Announcements

- **Upcoming deadlines**
  - Project Milestone 1 due on **Wednesday, October 18 at 8pm**
  - Quiz 5 due Thursday, October 19 at 8pm

Dimensionality Reduction

- We can write each input x as

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_d \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

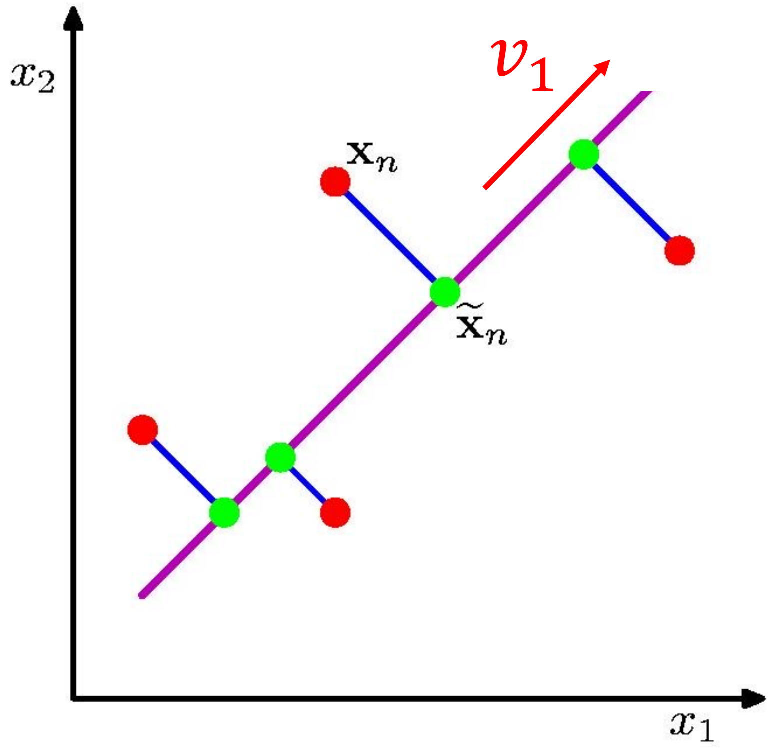
projections original axes

- We aim to approximate x using a new basis $\{v_i\}_i$ (of unit norm):

$$x \approx \tilde{f}(x) = f(x)_1 v_1 + f(x)_2 v_2 + \cdots + f(x)_{d'} v_{d'}$$

1D Case

- **Simplest case:** If $d' = 1$, then we want $x \approx f(x)_1 v_1$
- Given v_1, we can take $f(x)_1 = x^\top v_1$
  - Minimizes MSE of $\|x - f(x)_1 v_1\|$
  - Then, we have $\tilde{f}(x) = (x^\top v_1) v_1$
  - I.e., orthogonal projection
  - Assuming $\|v_1\|_2 = 1$



1D Case

- In this case, the loss is

$$L(v_1; Z) = \frac{1}{n}\sum_{i=1}^{n}\left\|x_i - \left(x_i^\top v_1\right)v_1\right\|_2^2$$

1D Case

- Since we have assumed $\|v_1\|_2 = 1$, we have

$$\left\| x_i - (x_i^\top v_1) v_1 \right\|_2^2$$

1D Case

- Since we have assumed $\|v_1\|_2 = 1$, we have

$$
\begin{aligned}
\left\|x_i - (x_i^\top v_1)v_1\right\|_2^2 &= \|x_i\|_2^2 - 2x_i^\top (x_i^\top v_1)v_1 + \left\|(x_i^\top v_1)v_1\right\|_2^2 \\
&= \|x_i\|_2^2 - 2(x_i^\top v_1)^2 + (x_i^\top v_1)^2 \|v_1\|_2^2 \\
&= \|x_i\|_2^2 - (x_i^\top v_1)^2 \\
&= \|x_i\|_2^2 - v_1^\top x_i x_i^\top v_1
\end{aligned}
$$

1D Case

- Thus, we have

$$\sum_{i=1}^{n} \left\| x_i - \left(x_i^\top v_1\right) v_1 \right\|_2^2$$

1D Case

- Thus, we have

$$\sum_{i=1}^{n} \left\| x_i - (x_i^\top v_1) v_1 \right\|_2^2 = \sum_{i=1}^{n} \|x_i\|_2^2 - v_1^\top x_i x_i^\top v_1$$

$$= \text{const} - \sum_{i=1}^{n} v_1^\top x_i x_i^\top v_1$$

$$= \text{const} - v_1^\top \left(\sum_{i=1}^{n} x_i x_i^\top \right) v_1$$

$$= \text{const} - n v_1^\top C v_1$$

General Case

PCA(Z):

$Z \leftarrow \{\, x - \text{Mean}(Z) \mid x \in Z \,\}$

$C \leftarrow \frac{1}{n} \sum_{i=1}^{n} x_i x_i^\top$

for $j \in \{1, \dots, d'\}$:

$v_j \leftarrow \text{Eigenvector}(C, j)$

return $f: x \mapsto [x^\top v_1 \quad \cdots \quad x^\top v_{d'}]^\top$

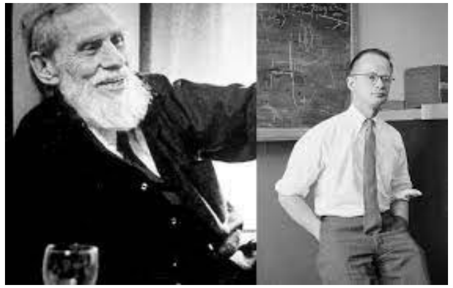
Lecture 13: Neural Networks

CIS 4190/5190

Fall 2023

Brief History of Neural Networks

- **1943:** Perceptron model (McCulloch & Pitts)
  - Intended as theoretical model of biological neurons
  - Linear classifiers! (Specialized learning algorithm)



- **1958:** Implementation as Mark I Perceptron (Rosenblatt)
  - Demonstrated capabilities of handwritten letter recognition

- **1969:** Perceptrons cannot learn XOR (Minsky & Papert)
  - Highly controversial (may have helped cause "AI winter")

Brief History of Neural Networks

- **1985:** Representation learning (Rumelhart, Hinton, & Williams)
  - Interpret intermediate computations of neural networks
- **1989:** Convolutional neural networks (Lecun)
  - Convert handcrafted convolutional filters into learnable parameters
- **1995:** Long short-term memory (Hochreiter & Schmidhuber)
  - Refinement of neural networks designed to predict sequences
  - Complex design demonstrates flexibility of neural networks

Brief History of Neural Networks

- **1998:** Convolutional neural networks for MNIST (Lecun)
  - Human-level performance on handwritten digit recognition
- **2012:** ImageNet breakthrough (Krizhevsky, Sutskever, & Hinton)
  - Reduced error on image classification by 50%
- **2017:** Transformer architecture (Vaswani et al.)
- **2018:** Turing award (Bengio, Hinton, & Lecun)

- **To be continued?**

What Changed?

- **More compute:** GPUs
- **More data:** ImageNet, Wikipedia/Web, etc.
  - New applications
- **Better optimization algorithms:** Mini-batch SGD, acceleration, etc.
- **Accumulation of “folk knowledge”:** Parameter initialization, etc.
  - Encoded in open source software packages
- **Modern perspective:** “Differentiable programming”
- **Lots of investment from tech companies**

Agenda

- **Model family**
  - Custom model family rather than a single model family

- **Optimization**
  - Backpropagation algorithm for computing gradient

A Simple Neural Network

- **Feedforward neural network model family (for regression):**

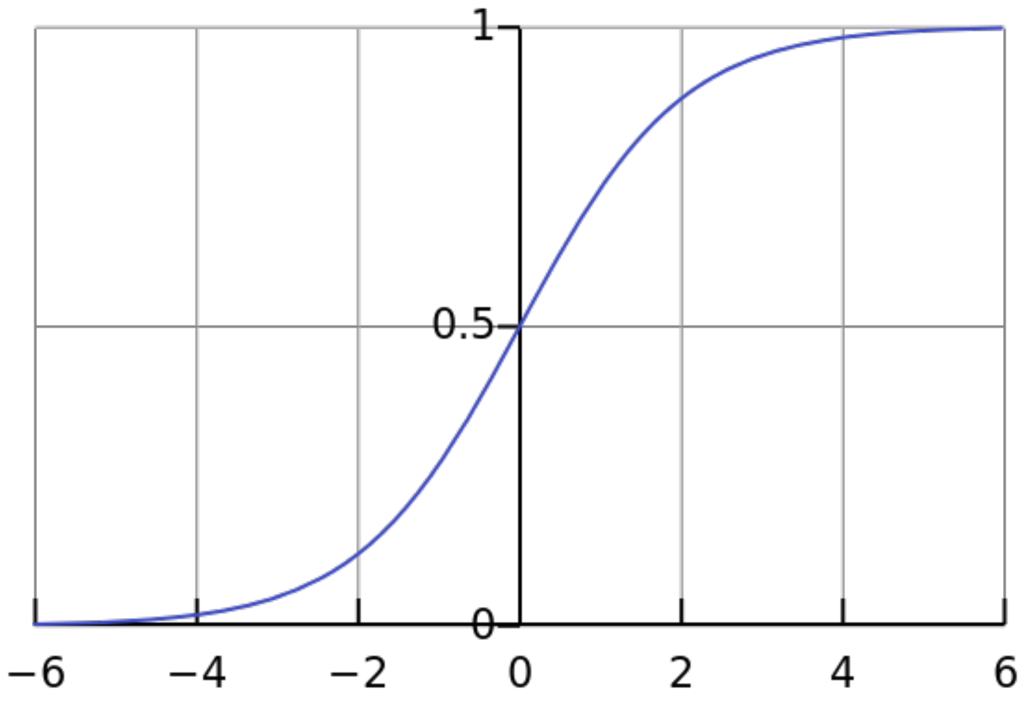
$$f_{W,\beta}(x) = \beta^\top g(Wx)$$

- **Parameters:** Matrix $W \in \mathbb{R}^{k \times d}$ and vector $\beta \in \mathbb{R}^k$
  - k is a hyperparameter called the **number of hidden neurons**

- Here, $g: \mathbb{R} \to \mathbb{R}$ is a given **activation function**
  - It is applied componentwise in $f_{W,\beta}$ (i.e., $g\left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right) = \begin{bmatrix} g(z_1) \\ g(z_2) \end{bmatrix}$)
  - **Example:** $g(z) = \sigma(z)$ (where σ is the sigmoid function)

A Simple Neural Network

- Possible choice of activation function: $g(z) = \sigma(z)$



A Simple Neural Network

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$$f_{W,\beta}(x) =$$

A Simple Neural Network

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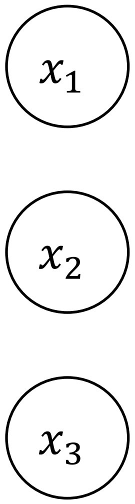
$$f_{W,\beta}(x) = \qquad\qquad x$$



A Simple Neural Network

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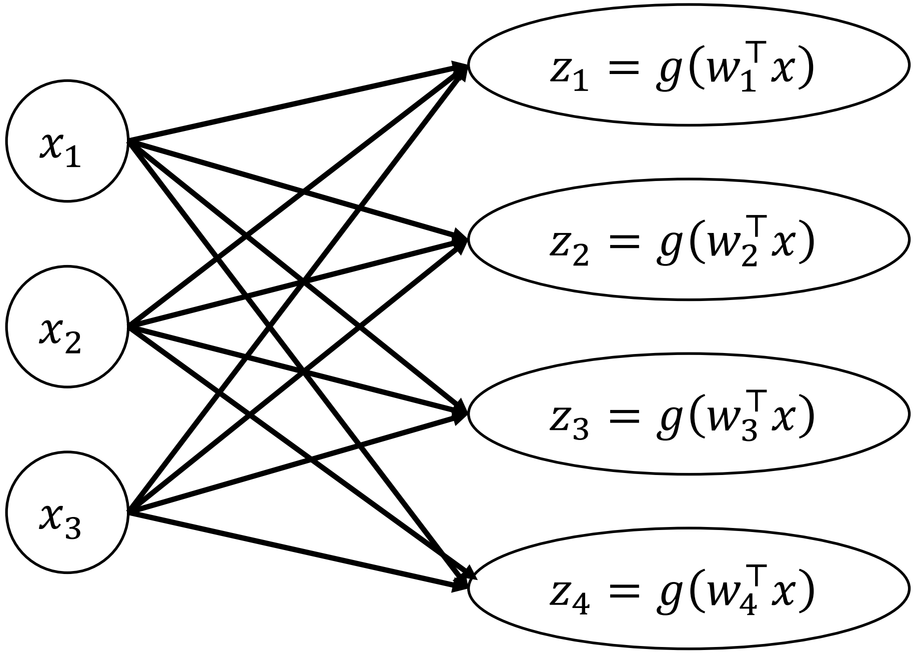
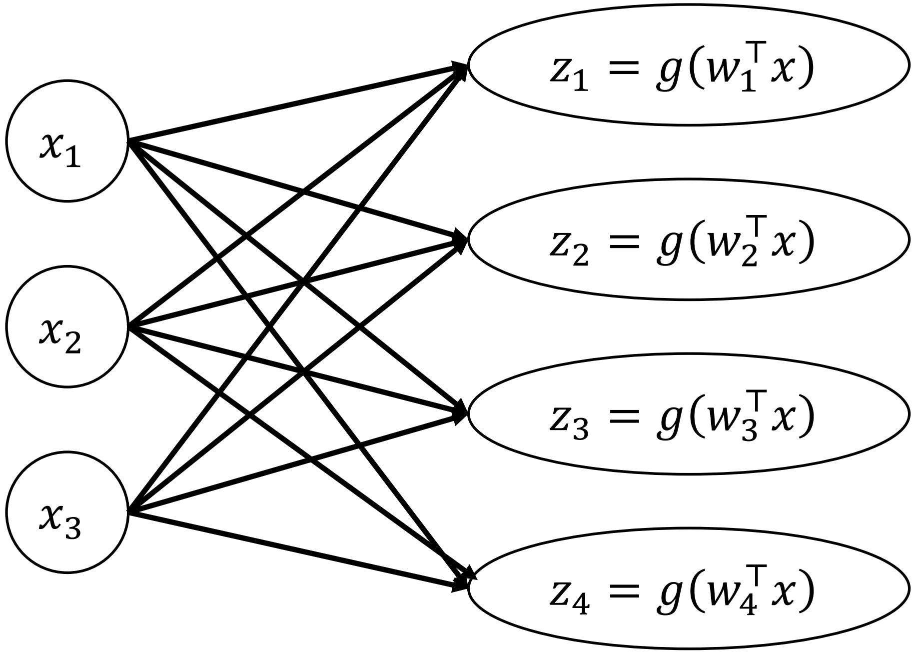


A Simple Neural Network

- **Feedforward neural network model family (for regression):**

$$f_{W,\beta}(x) = \quad g(Wx)$$



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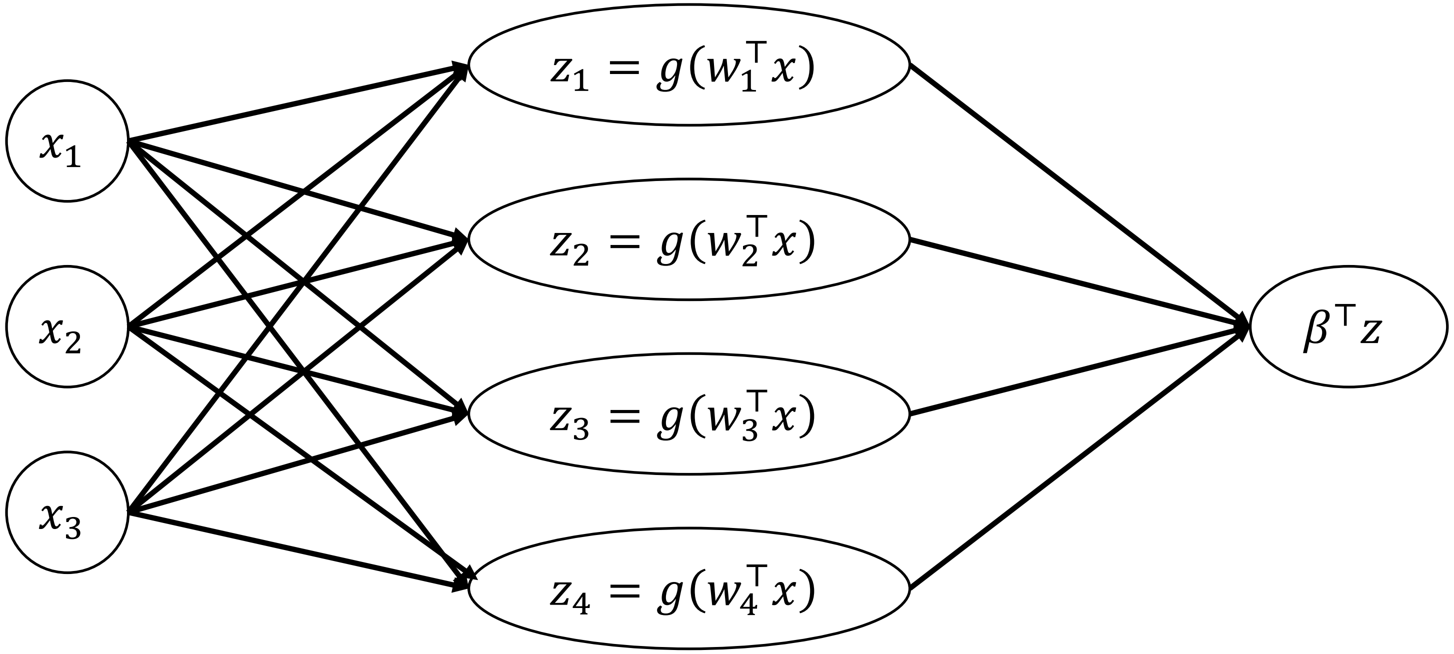

A Simple Neural Network

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$$f_{W,\beta}(x) = \beta^\top g(Wx)$$

- What happens if g is linear?

A Simple Neural Network

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- What happens if g is linear? Recovers linear functions!
  - Special case of identity:

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- Using a nonlinearity is important!
  - **In general:** Linear regression over "features" $g(Wx)$

What About Classification?

- **Recall:** For logistic regression, we choose the likelihood to be

$$p_{\beta}(Y = 1 \mid x) = \frac{1}{1 + e^{-\beta^{\top} x}}$$

What About Classification?

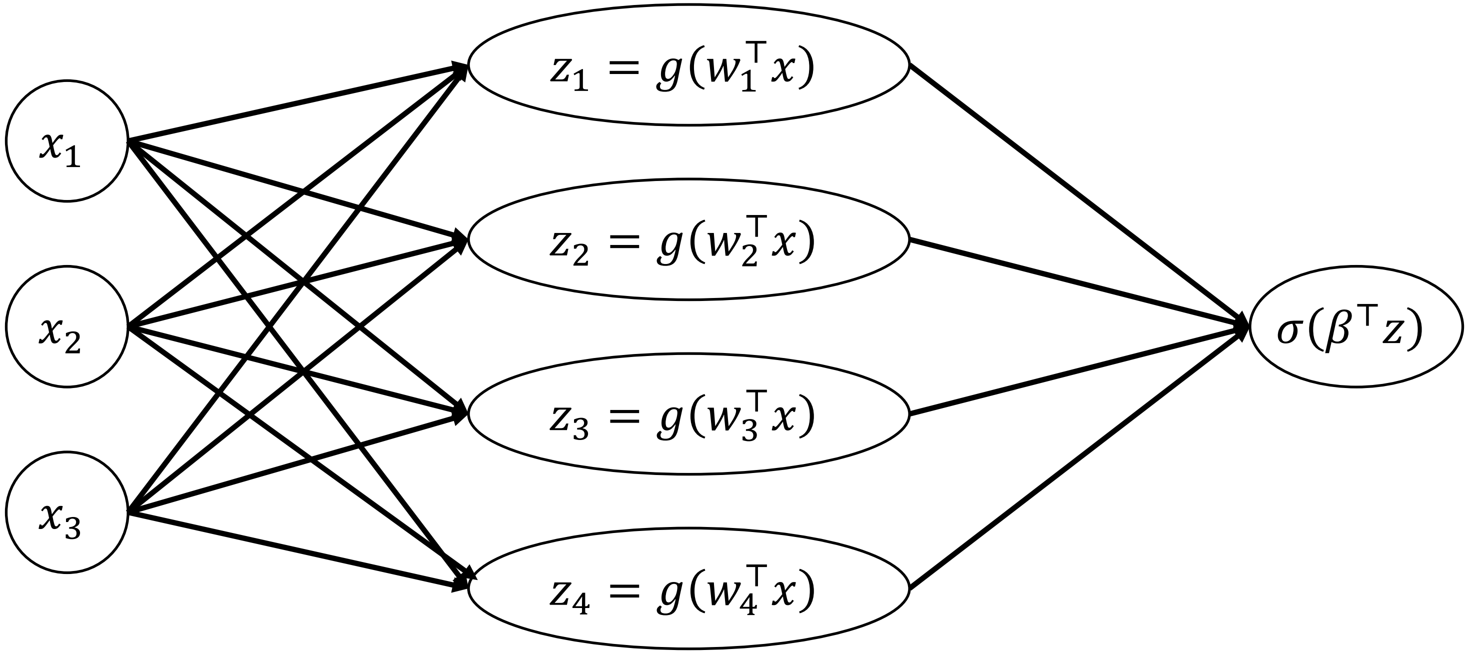
- **Recall:** For logistic regression, we choose the likelihood to be

$$p_\beta(Y = 1 \mid x) = \sigma(\beta^\top x)$$

What About Classification?

- **For binary classification:**

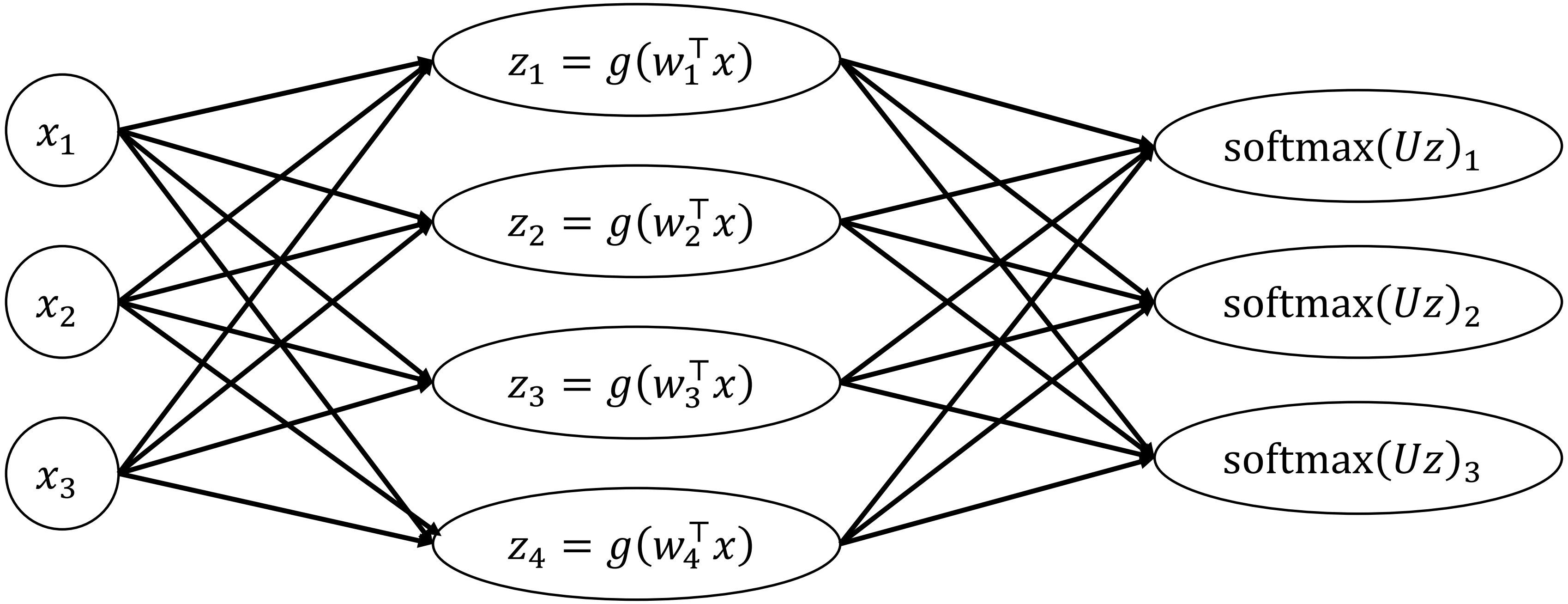
$$p_{W,\beta}(Y = 1 \mid x) = \sigma\left(\beta^\top g(Wx)\right)$$



What About Classification?

- **For multi-class classification:**

$$p_{W,U}(Y = y \mid x) = \text{softmax}\big(Ug(Wx)\big)_y$$



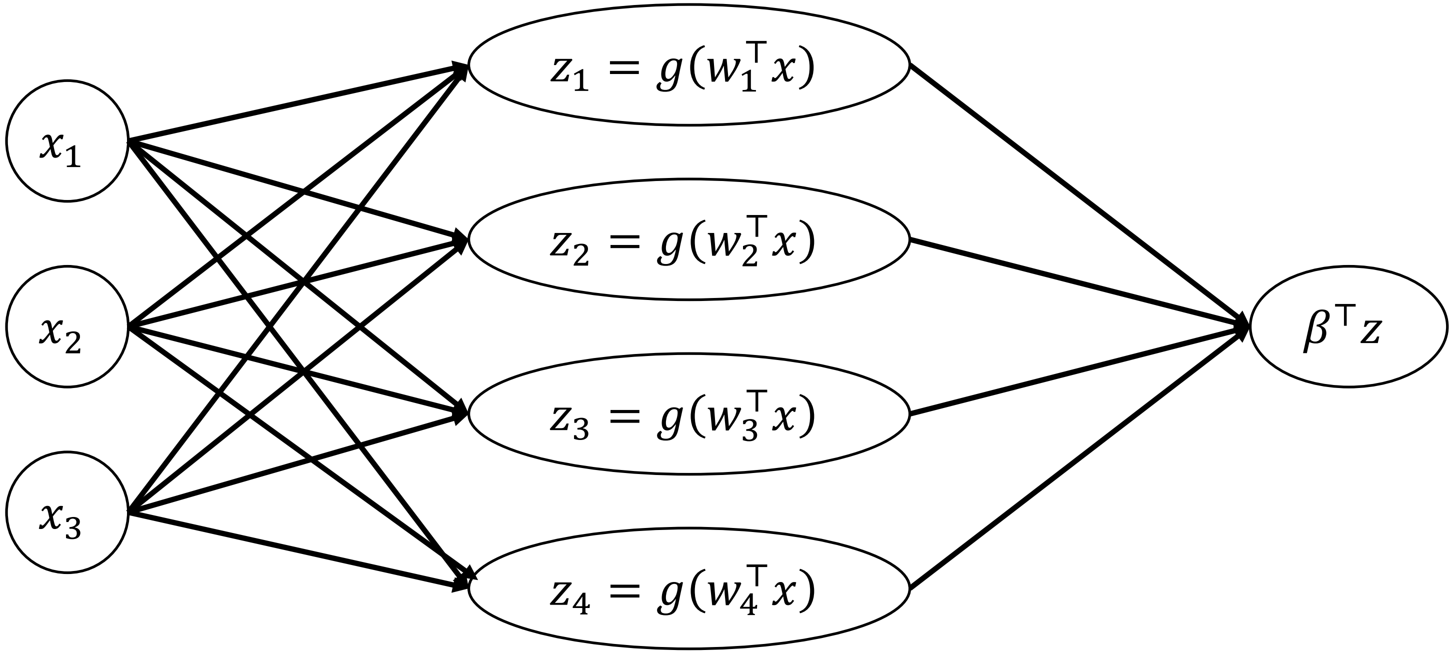
Historical vs. Modern View

- **Historical view:** Specific model families
  - Feedforward neural networks, convolutional neural networks, etc.
  - Each new model family (“architecture”) requires a custom implementation
- **Modern view:** Design model families by composing building blocks
  - Building blocks are “layers”
  - Layers can be **programmatically** composed together (by composing, concatenating, etc.) to form different model families

Historical View

- **Feedforward neural network model family (for regression):**

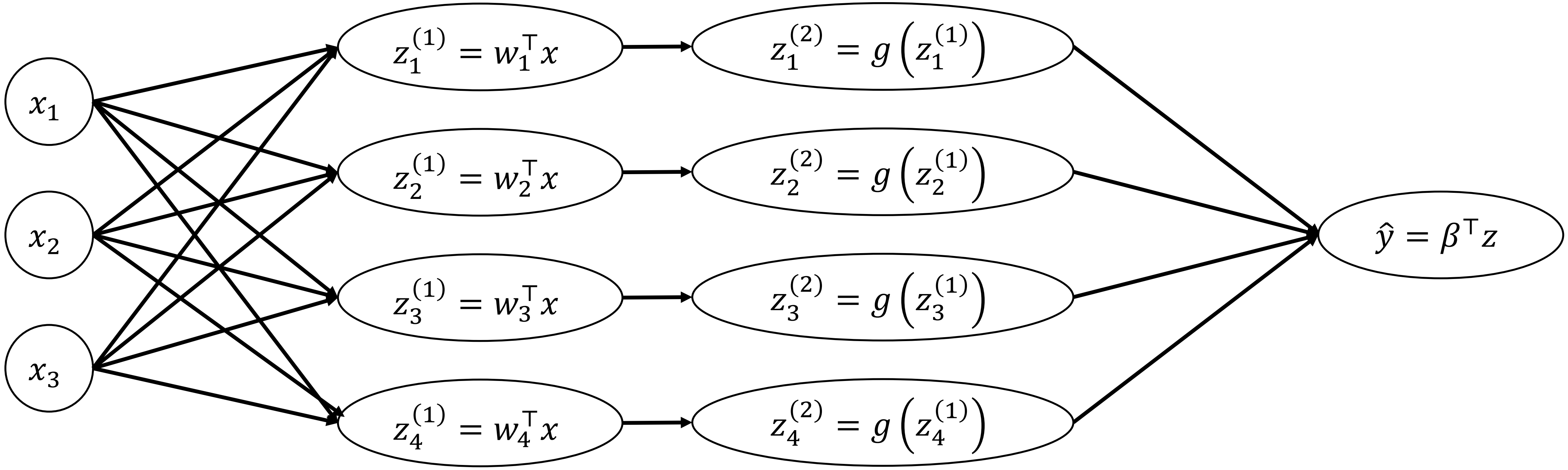
$$f_{W,\beta}(x) = \beta^\top g(Wx)$$



Modern View

- **Feedforward neural network model family (for regression):**

$$f_{W,\beta}(x) = f_\beta\left(g(f_W(x))\right) = f_\beta \circ g \circ f_W(x)$$



Modern View

- Each **layer** is a parametric function $f_{W_j}: \mathbb{R}^k \to \mathbb{R}^h$ for some k, h

- Compose sequentially to form model family:

$$f_W(x) = f_{W_m}\left(\ldots\left(f_{W_1}(x)\right)\ldots\right)$$

- We will use the following notation:

$$f_W = f_{W_m} \circ \cdots \circ f_{W_1}$$

Modern View

- Each **layer** is a parametric function $f_{W_j}\colon \mathbb{R}^k \to \mathbb{R}^h$ for some k, h

- Can compose layers in other ways, e.g., concatenation:

$$f_W(x) = f_{W_1}(x) \oplus f_{W_2}(x)$$

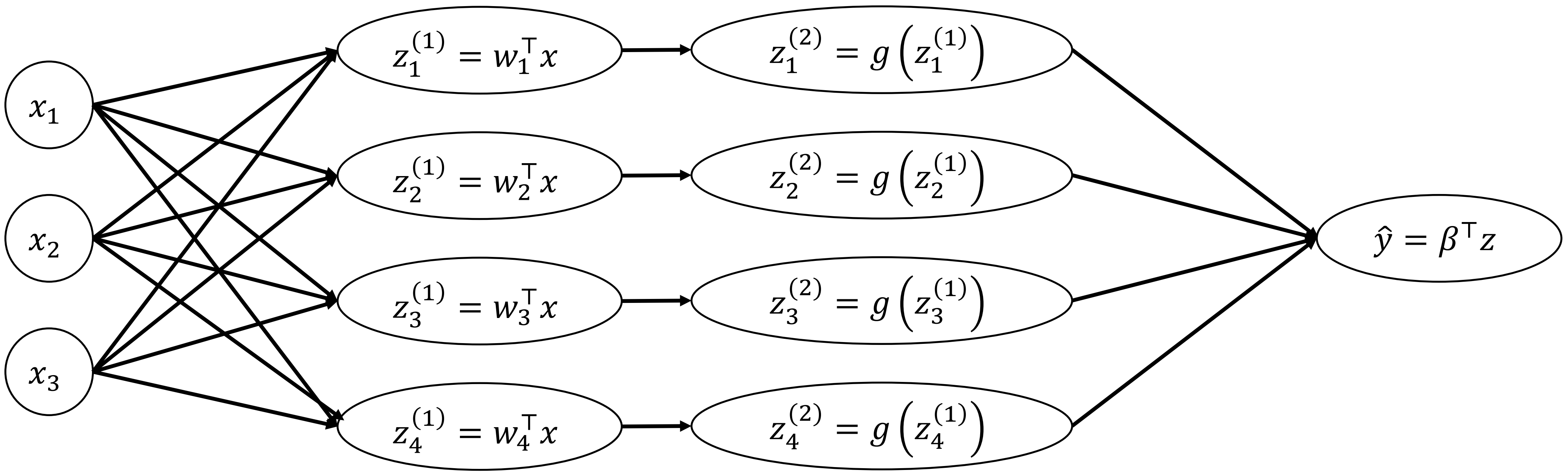
- Here, we have defined

$$[z_1 \quad \cdots \quad z_d]^\top \oplus \left[z'_1 \quad \cdots \quad z'_{d'}\right]^\top = \left[z_1 \quad \cdots \quad z_d \quad z'_1 \quad \cdots \quad z'_{d'}\right]^\top$$

Modern View

- **Feedforward neural network model family (for regression):**

$$f_{W,\beta}(x) = f_\beta \circ g \circ f_W(x)$$



Modern View

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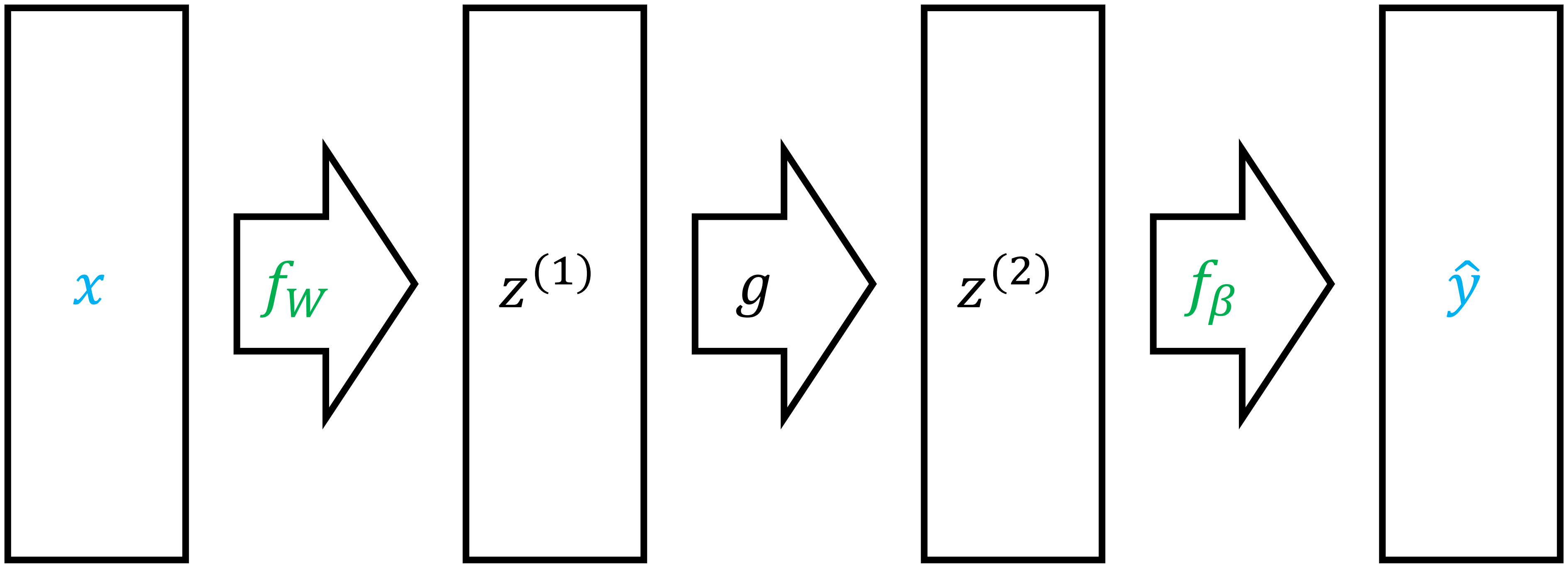
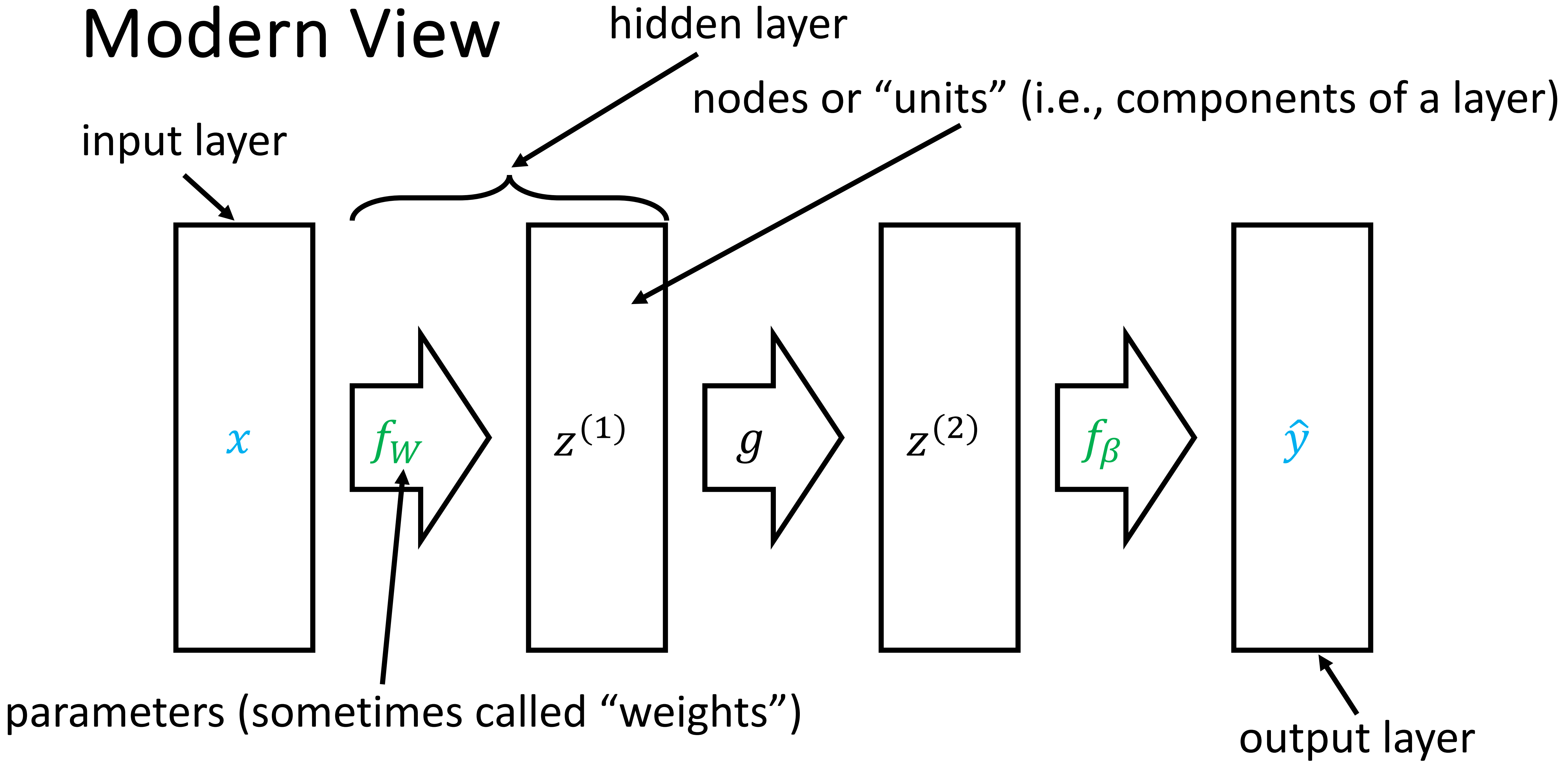


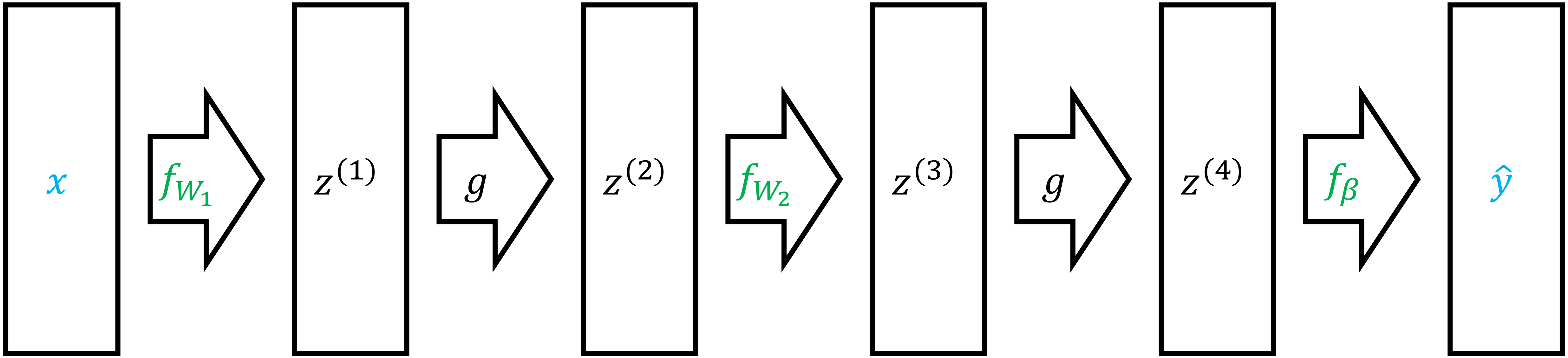
Modern View


hidden layer
nodes or "units" (i.e., components of a layer)
input layer
x
f_W
$z^{(1)}$
g
$z^{(2)}$
f_β
$\hat{y}$
parameters (sometimes called "weights")
output layer

Modern View

- **Neural network with two hidden linear layers:**

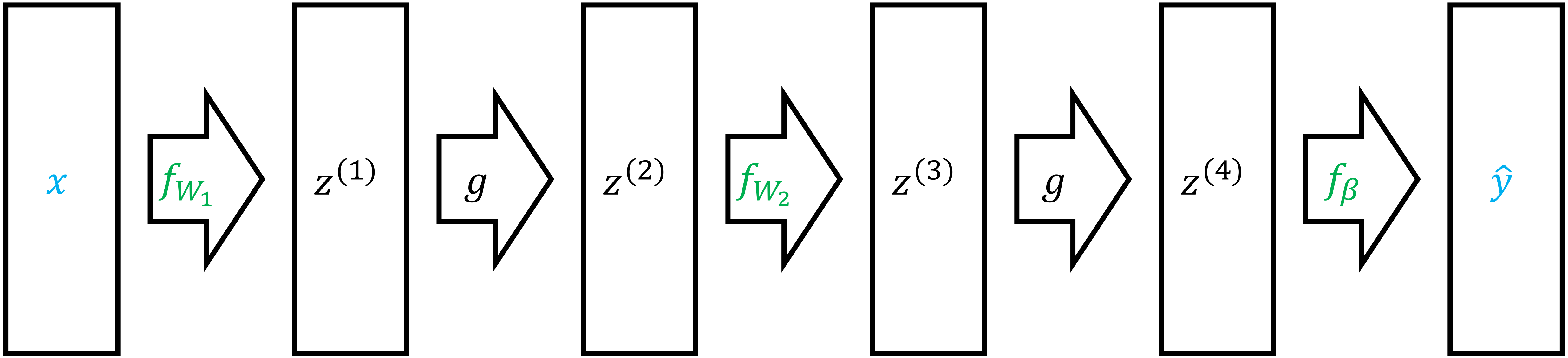
$$f_{W_1,W_2,\beta}(x) = f_\beta \circ g \circ f_{W_2} \circ g \circ f_{W_1}(x)$$



Modern View

- **Neural network with two hidden linear layers:**

$$f_{W_1,W_2,\beta}(x) = f_\beta\left(g\left(f_{W_2}\left(g\left(f_{W_1}(x)\right)\right)\right)\right)$$



Learn successively more “high-level” representations

Computing AND

x_1	x_2	x_1 AND x_2
0	0	0
0	1	0
1	0	0
1	1	1

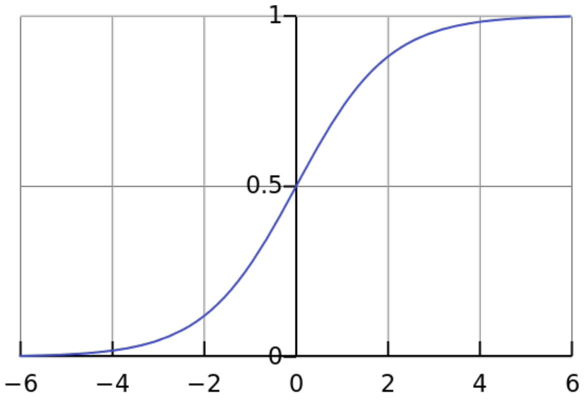
Computing AND

x_1	x_2	x_1 AND x_2
0	0	0
0	1	0
1	0	0
1	1	1

$$f_\beta(x) = \sigma(-30 + 20x_1 + 20x_2)$$

Computing AND

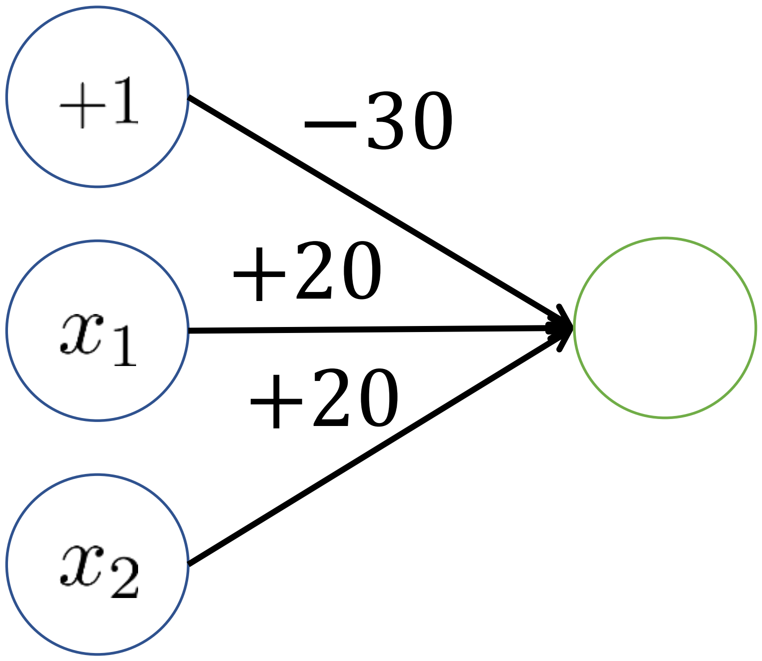
x_1	x_2	x_1 AND x_2	$f_\beta(x)$
0	0	0	$\sigma(-30) \approx 0$
0	1	0	$\sigma(-10) \approx 0$
1	0	0	$\sigma(-10) \approx 0$
1	1	1	$\sigma(10) \approx 1$



$$f_\beta(x) = \sigma(-30 + 20x_1 + 20x_2)$$

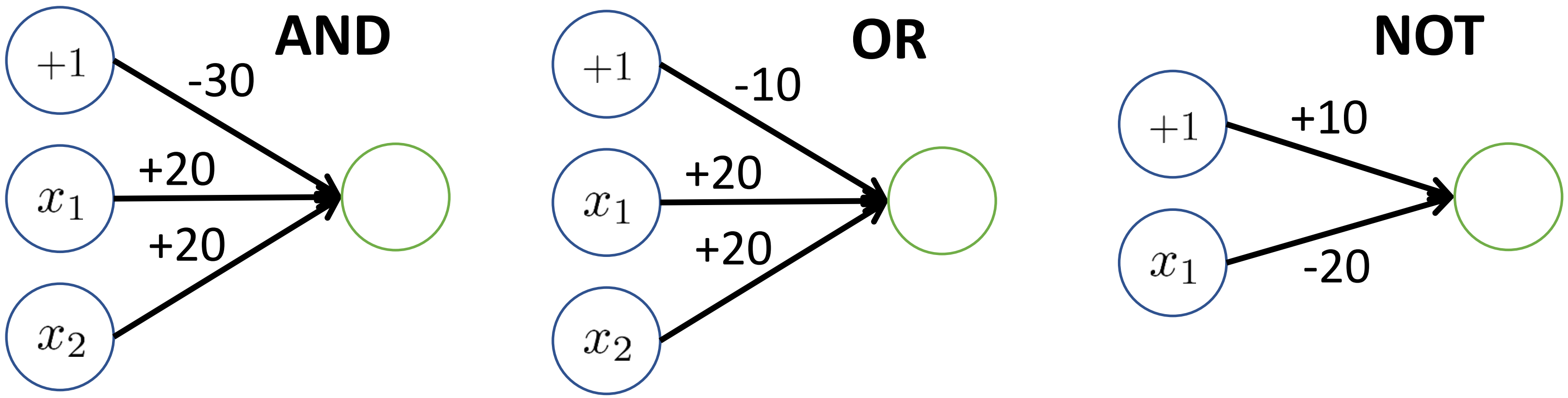
$\sigma(x)$

Computing AND

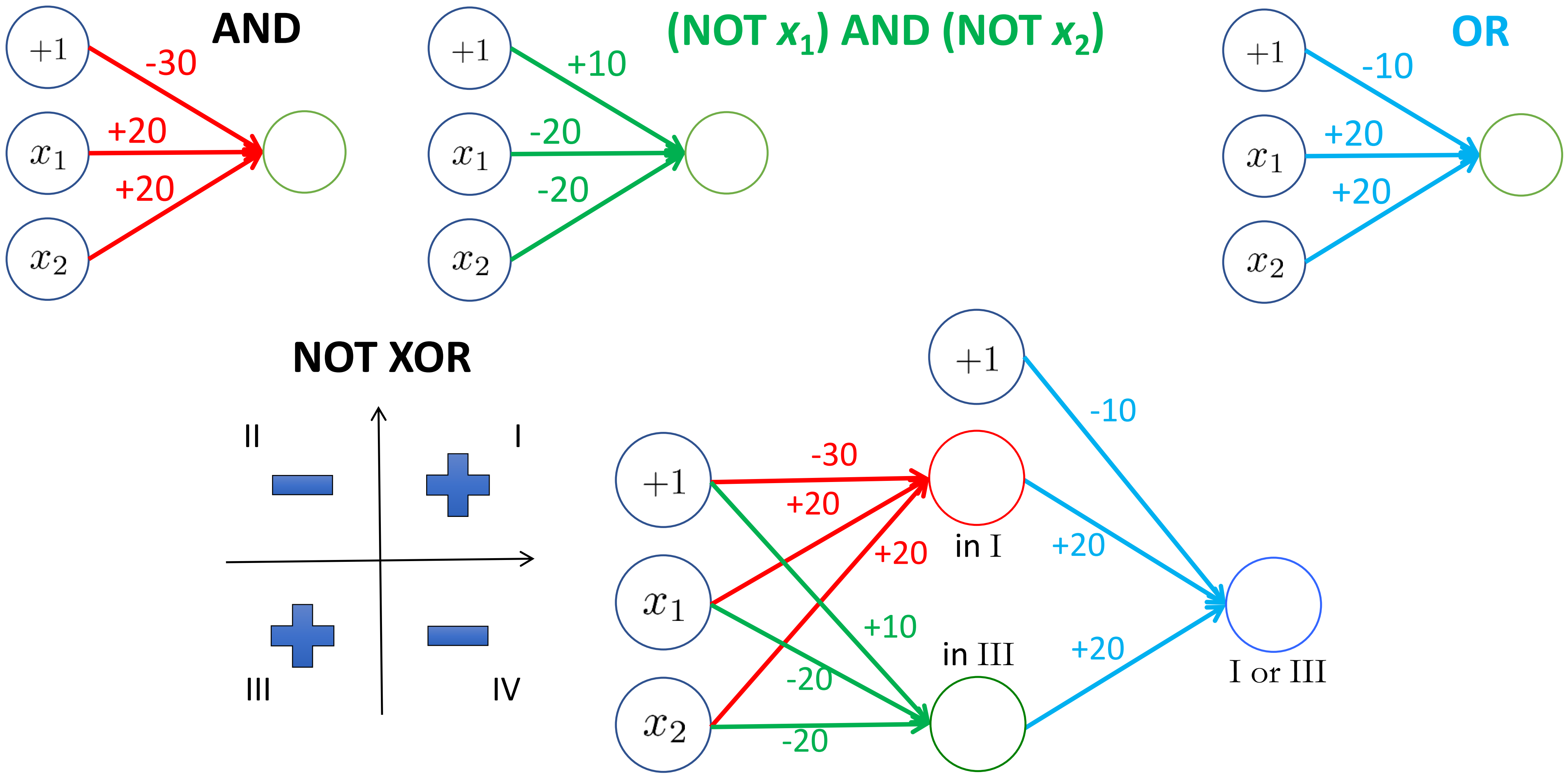


$$f_\beta(x) = \sigma(-30 + 20x_1 + 20x_2)$$

Computing Boolean Functions

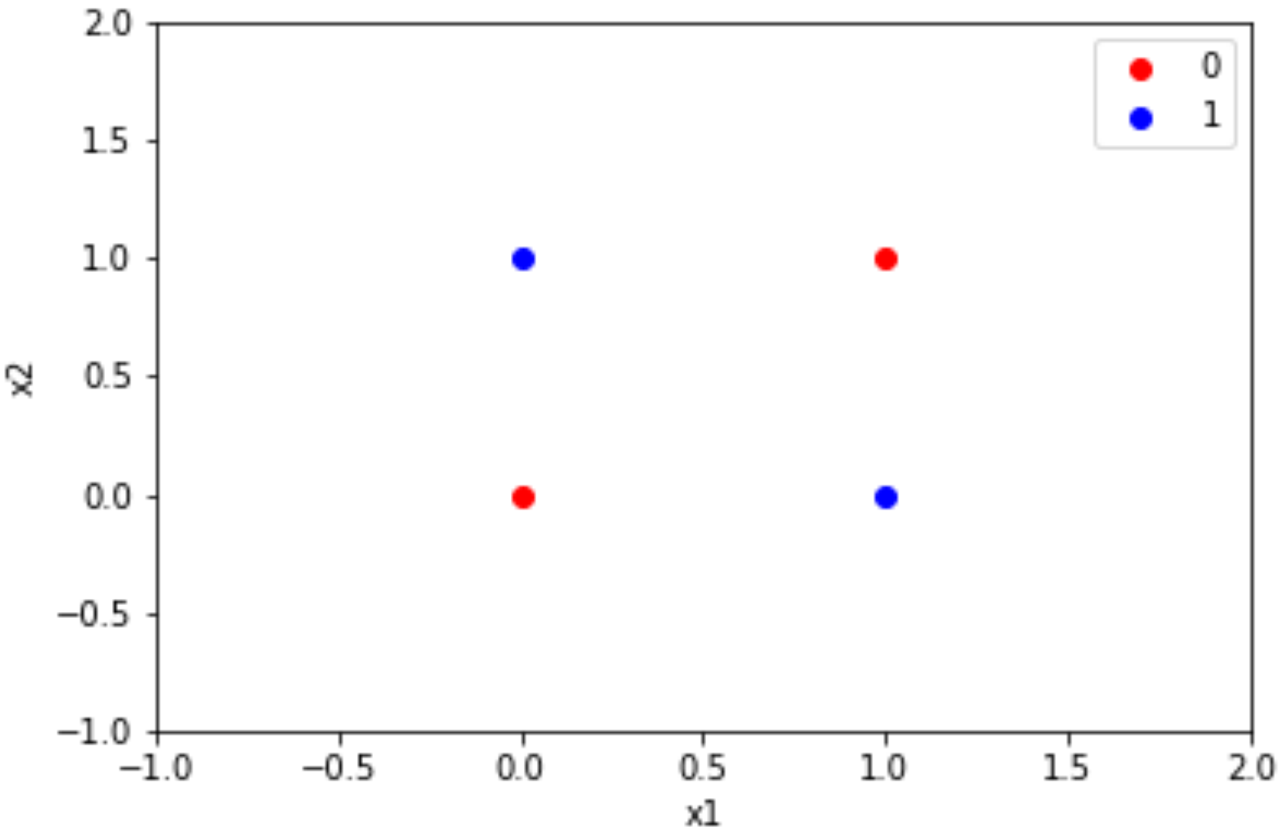


Computing Boolean Functions

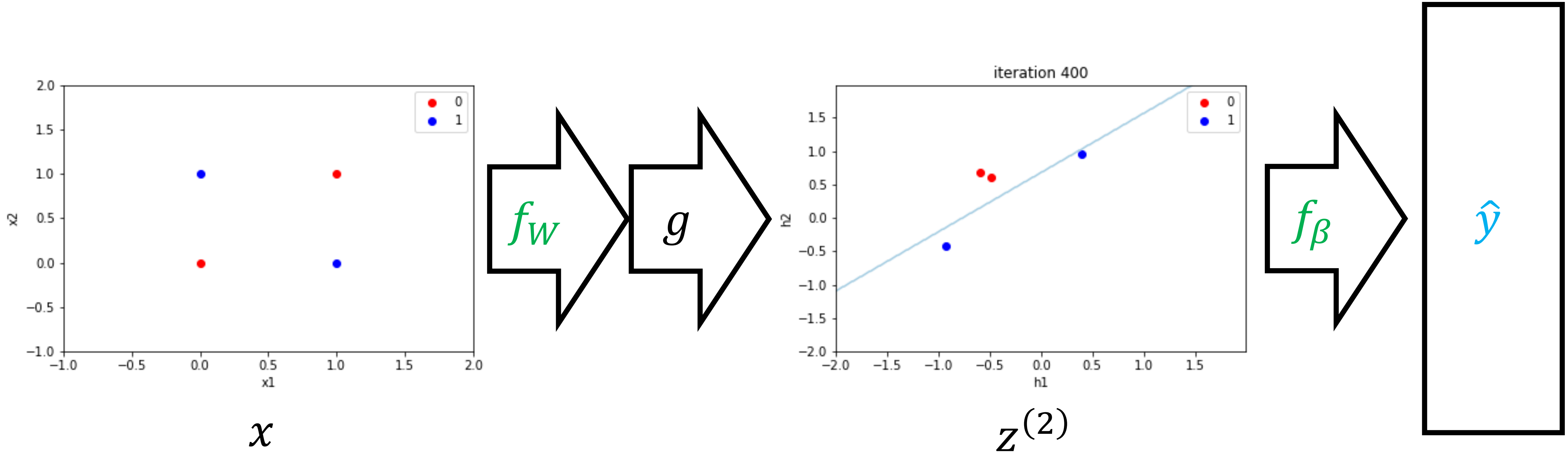


Based on slide and example by Andrew Ng

Computing XOR



Computing XOR



Computing XOR

- **Before:**
  - Linear regression + feature engineering
  - Include quadratic features to compute XOR

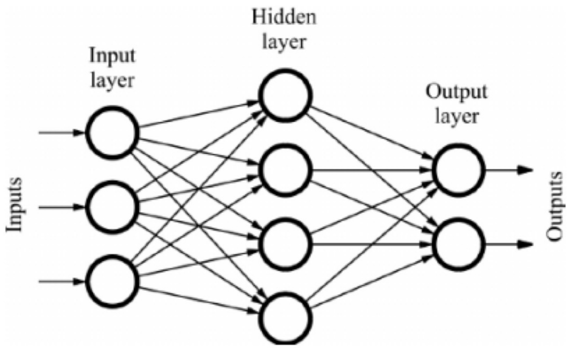
- **Neural networks:**
  - Design architecture to capture function
  - Automatically learn good "features" $z^{(2)} = g\left(f_W(x)\right)$ perform linear regression on these features $f_{W,\beta}(x) = \beta^\top z^{(2)}$
  - Called **representation learning**

Neural Networks

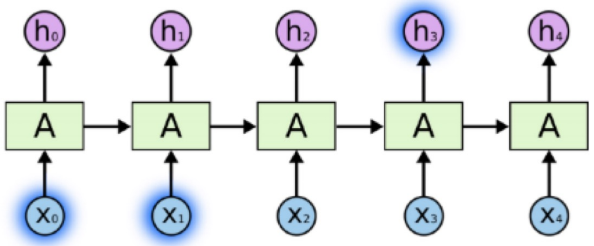
- **Pros**
  - **“Meta” strategy:** Enables users to **design** model family
  - Design model families that capture **symmetries/structure** in the data (e.g., read a sentence forwards, translation invariance for images, etc.)

Common Layers

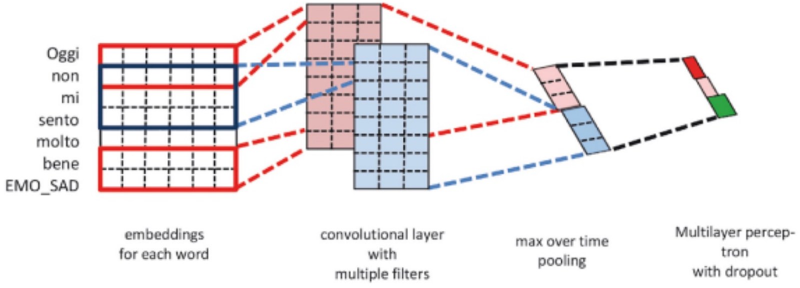
Feed-forward NNs



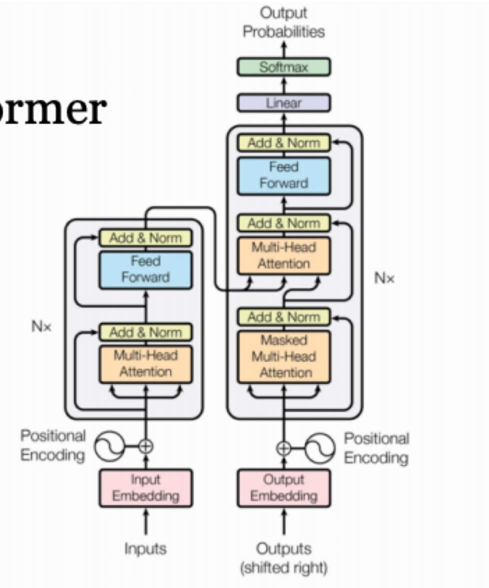
Recurrent NNs



Convolutional NNs



Transformer



Always coupled with word embeddings…

Neural Networks

- **Pros**
  - **“Meta” strategy:** Enables users to **design** model family
  - Design model families that capture **symmetries/structure** in the data (e.g., read a sentence forwards, translation invariance for images, etc.)
  - “Representation learning” (automatically learn features for certain domains)
  - More parameters!

- **Cons**
  - Very hard to train! (Non-convex loss functions)
  - Lots of parameters → need lots of data!
  - Lots of design decisions

Agenda

- **Model family**
  - Custom model family rather than a single model family

- **Optimization**
  - Backpropagation algorithm for computing gradient

Optimization Algorithm

- Based on gradient descent, with a few tweaks
  - **Note:** Loss is nonconvex, but gradient descent works well in practice

- **Key challenge:** How to compute the gradient?
  - **Strategy so far:** Work out gradient for every model family
  - **New strategy:** Algorithm for computing gradient of an arbitrary programmatic composition of layers
  - This algorithm is called **backpropagation**

Gradient Descent

- $W_1 \leftarrow \text{Initialize}(\,)$
- **for** $t \in \{1,2,\dots\}$ **until** convergence:

$$W_{t+1,j} \leftarrow W_{t,j} - \frac{\alpha}{n} \cdot \sum_{i=1}^{n} \nabla_{W_j} L\big(f_{W_t}(x_i), y_i\big) \quad \text{(for each } j\text{)}$$

- **return** f_{W_t}

Backpropagation

- **Input**
  - Example-label pair (x, y)
  - Arbitrary model $f_{W_m} \circ \cdots \circ f_{W_1}$
  - Loss $L(\hat{y}, y)$ for predicted label $\hat{y}$ and true label y
  - Derivative $\nabla_{\hat{y}} L(\hat{y}, y)$ (as a function)
  - Derivatives $D_{W_j} f_{W_j}(z)$ and $D_z f_{W_j}(z)$ (e.g., as a function)

- **Output:** $\nabla_{W_j} L(f_W(x), y)$

Recall: Multi-Dimensional Derivatives

- **Given:**
  - Function $f_W(z)$ mapping parameters $W \in \mathbb{R}^d$ and input vector $z \in \mathbb{R}^k$ to a vector $f_W(z) \in \mathbb{R}^h$
  - Current parameters W and z

- The **derivative** of f_W at W and z with respect to z is a matrix

$$D_z f_W(z) \in \mathbb{R}^{h \times k}$$

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- **Intuition:** The linear function that best approximates f_W at W and z:

$$f_{W+dW}(z + dz) \approx f_W(z) + D_z f_W(z) dz + D_W f_W(z) dW$$

Backpropagation Example

- **Gradient of MSE loss (for regression):**

$$\nabla_W L(W, \beta; Z) = \nabla_W \frac{1}{n} \sum_{i=1}^{n} \left(f_{W,\beta}(x_i) - y_i\right)^2$$
$$= \frac{2}{n} \sum_{i=1}^{n} \left(f_{W,\beta}(x_i) - y_i\right) D_W f_{W,\beta}(x_i)$$

$$\nabla_\beta L(W, \beta; Z) = \nabla_\beta \frac{1}{n} \sum_{i=1}^{n} \left(f_{W,\beta}(x_i) - y_i\right)^2$$
$$= \frac{2}{n} \sum_{i=1}^{n} \left(f_{W,\beta}(x_i) - y_i\right) D_\beta f_{W,\beta}(x_i)$$

Backpropagation Example

- **Derivative of neural network:**

$$D_\beta f_{W,\beta}(x) = D_\beta\left(f_\beta \circ g \circ f_W\right)(x)$$
$$= D_\beta f_\beta\left(g \circ f_W(x)\right)$$

$$D_W f_{W,\beta}(x) = D_W\left(f_\beta \circ g \circ f_W\right)(x)$$
$$= D_z f_\beta\left(g \circ f_W(x)\right) D_W(g \circ f_W)(x)$$
$$= D_z f_\beta\left(g \circ f_W(x)\right) D_z g\left(f_W(x)\right) D_W f_W(x)$$

Backpropagation

- **General case:** Consider a neural network

$$f_W(x) = f_{W_m} \circ f_{W_{m-1}} \circ \cdots \circ f_{W_1}(x)$$

$$z^{(j)} = f_{W_j} \circ \cdots \circ f_{W_1}(x) = \begin{cases} x & \text{if } j = 0 \\ f_{W_j}\left(z^{(j-1)}\right) & \text{if } j > 0 \end{cases}$$

- We have

$$D_{W_m} f_W(x)$$

Backpropagation

- **General case:** Consider a neural network

$$f_W(x) = f_{W_m} \circ f_{W_{m-1}} \circ \cdots \circ f_{W_1}(x)$$

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Backpropagation

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- We have

$$D_{W_m} f_W(x) = D_{W_m} f_{W_m}\left(z^{(m-1)}\right)$$

Backpropagation

- **General case:** Consider a neural network

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$$z^{(j)} = f_{W_j} \circ \cdots \circ f_{W_1}(x) = \begin{cases} x & \text{if } j = 0 \\ f_{W_j}\left(z^{(j-1)}\right) & \text{if } j > 0 \end{cases}$$

- We have

$$D_{W_{m-1}} f_W(x)$$

Backpropagation

- **General case:** Consider a neural network

$$f_W(x) = f_{W_m} \circ f_{W_{m-1}} \circ \cdots \circ f_{W_1}(x)$$

$$z^{(j)} = f_{W_j} \circ \cdots \circ f_{W_1}(x) = \begin{cases} x & \text{if } j = 0 \\ f_{W_j}\left(z^{(j-1)}\right) & \text{if } j > 0 \end{cases}$$

- We have

$$D_{W_{m-1}} f_W(x) = D_z f_{W_m}\left(z^{(m-1)}\right) D_{W_{m-1}} f_{W_{m-1}}\left(z^{(m-2)}\right)$$

Backpropagation

- **General case:** Consider a neural network

$$f_W(x) = f_{W_m} \circ f_{W_{m-1}} \circ \cdots \circ f_{W_1}(x)$$

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- We have

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Backpropagation

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$$f_W(x) = f_{W_m} \circ f_{W_{m-1}} \circ \cdots \circ f_{W_1}(x)$$

$$z^{(j)} = f_{W_j} \circ \cdots \circ f_{W_1}(x) = \begin{cases} x & \text{if } j = 0 \\ f_{W_j}\left(z^{(j-1)}\right) & \text{if } j > 0 \end{cases}$$

- We have

$$D_{W_{m-2}} f_W(x)$$

Backpropagation

- **General case:** Consider a neural network

$$f_W(x) = f_{W_m} \circ f_{W_{m-1}} \circ \cdots \circ f_{W_1}(x)$$

$$z^{(j)} = f_{W_j} \circ \cdots \circ f_{W_1}(x) = \begin{cases} x & \text{if } j = 0 \\ f_{W_j}\left(z^{(j-1)}\right) & \text{if } j > 0 \end{cases}$$

- We have

$$D_{W_{m-2}} f_W(x) = D_z f_{W_m}\left(z^{(m-1)}\right) D_z f_{W_{m-1}}\left(z^{(m-2)}\right) D_{W_{m-2}} f_{W_{m-2}}\left(z^{(m-3)}\right)$$

Backpropagation

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$$f_W(x) = f_{W_m} \circ f_{W_{m-1}} \circ \cdots \circ f_{W_1}(x)$$

$$z^{(j)} = f_{W_j} \circ \cdots \circ f_{W_1}(x) = \begin{cases} x & \text{if } j = 0 \\ f_{W_j}\left(z^{(j-1)}\right) & \text{if } j > 0 \end{cases}$$

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Backpropagation

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Backpropagation

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$$D_{W_{m-2}} f_W(x) = D_z f_{W_m}\left(z^{(m-1)}\right) D_z f_{W_{m-1}}\left(z^{(m-2)}\right) D_{W_{m-2}} f_{W_{m-2}}\left(z^{(m-3)}\right)$$

Backpropagation

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- We have

$$D_{W_j} f_W(x)$$

Backpropagation

- **General case:** Consider a neural network

$$f_W(x) = f_{W_m} \circ f_{W_{m-1}} \circ \cdots \circ f_{W_1}(x)$$

$$z^{(j)} = f_{W_j} \circ \cdots \circ f_{W_1}(x) = \begin{cases} x & \text{if } j = 0 \\ f_{W_j}\left(z^{(j-1)}\right) & \text{if } j > 0 \end{cases}$$

- We have

$$D_{W_j} f_W(x) = D_z f_{W_m}\left(z^{(m-1)}\right) \ldots D_z f_{W_{j+1}}\left(z^{(j)}\right) D_{W_j} f_{W_j}\left(z^{(j-1)}\right)$$

Backpropagation

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Backpropagation

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Backpropagation

- We have

$$D_{W_j} f_W(x) = \underbrace{D_z f_{W_m}\left(z^{(m-1)}\right) \ldots D_z f_{W_{j+1}}\left(z^{(j)}\right)} D_{W_j} f_{W_j}\left(z^{(j-1)}\right)$$

Portions shared across terms

Denote it by $D^{(j)}$

Backpropagation Algorithm

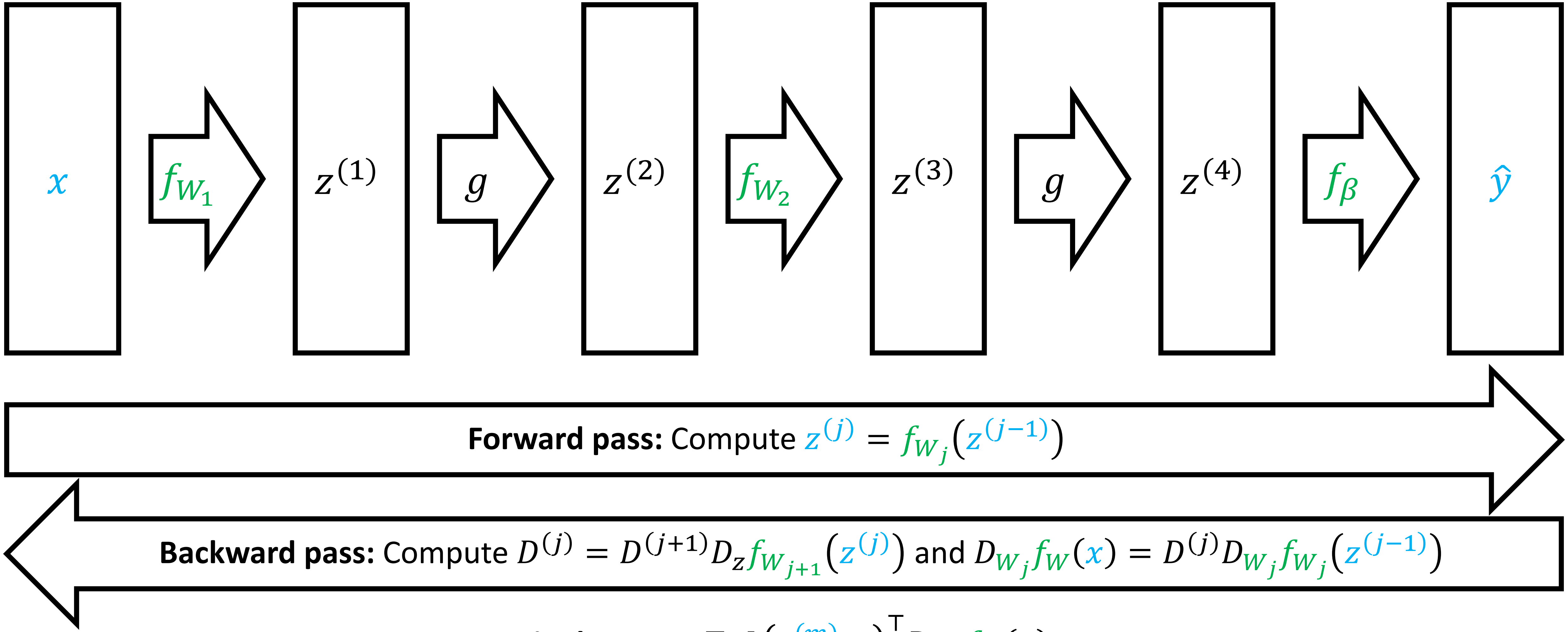
- Compute recursively starting from $j = m$ to $j = 1$:

$$D^{(j)} = D_z f_{W_m}\left(z^{(m-1)}\right) \dots D_z f_{W_{j+1}}\left(z^{(j)}\right)$$

$$= \begin{cases} 1 & \text{if } j = m \\ D^{(j+1)} D_z f_{W_{j+1}}\left(z^{(j)}\right) & \text{if } j < m \end{cases}$$

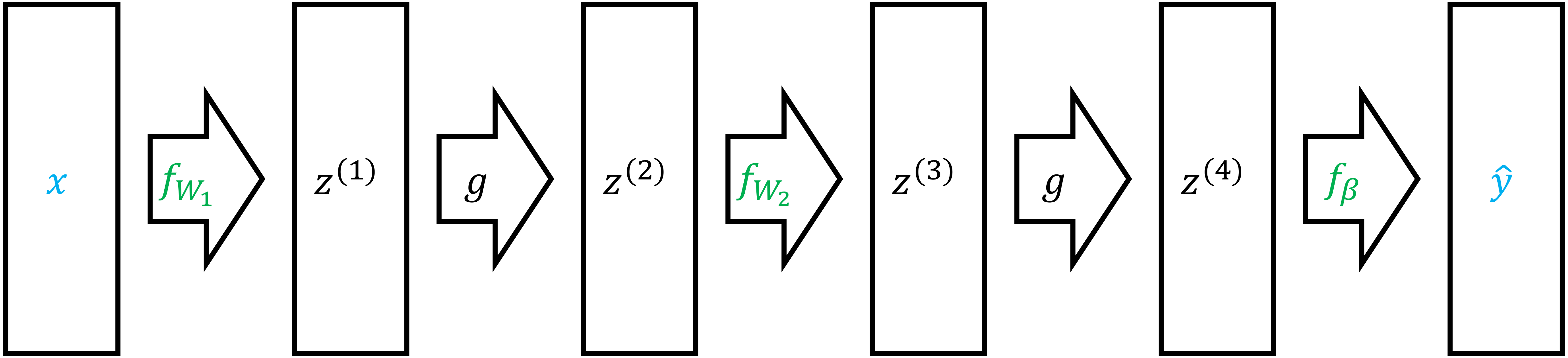
$$D_{W_j} f_W(x) = D^{(j)} D_{W_j} f_{W_j}\left(z^{(j-1)}\right)$$

Backpropagation

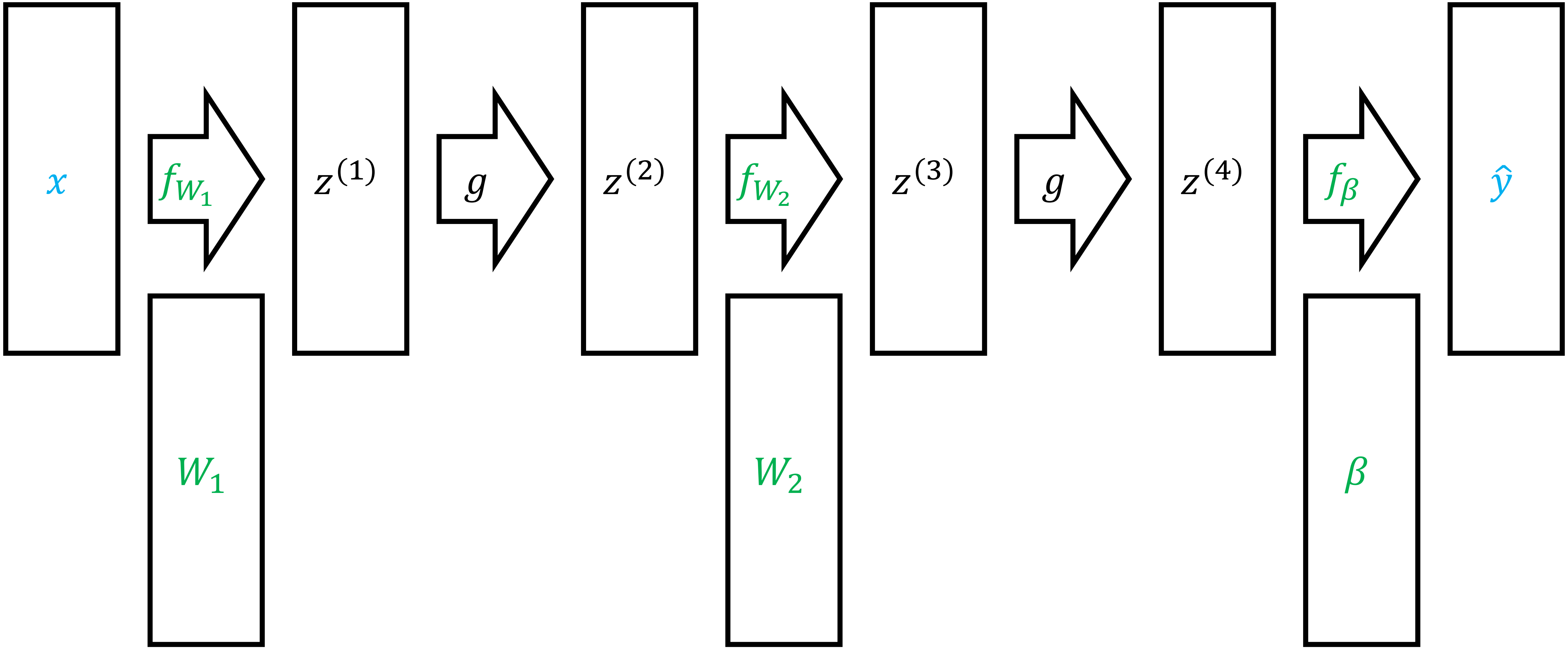


Final output: $\nabla_{\hat{y}} L\left(z^{(m)}, y\right)^\top D_{W_j} f_W(x)$

Backpropagation



Backpropagation



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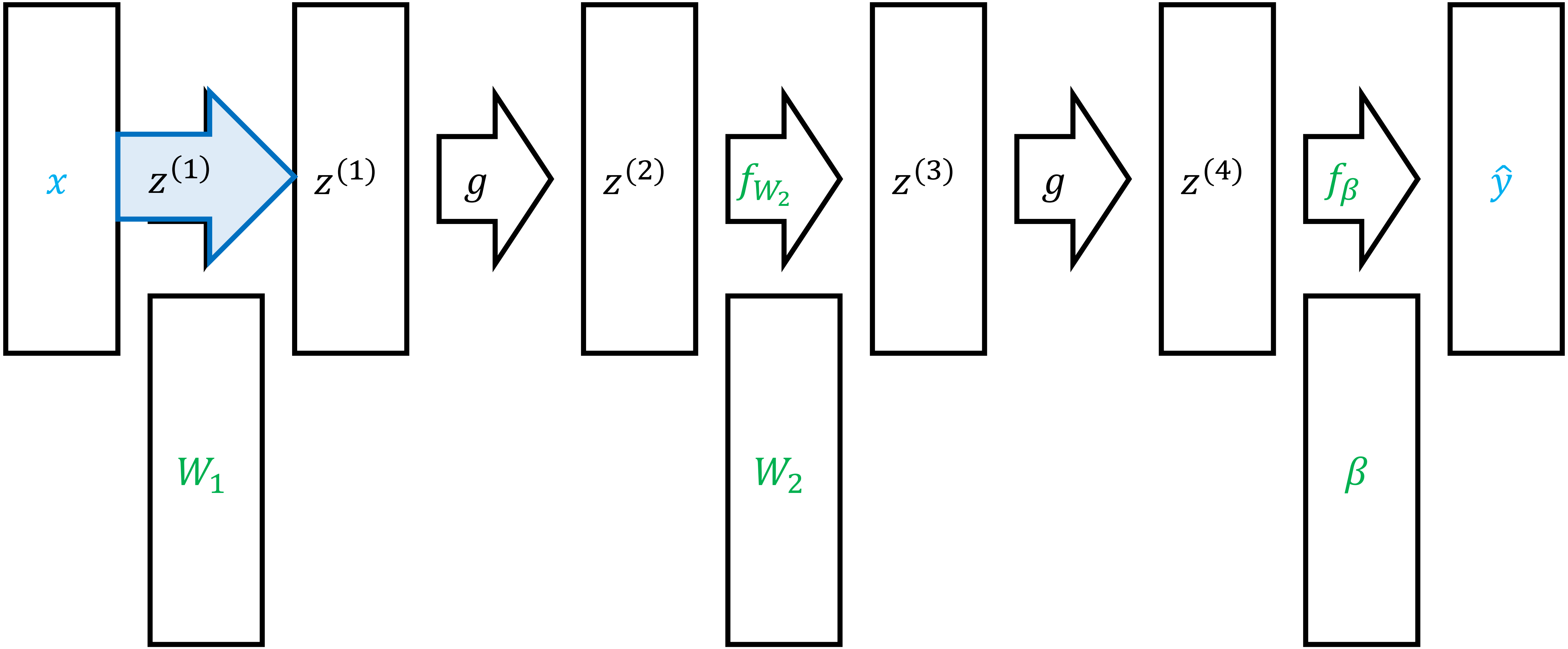
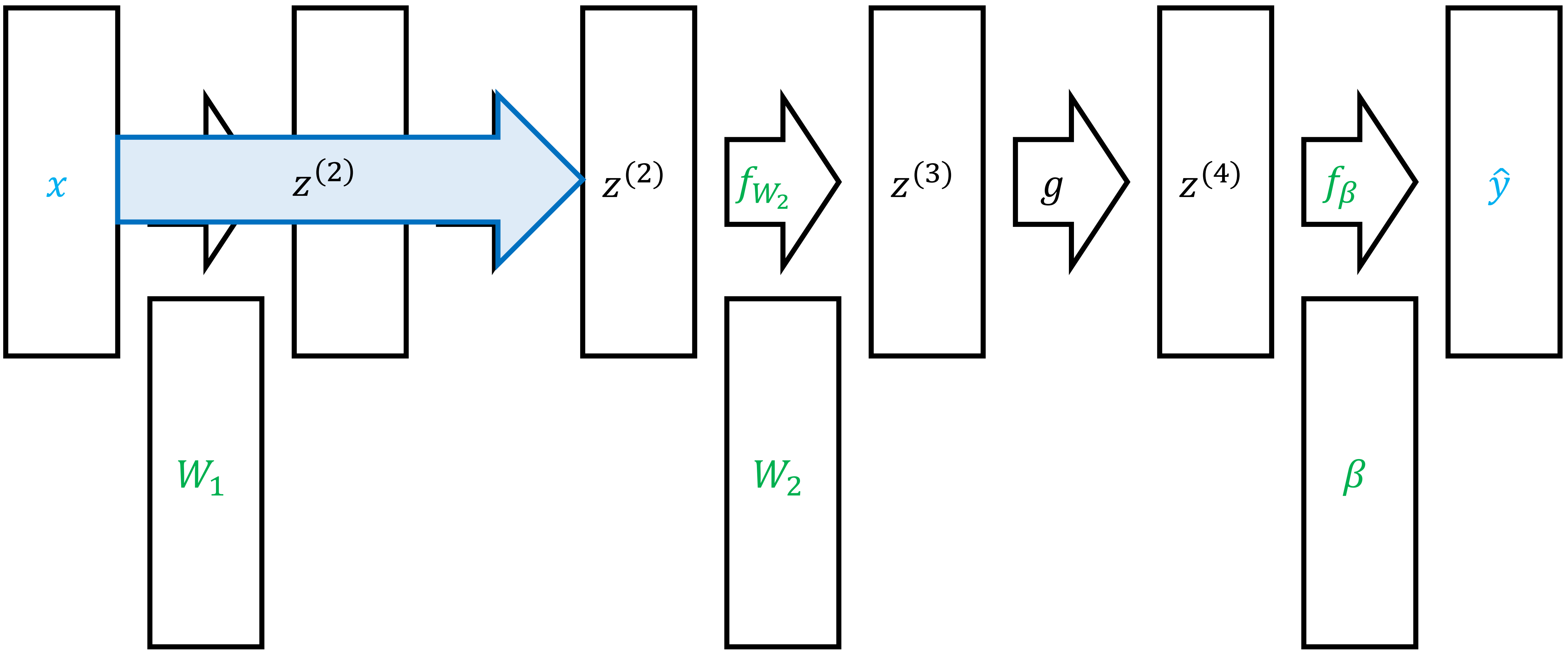
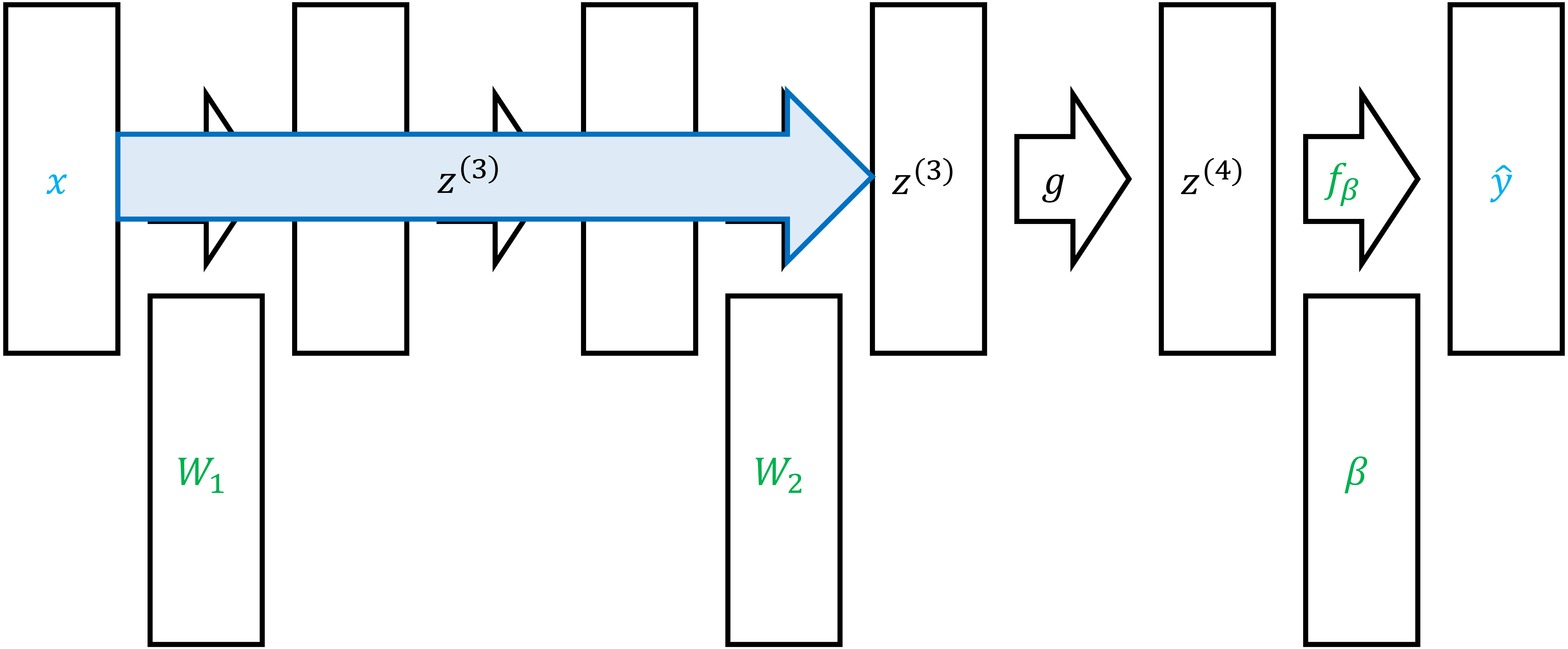


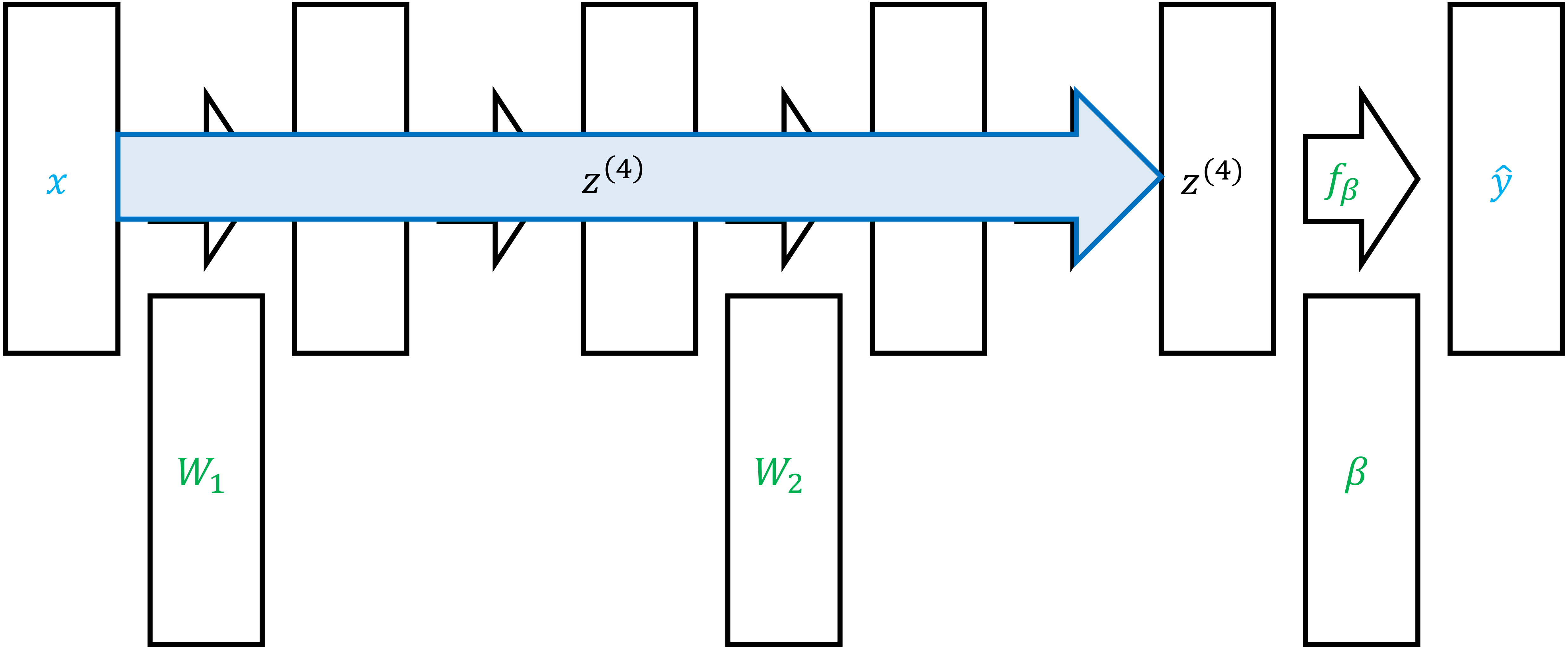
Backpropagation



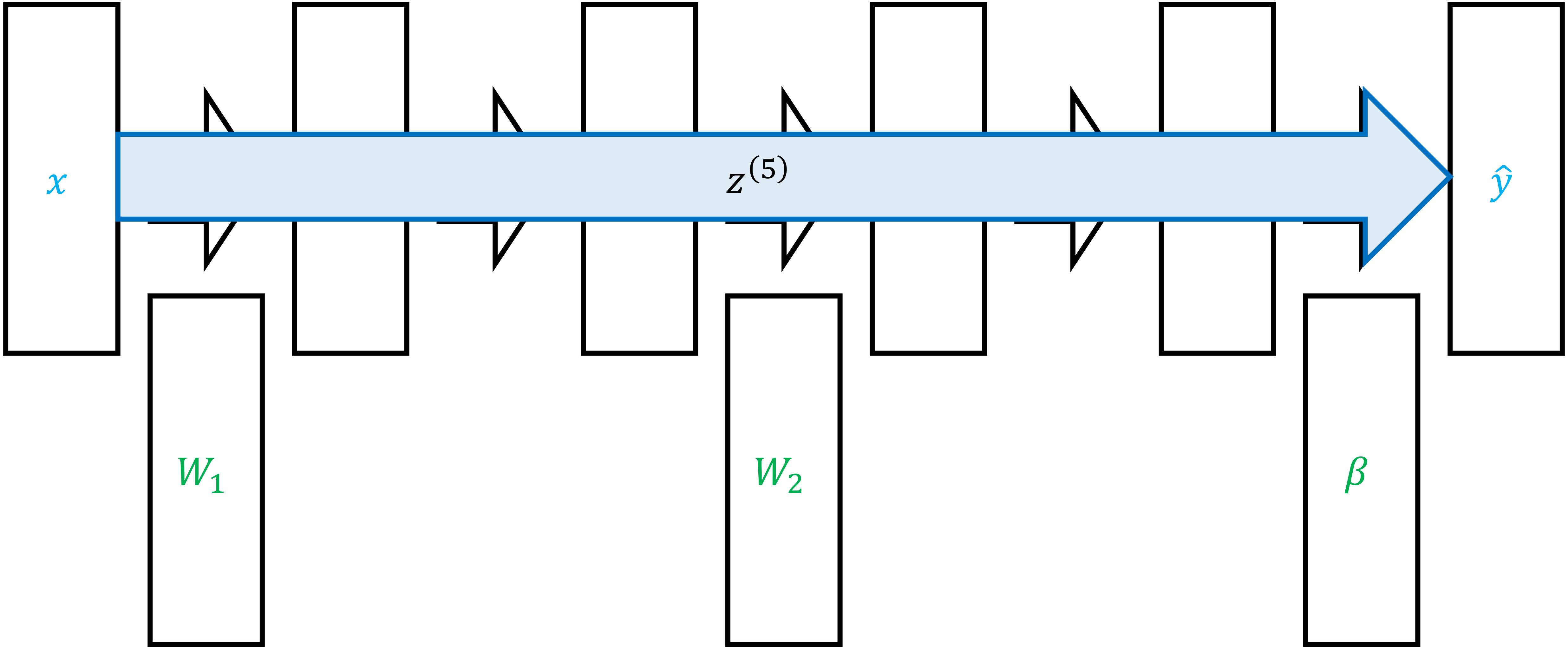
Backpropagation



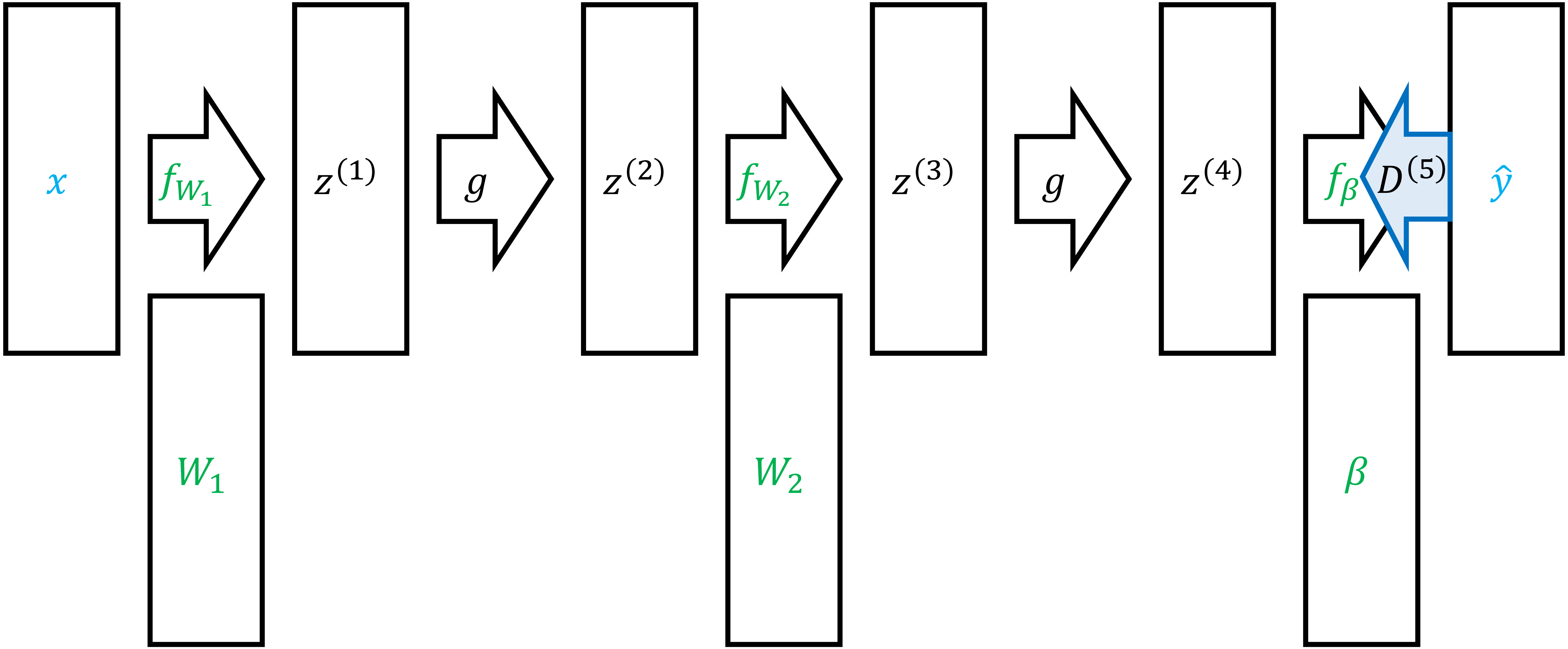
Backpropagation



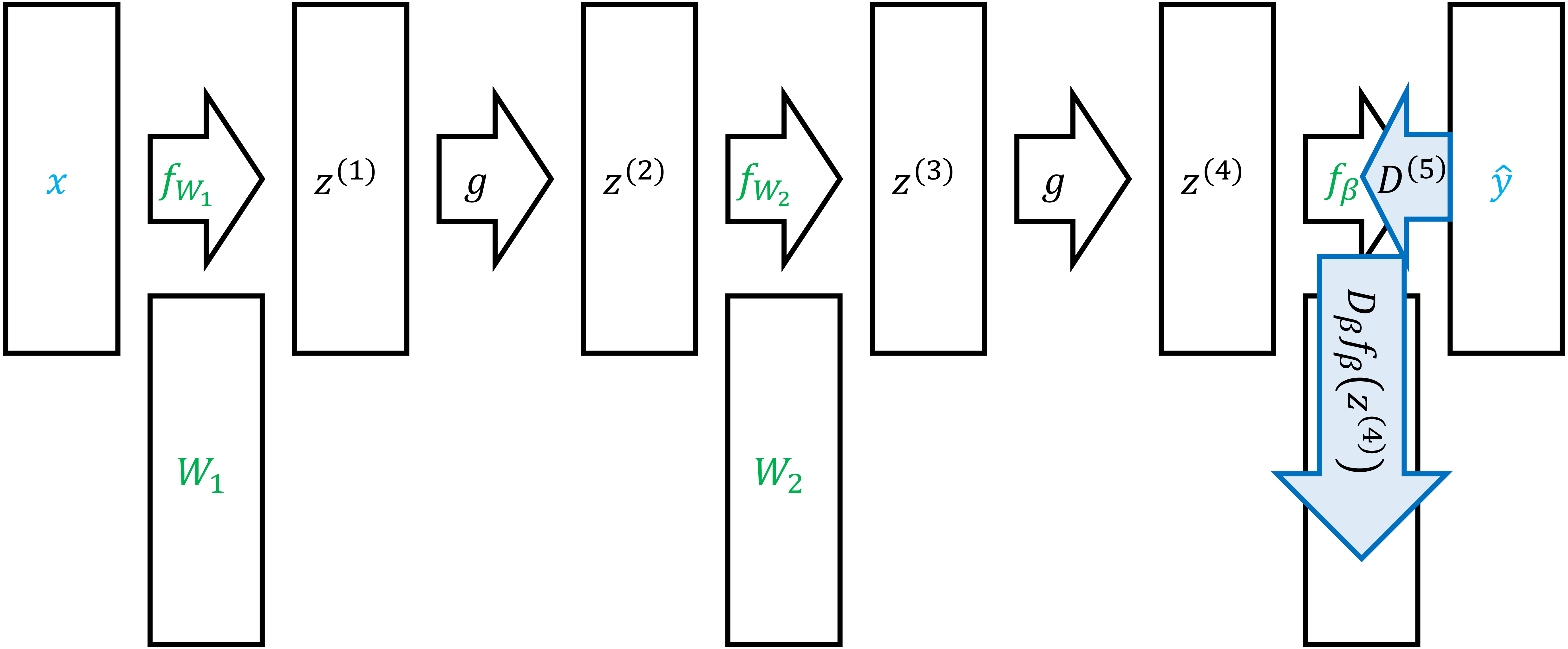
Backpropagation



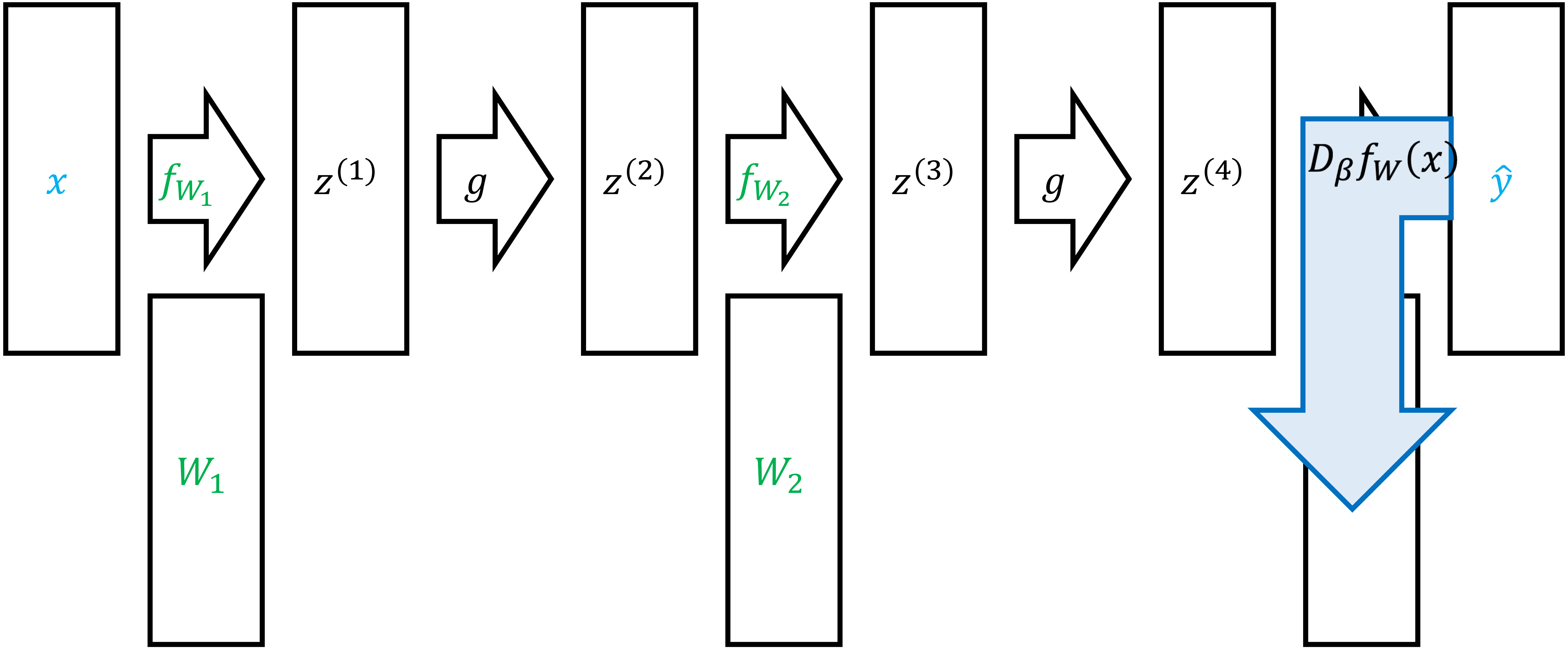
Backpropagation



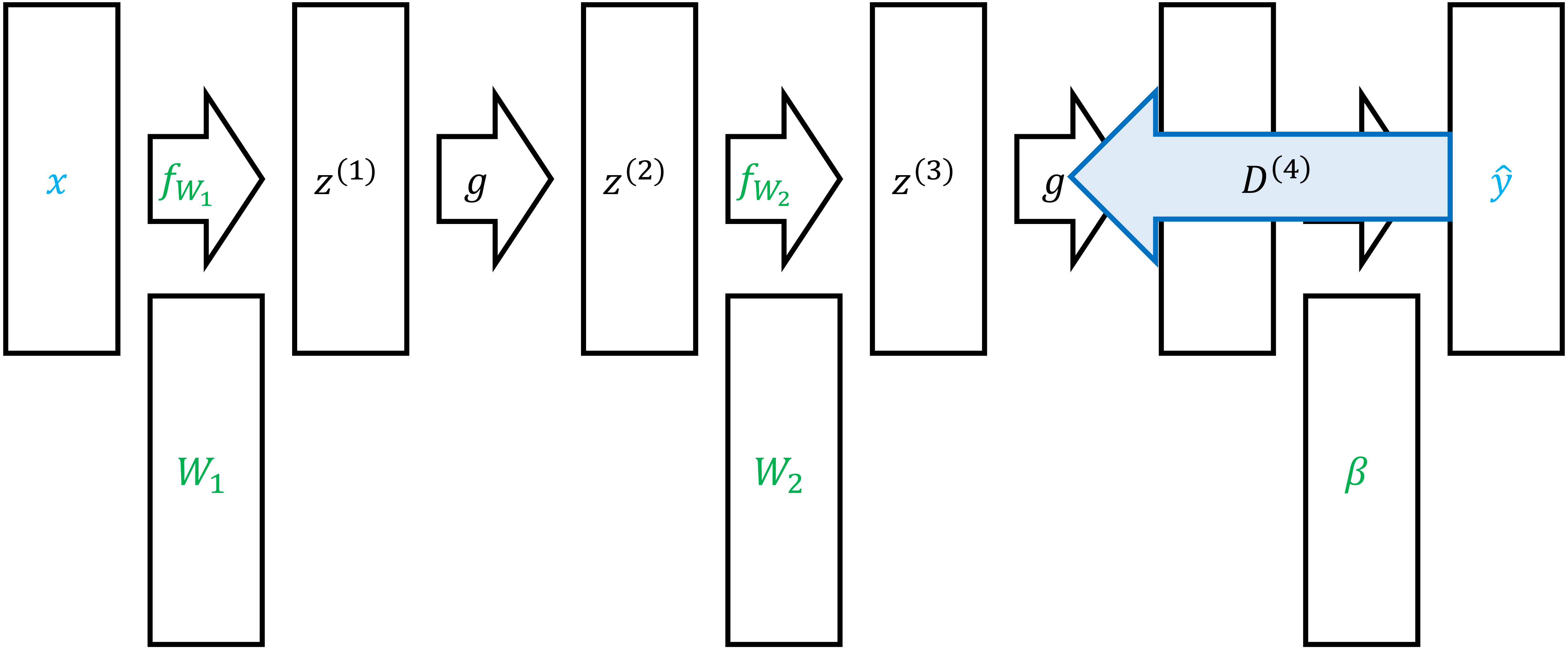
Backpropagation



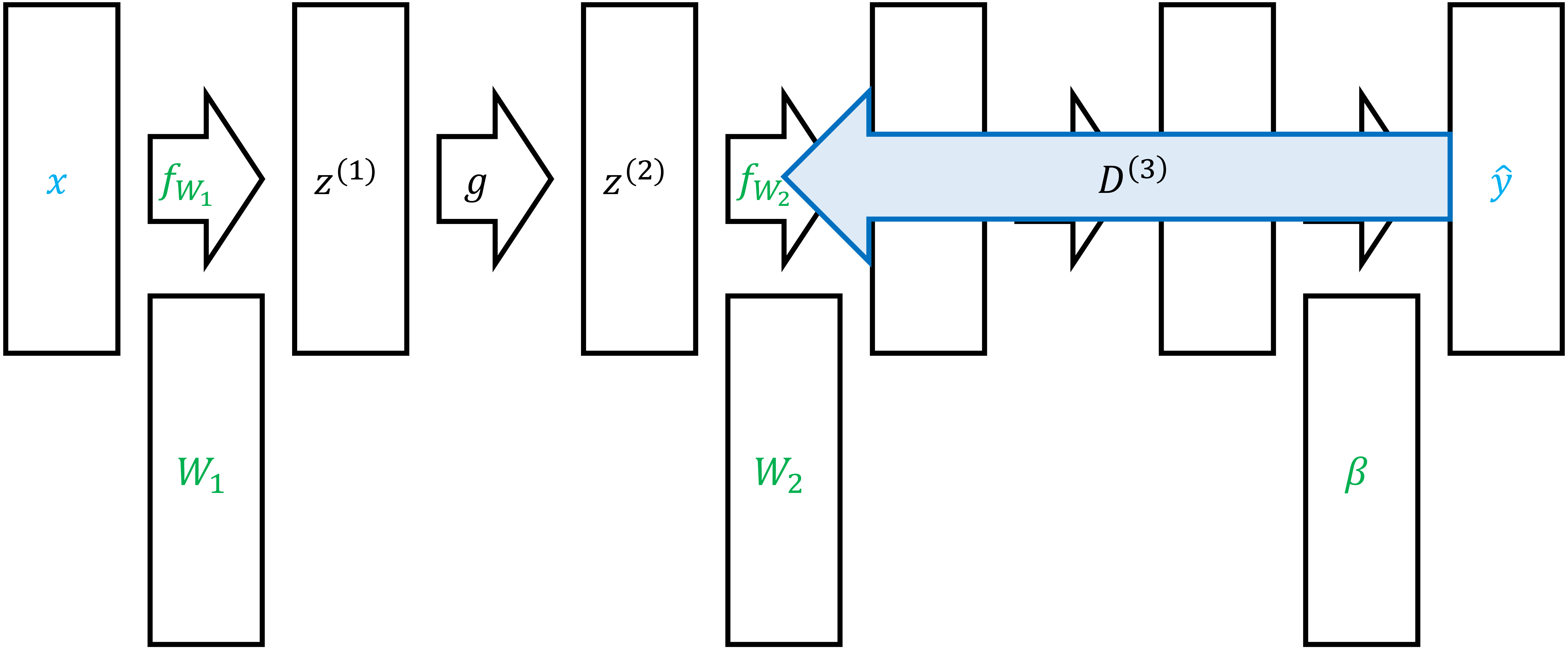
Backpropagation



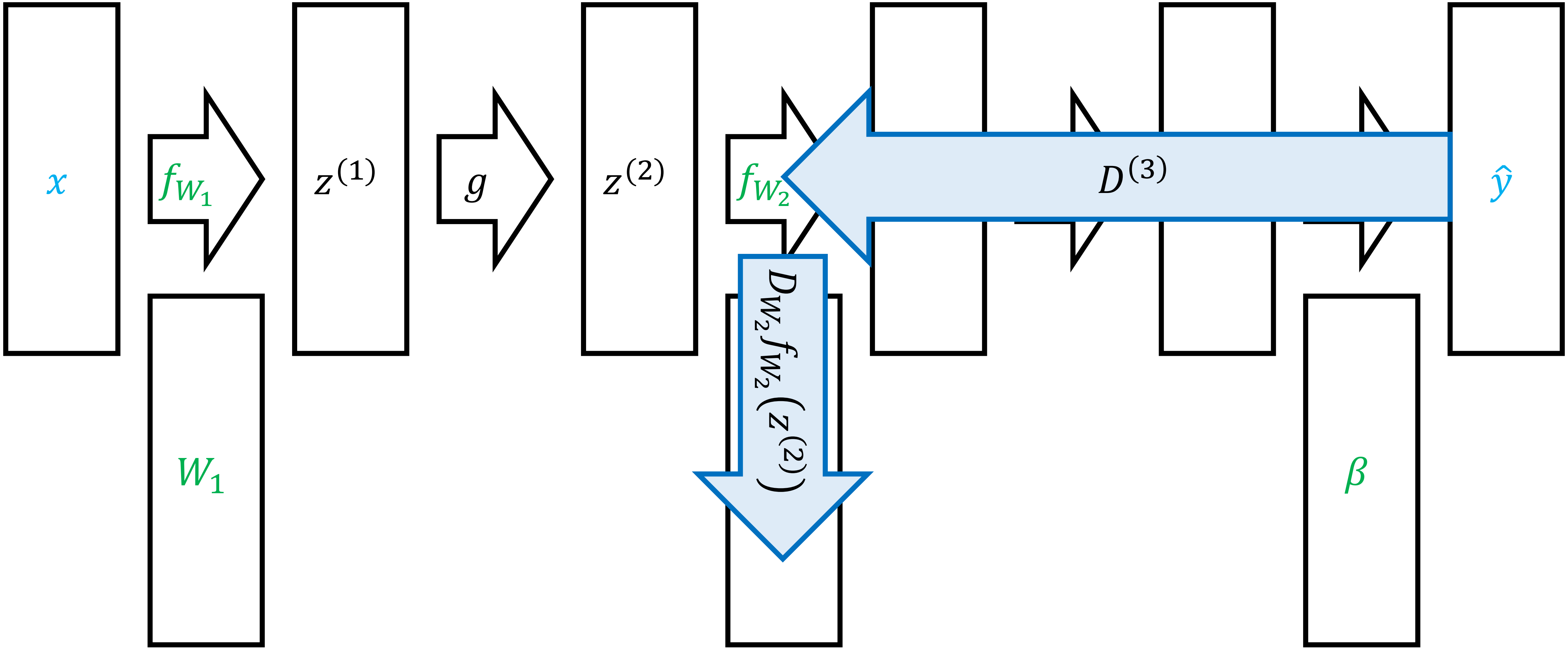
Backpropagation



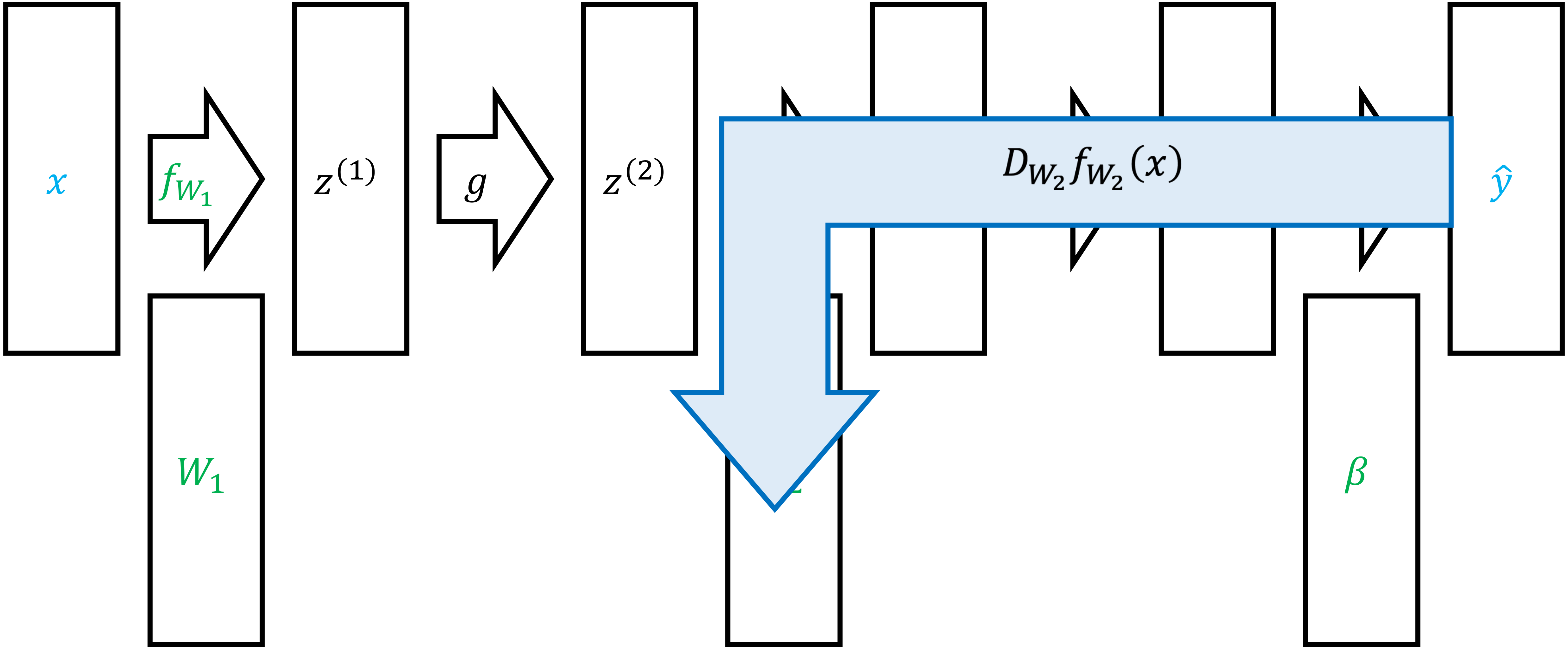
Backpropagation



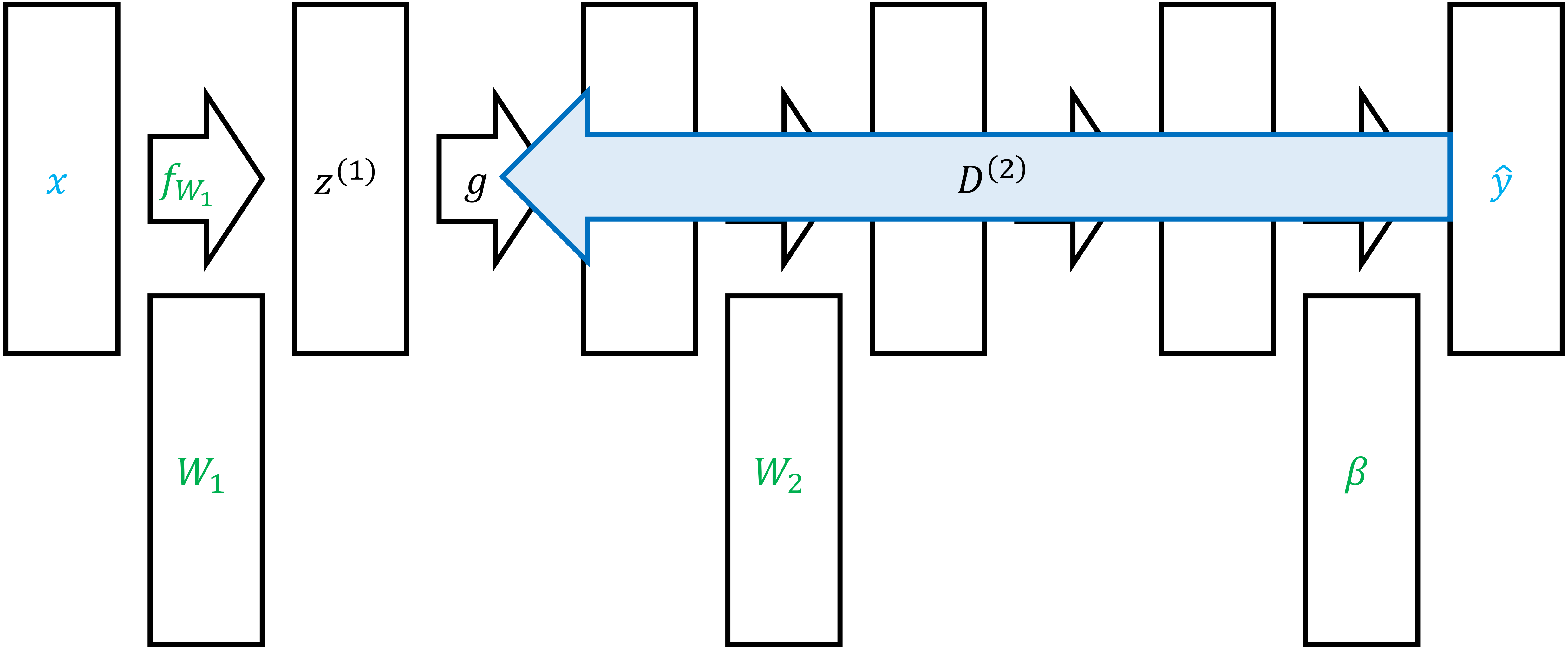
Backpropagation



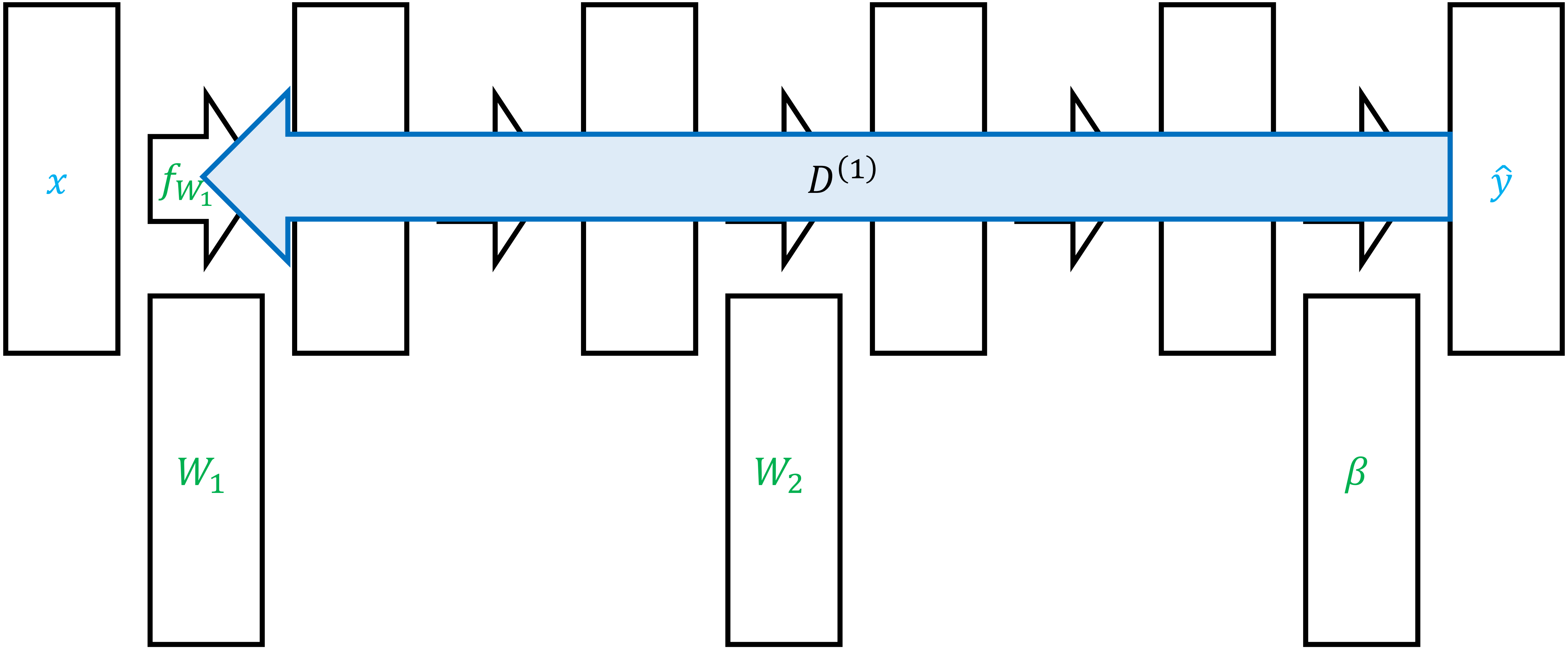
Backpropagation



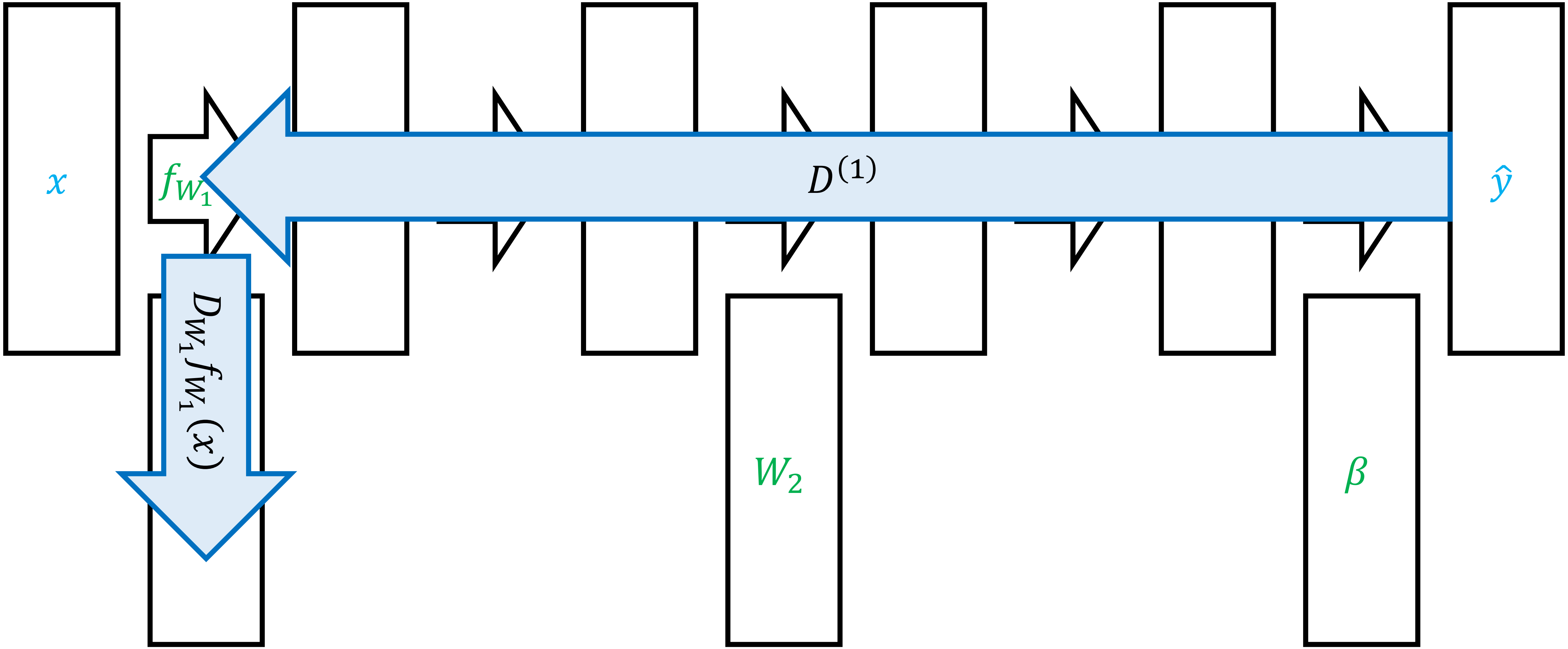
Backpropagation



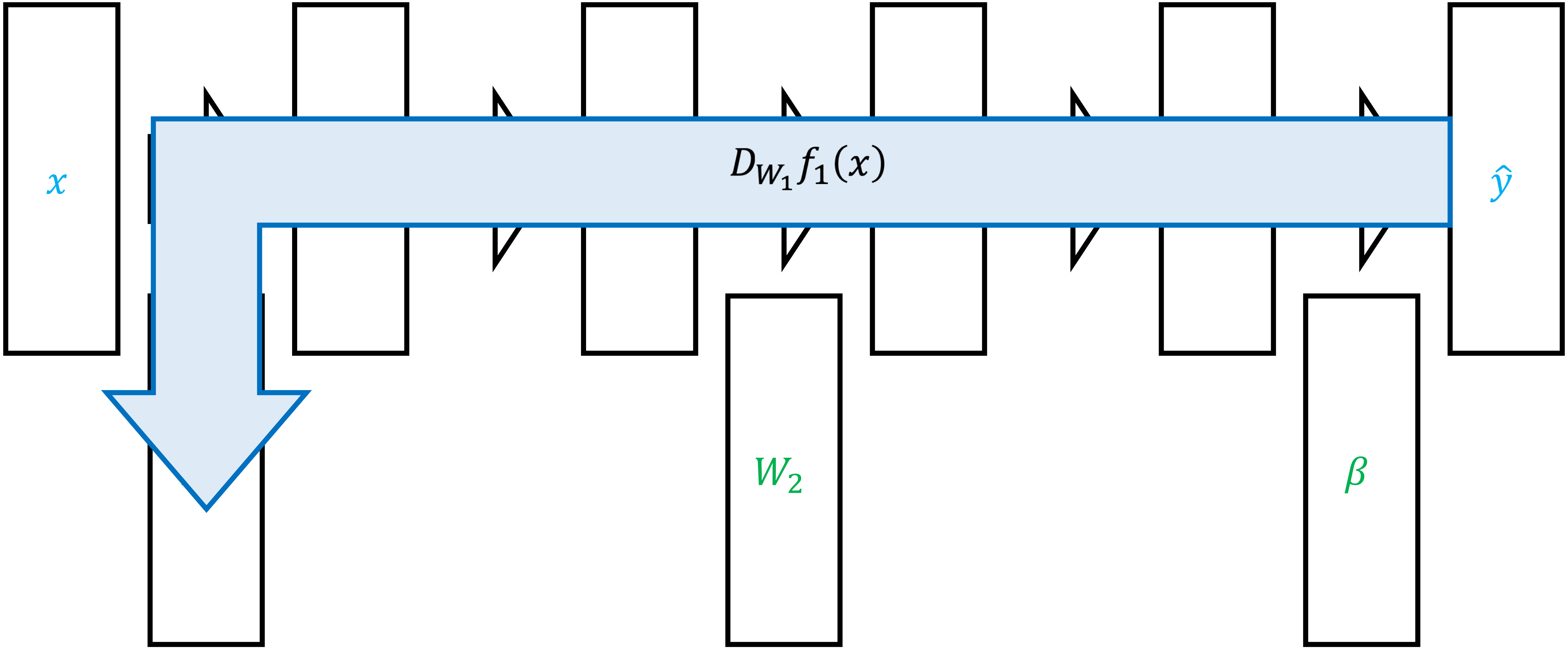
Backpropagation



Backpropagation



Backpropagation



Backpropagation Algorithm

- **Forward pass:** Compute forwards from $j = 0$ to $j = m$
  - $z^{(j)} = \begin{cases} x & \text{if } j = 0 \\ f_{W_j}\left(z^{(j-1)}\right) & \text{if } j > 0 \end{cases}$

- **Backward pass:** Compute backwards from $j = m$ to $j = 1$
  - $D^{(j)} = \begin{cases} 1 & \text{if } j = m \\ D^{(j+1)} D_z f_{W_{j+1}}\left(z^{(j)}\right) & \text{if } j < m \end{cases}$
  - $D_{W_j} f_W(x) = D^{(j)} D_{W_j} f_{W_j}\left(z^{(j-1)}\right)$

- **Final output:** $\nabla_{W_j} L(f_W(x), y)^\top = \nabla_{\hat{y}} L\left(z^{(m)}, y\right)^\top D_{W_j} f_W(x)$ for each j

Gradient Descent

- $W_1 \leftarrow \text{Initialize}(\,)$
- **for** $t \in \{1,2,\ldots\}$ **until** convergence:

$$W_{t+1,j} \leftarrow W_{t,j} - \frac{\alpha}{n} \cdot \sum_{i=1}^{n} \nabla_{W_j} L\big(f_{W_t}(x_i), y_i\big) \quad \text{(for each } j\text{)}$$

- **return** f_{W_t}

Gradient Descent

- $W_1 \leftarrow \mathrm{Initialize}(\,)$
- **for** $t \in \{1,2,\ldots\}$ **until** convergence:
  - Compute gradients $\nabla_{W_j} L\big(f_{W_t}(x_i), y_i\big)$ using backpropagation
  - Update parameters:

$$W_{t+1,j} \leftarrow W_{t,j} - \frac{\alpha}{n} \cdot \sum_{i=1}^{n} \nabla_{W_j} L\big(f_{W_t}(x_i), y_i\big) \quad (\text{for each } j)$$

- **return** f_{W_t}