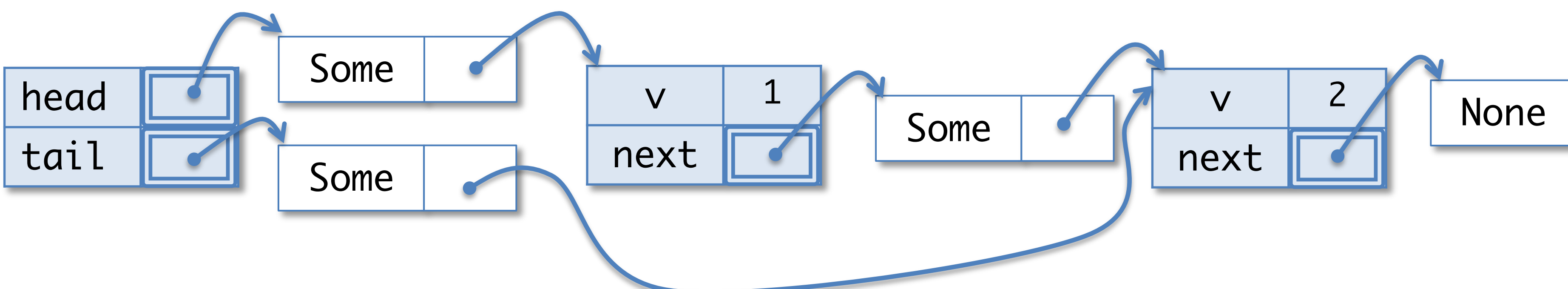
Queues in the Heap



An empty queue

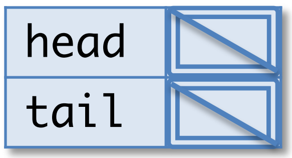


A queue with one element

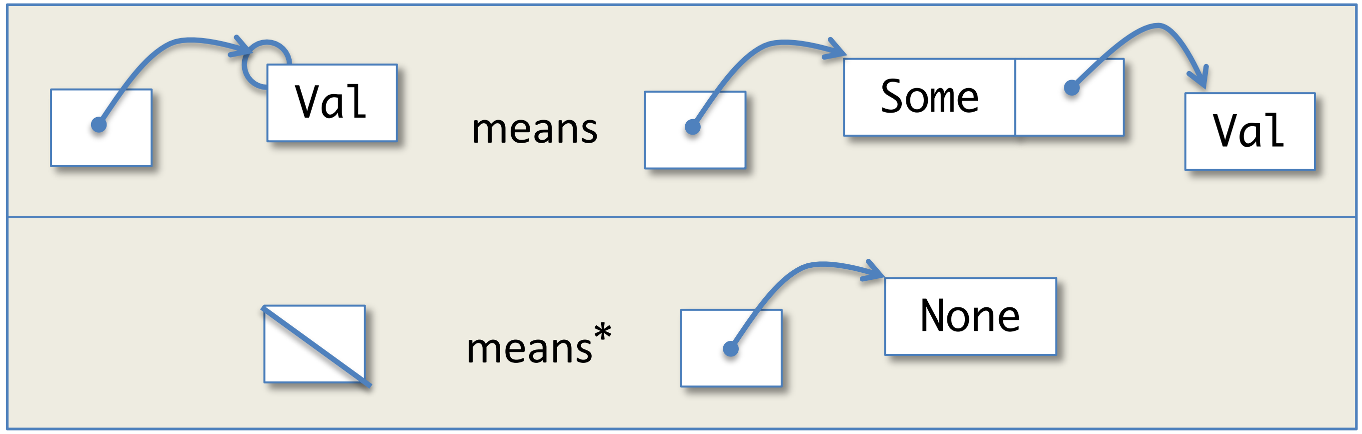


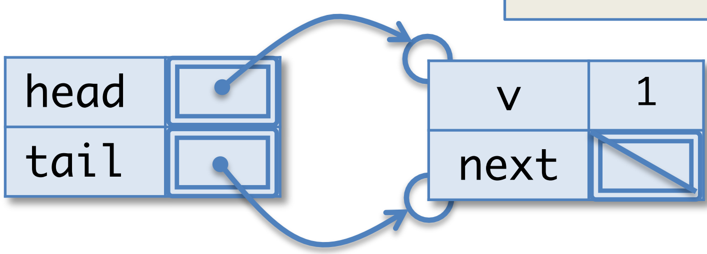
A queue with two elements

Visual Shorthand: Abbreviating Options



An empty queue





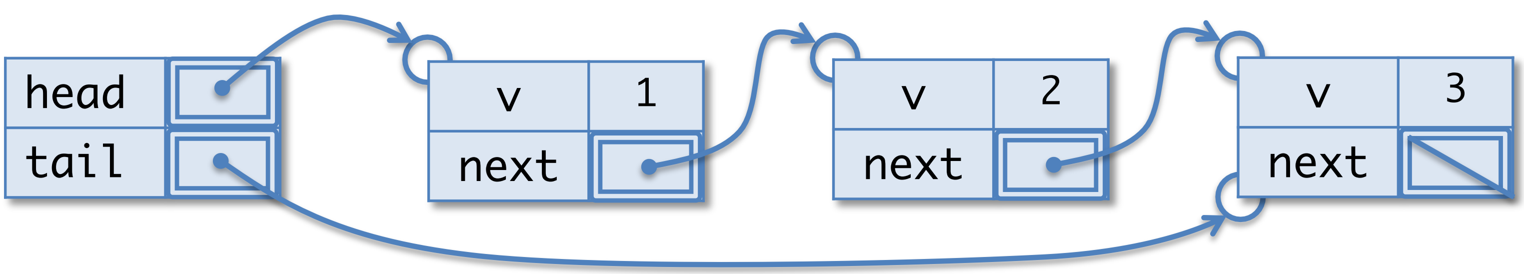
A queue with one element

*Note: Ocaml can optimize "nullary" constructors like Nil, None, Empty so that they aren't allocated in the heap. This is why

`None == None`

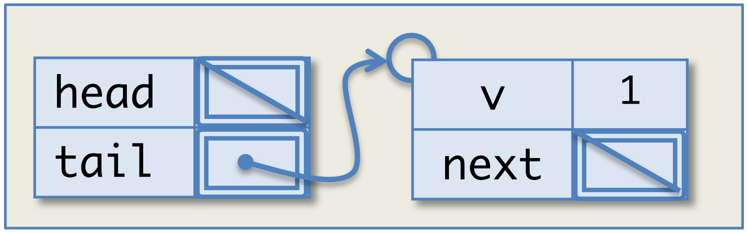
even though `not ((Some x) == (Some x))`.

Be careful with equality and options.

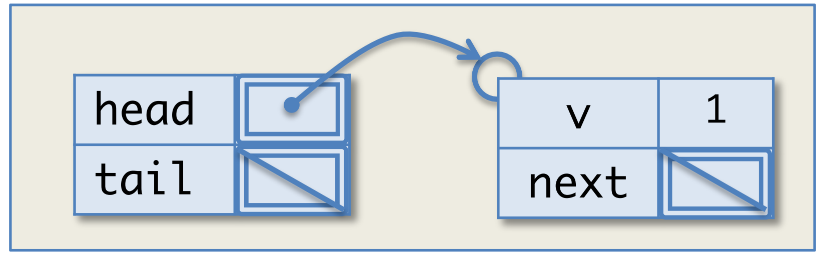


A queue with three elements

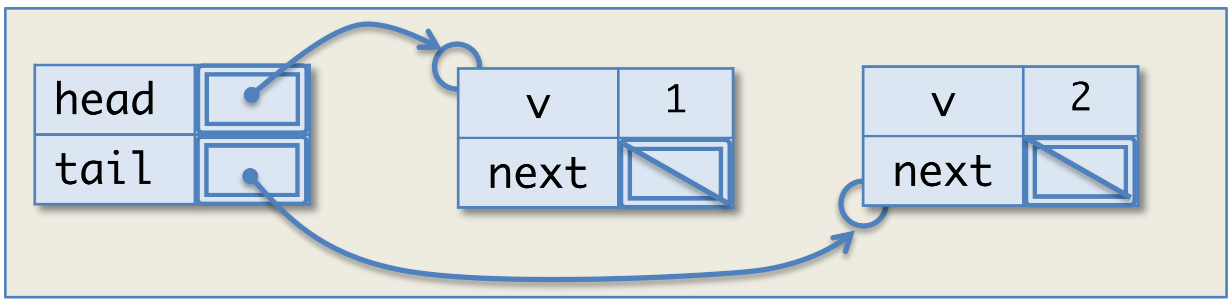
“Bogus” values of type `int queue`



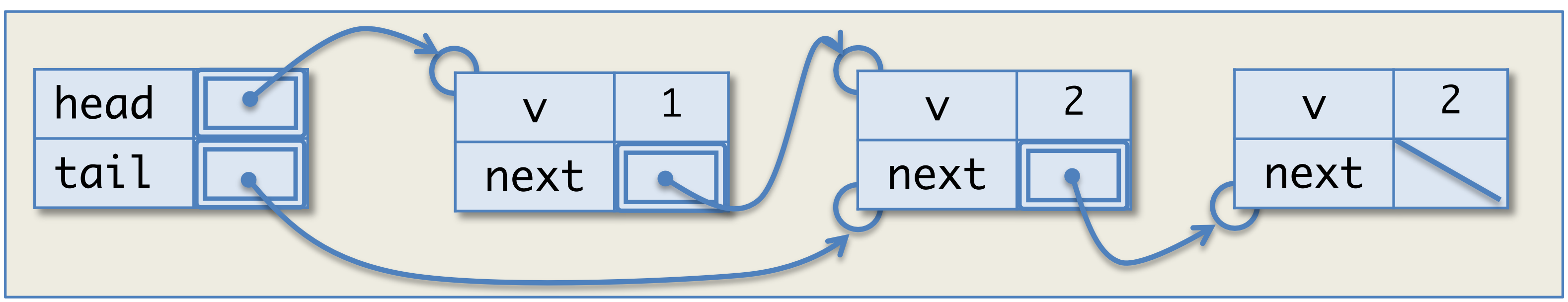
head is None, tail is Some



head is Some, tail is None



tail is not reachable from the head



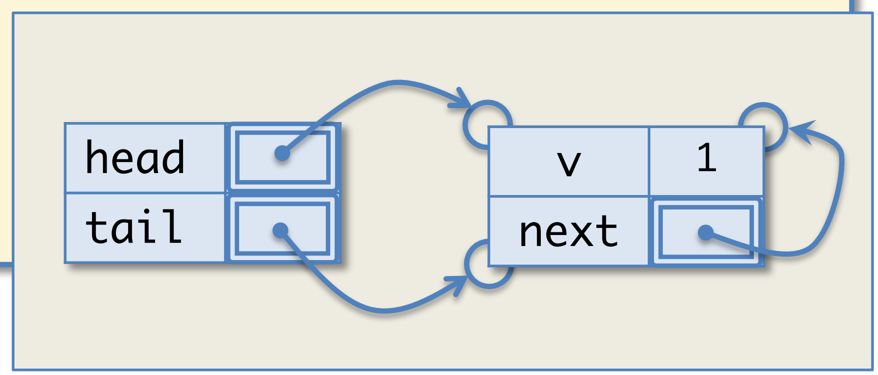
tail doesn’t point to the last element of the queue

Given the queue datatype shown below, is it possible to create a *cycle* of references in the heap. (i.e. a way to get back to the same place by following references.)

```
type 'a qnode = {
      v: 'a;
      mutable next : 'a qnode option
}

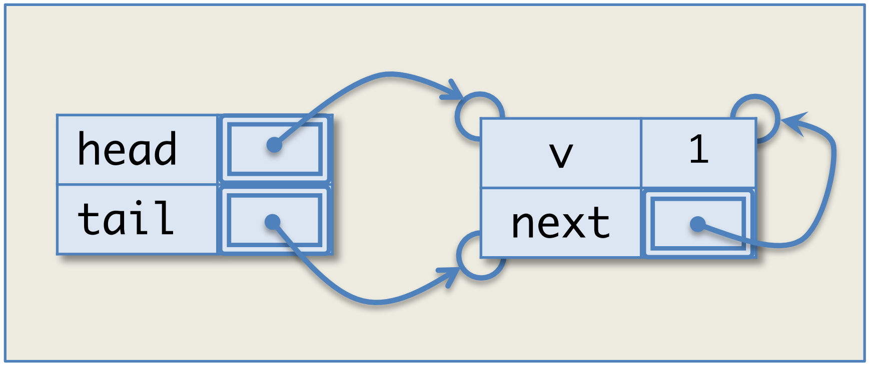
type 'a queue = { mutable head : 'a qnode option;
                  mutable tail : 'a qnode option }
```

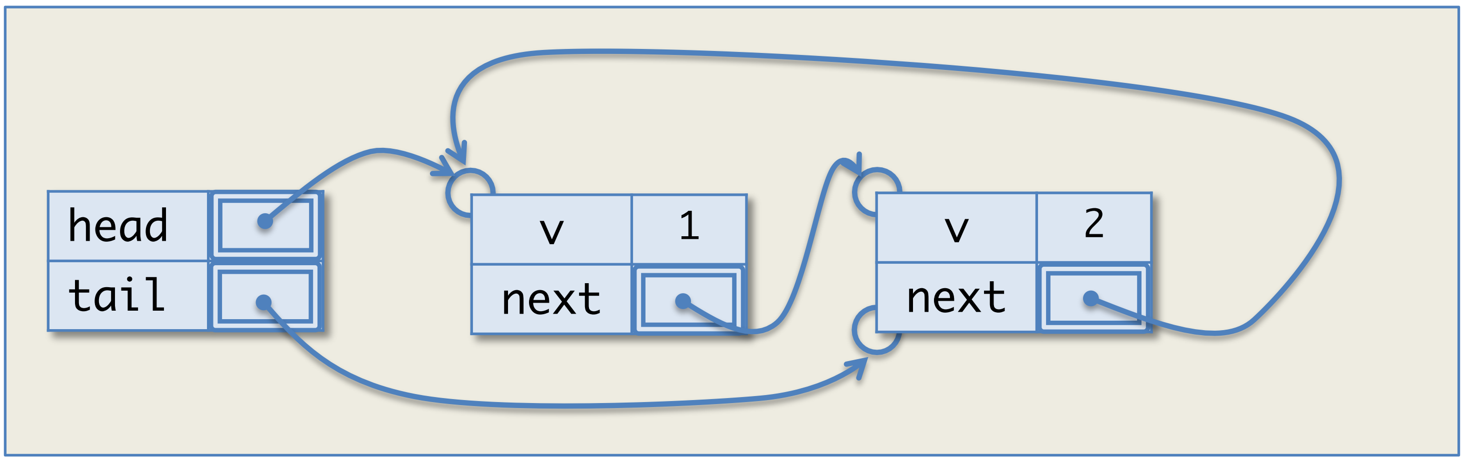
1. yes
2. no
3. not sure



Answer: 1

Cyclic `int queue` values





(And infinitely many more…)

Linked Queue Invariants

- Just as we imposed some restrictions on which trees count as legitimate Binary Search Trees, Linked Queues must also satisfy representation *invariants*:

> Either:
> (1) `head` and `tail` are both `None` (i.e. the queue is empty)
> or
> (2) `head` is `Some n1`, `tail` is `Some n2` and
> - `n2` is reachable from `n1` by following 'next' pointers
> - `n2.next` is `None`

- We can prove that these properties suffice to rule out all of the "bogus" examples.
- Each queue operation may assume that these invariants hold of its inputs, and must ensure that the invariants hold when it's done.