Solution Set

CIS 1210—Data Structures and Algorithms—Spring 2025

Minimum Spanning Trees—Tuesday, April 1 / Wednesday, April 2

Readings

- Lecture Notes Chapter 21: Minimum Spanning Trees
- Lecture Notes Chapter 22: Union Find

Review: Minimum Spanning Trees (MSTs)

A spanning tree of a graph $G = (V, E)$ is a subgraph that is a tree and which "spans" (includes) all vertices in V. Given an undirected, connected, and weighted graph G, a minimum spanning tree (MST) of G is a spanning tree whose edges have a minimum sum out of all possible spanning trees of G.

Prim's Algorithm

At a high-level, Prim's algorithm starts with a single vertex and grows the MST by greedily adding the vertex that can be added the most "cheaply" to the partial tree. In other words, at each step, we greedily add the vertex that has the minimum weight edge to some vertex in the partial tree. As seen in the pseudocode, Prim's algorithm is very similar to Dijkstra's, so we can see that the runtime is $O(m \log n)$ since the input graph is connected and thus $n = O(m)$.

Kruskal's Algorithm

At a high-level, Kruskal's algorithm finds an MST by first starting on the graph with no edges. We sort the edges in increasing order of weight and then iterate through the edges in this order, adding an edge to the MST if it does not create a cycle since we want the minimum total sum. In contrast to Prim's, Kruskal's starts with every vertex as its own tree and then gradually connects these "subtrees" into an MST.

Since Kruskal's relies on the ability to detect cycles efficiently, as seen in the pseudocode, we use the **Union-Find** data structure, where the MakeSet operation runs in $O(1)$ time and the Union/Find operations run in $O(\log n)$ time if we use Union by Rank. If we do not know anything about the edge weights, then the runtime of Kruskal's will be the time it takes to sort the edges, which is $O(m \log m) = O(m \log n)$.

Reverse Delete Algorithm

Reverse Delete leverages the Cycle Property by removing edges in decreasing order of weight, ensuring that only the heaviest edges in any cycle are deleted, which guarantees they are not part of any MST—though it's slower due to the repeated connectivity checks, making its runtime $O(m^2)$ compared to Prim's and Kruskal's.

Correctness

The correctness and intuition behind Prim's and Kruskal's is derived from the cut property. The correctness of another MST algorithm, Reverse-Delete, is derived from the cycle property. Both are summarized below:

Cut Property: Let $e = (u, v)$ be a minimum cost edge "crossing the cut" with one end in S and the other in $V - S$. Then some MST of G contains e. Moreover, if $e = (u, v)$ is the unique minimum cost edge crossing this cut, then e belongs to *every* MST of G.

Cycle Property: Let C be any cycle in G and let $e = (u, v)$ be a heaviest edge in C. Then e does not belong to some MST of G. Moreover, if $e = (u, v)$ is the unique heaviest edge on the cycle, then e belongs to *no* MST of G.

Problem 1: True or False

1. Kruskal's and Prim's algorithms work on a graph with negative edge weights.
2. Suppose that we have an MST T of a graph G but are told that an edge not in T has a lower weight than originally specified and so T is now an invalid MST. It is guaranteed that we can fix our tree by removing an edge and adding a different one.

Solution

1. **True.** Yes, Kruskal's algorithm still works on a graph with negative weights because it relies on a "pre-processing" step where edges are sorted in increasing order by their weights — this sorting is still done correctly even if some edge weights are negative. Kruskal's operates by going through the sorted edge weights in increasing order while trying to add each edge to the MST, so it will still correctly select edges in order of increasing weight by selecting the more negative weighted edges first.

   Similarly, Prim's algorithm will also still work on a graph with negative edges because in each step, it greedily chooses the lowest weighted edge that crosses the cut, so it will still operate correctly by selecting the most negative weighted edge if necessary. Additionally, Prim's algorithm derives its correctness from the cut property, and note that the cut property still holds even if some edge weights in the graph are negative.

   Note: Although Prim's/ Kruskal's algorithm will still correctly find a MST when a graph has negative weights, observe that the output MST may not be the minimum spanning subgraph, since we could potentially add more negative weighted edges to the output MST to get a "more" minimum spanning subgraph that is not a tree.

2. **True.** Since we are given that our MST T is invalid because of this change, observe that the modified edge e does not exist in T or else T would still be a valid MST. Furthermore, e must exist in all possible MSTs of the new graph. So, add e to the original MST T. Since all trees are maximally acyclic, in our new graph of $T + \{e\}$, we have formed a cycle C that contains the edge e. By cycle property, a maximum-weight edge in C does not exist in some MST. Note that since the modified edge e is necessary, a maximum-weight edge e' must also be distinct from e. Therefore, removing e' from the graph of $T + \{e\}$ yields a valid MST again.

Problem 2

Suppose we have some MST, T, in a graph G with positive edge weights. Construct a graph G' where for any weight $w(e)$ for edge e in G, we have weights $w(e)^2$ in G'. Is T still a MST in G'? Prove your answer.

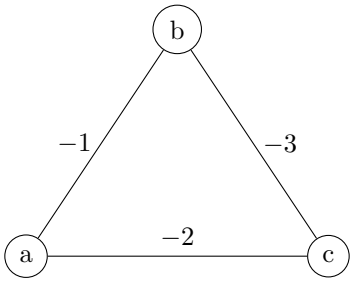
If G also had some negative edge weights, would your answer change from above change? Prove your answer.

Solution

If G only has positive weights, then after squaring edge weights, T remains a valid MST in the transformed graph G'. We will prove this via contradiction. Assume the claim does not hold and that T is no longer an MST in G'. Instead, let T' be an MST of G'. Because $T \neq T'$, it must be that for some cut in the graph, T chooses edge e while T' chooses some other edge e' such that $w(e')^2 < w(e)^2$. However, since T is a

proper MST in G, the edge e must have been a minimum weight edge spanning that cut in G, and therefore $w(e) \leq w(e')$. Note that since we have positive edge weights, it is not possible to have both $w(e')^2 < w(e)^2$ and $w(e) \leq w(e')$, so we have reached a contradiction. Therefore, squaring edge weights in a positively weighted graph is a valid transformation.

On the other hand, note that if we have negative edge weights, we can still have both $w(e')^2 < w(e)^2$ and $w(e) \leq w(e')$. Hence, the contradiction from above does not hold, and the claim is actually false. As a counterexample, consider the graph below. The MST T consists of the edges (b, c) and (a, c), with edge weights of -3 and -2, respectively. After squaring, however, we have edge weights of 1, 4, and 9, so the new MST T' consists of the edges (a, b) and (a, c), with edge weights of 1 and 4, respectively. Therefore, squaring edge weights when some are negative is not a valid transformation, as shown in this example where the MSTs have changed.



Again, generally any transformation/change to edge weights that preserves their relative ordering is valid.

Problem 3

Suppose we have a connected graph G where all edge weights are equal. Design an efficient algorithm to find an MST of G. What is the running time of your algorithm?

Solution

Algorithm: Run BFS/DFS on G and output the edges in the BFS/DFS tree.

Proof of Correctness: Observe that because all edge weights are equal, any spanning tree of G will be a minimum spanning tree. We know that running BFS/DFS on G will output a BFS/DFS tree; in particular, we have a DFS tree instead of a forest because G is connected. Additionally, since G is connected, we know that our BFS/DFS tree must be spanning because the traversals visit every vertex in the graph. Therefore, since the BFS/DFS tree is a spanning tree, it is also a minimum spanning tree, so our algorithm is correct.

Runtime Analysis: Since we only run BFS/DFS, the running time of our algorithm is $O(|V| + |E|)$.