Announcements

- Homework 3 due Wednesday at 8pm

Lecture 19: Reinforcement Learning

CIS 4190/5190

Spring 2025

Three Kinds of Learning

- **Supervised learning**
  - Given labeled examples (x, y), learn to predict y given x
- **Unsupervised learning**
  - Given unlabeled examples x, uncover structure in x
- **Reinforcement learning**
  - Learning from sequence of interactions with the environment

Sequential Decision Making

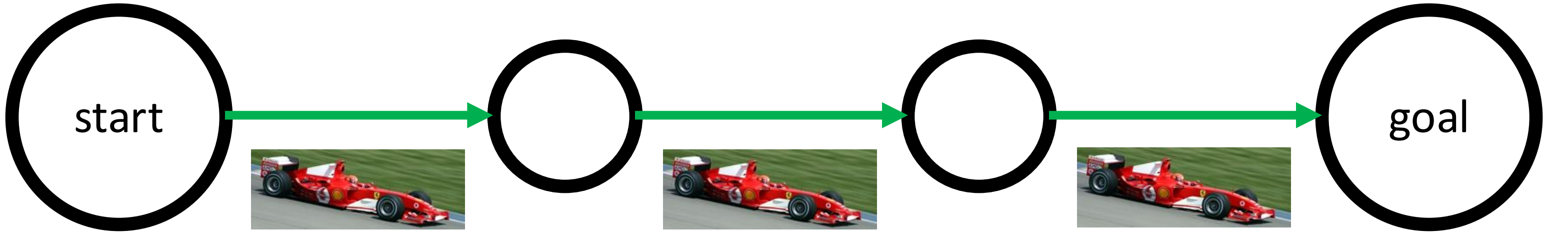
- Make a sequence of decisions to maximize a real-valued reward

- **Examples**
  - Driving a car
  - Making movie recommendations
  - Treating a patient over time
  - Navigating a webpage

Sequential Decision Making

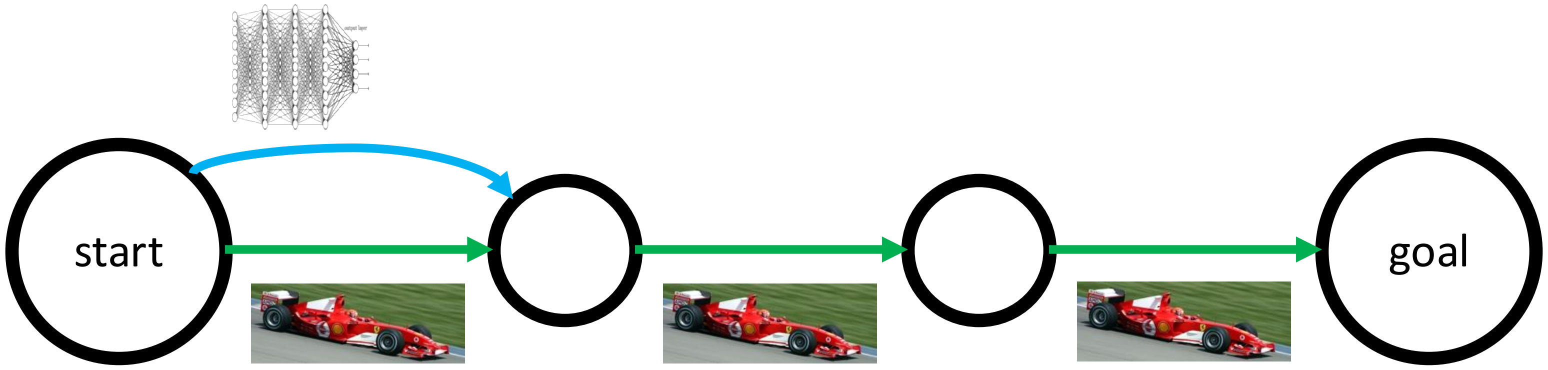
- Machine learning almost always aims to inform decision making
  - Only show user an image if it contains a pet
  - Help a doctor make a treatment decision
- Reinforcement learning is about **sequences** of decisions
- **Naïve strategy:** Predict future and optimize decisions accordingly
  - But decisions affect forecasts
  - If we show the user too many cats, they might get bored of cats!
- **Solution:** Jointly perform prediction and optimization

What makes RL hard?



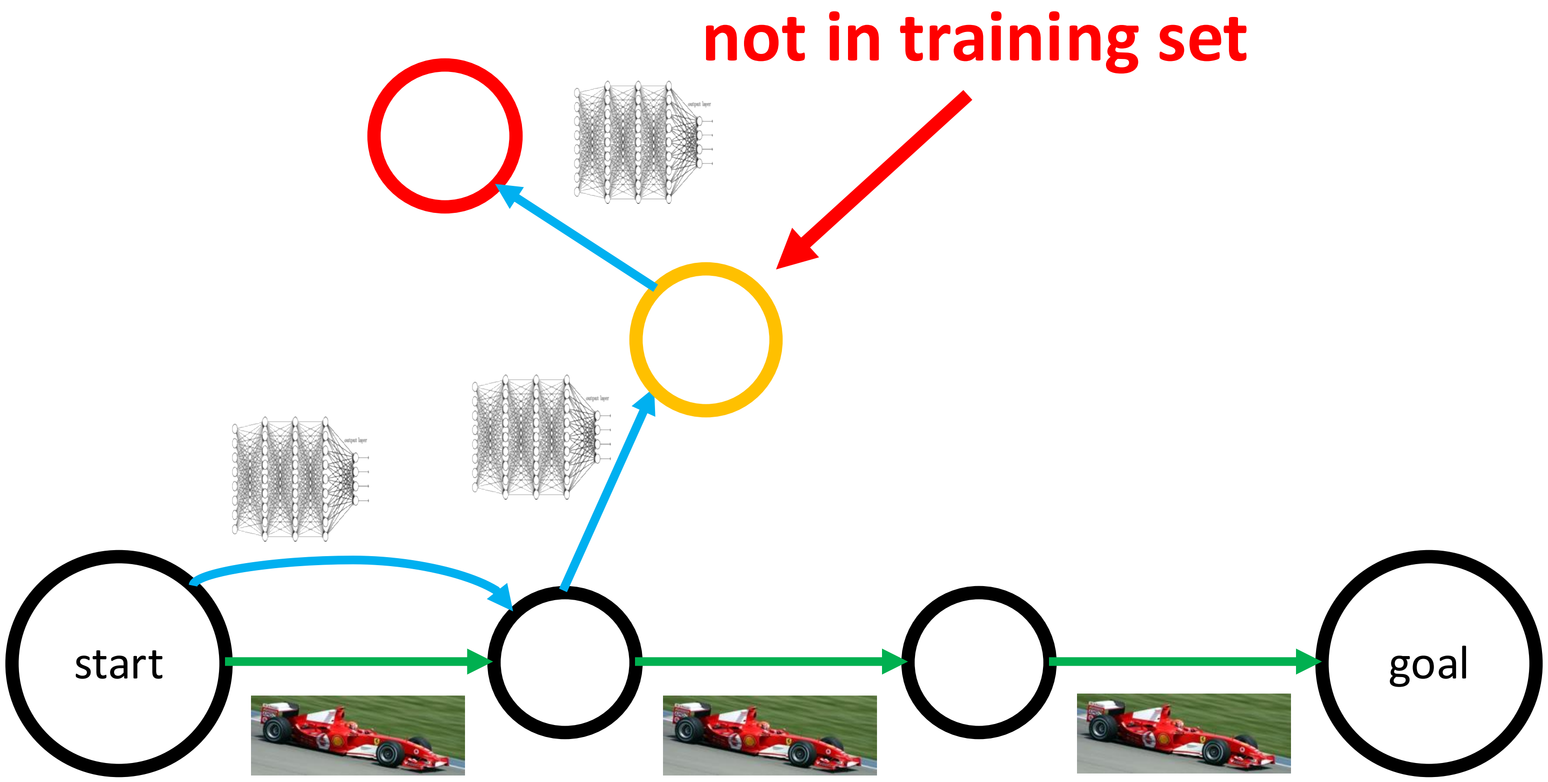
Ross & Bagnell 2011

What makes RL hard?



Ross & Bagnell 2011

What makes RL hard?



Ross & Bagnell 2011

What makes RL hard?

- Distribution shift is **fundamental** to the problem
    - **Repeat:** Improve policy → distribution shifts → improve policy → …
    - This is with a human expert in the loop! Without the expert, we must start off acting randomly

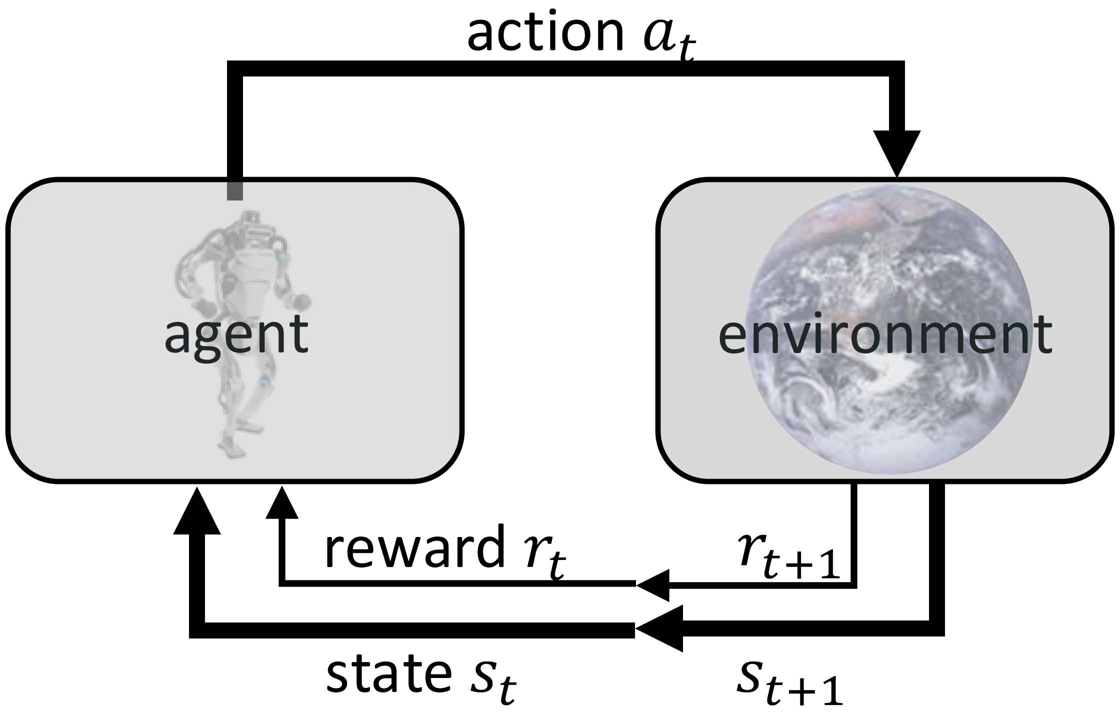
- Generally, using expert data where available is promising (called “imitation learning”)
    - **Caveat:** Limited by human performance (e.g., AlphaGo Zero significantly outperforms AlphaGo, which was pretrained on expert games)

Reinforcement Learning Problem

- **At each step** $t \in \{1, \dots, T\}$**:**
  - Observe **state** $s_t \in S$ and **reward** $r_t \in \mathbb{R}$
  - Take **action** $a_t = \pi(s_t) \in A$

- **Goal:** Learn a **policy** $\pi: S \to A$ that maximizes discounted reward sum:

$$R_T = \sum_{t=1}^{T} \gamma^t \cdot r_t$$



Reinforcement Learning Problem

state: joint angles
actions: motor torques
dynamics: robot physics
reward: average speed

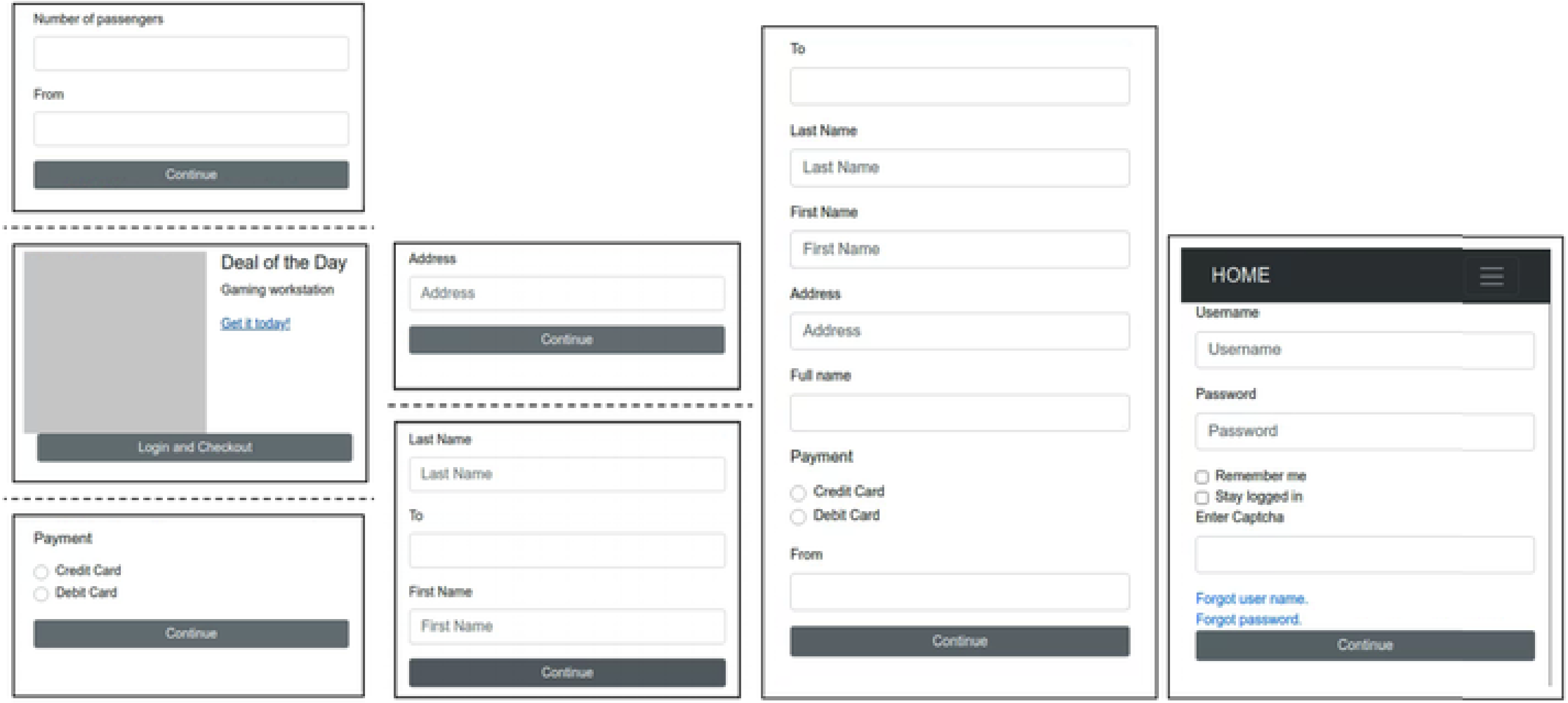
state: current stock
actions: how much to purchase
dynamics: demand at each store
reward: profit

Reinforcement Learning Successes

Image: Google via Getty Images

Playing board games and videogames

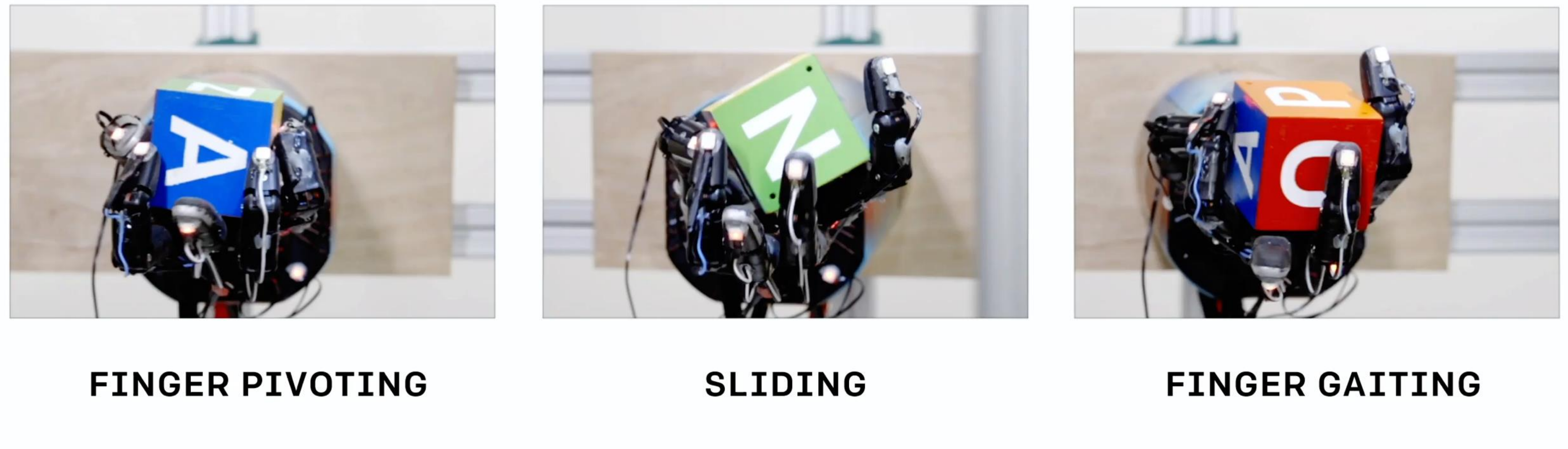
Reinforcement Learning Successes



(a) Early training (b) Mid training (c) Late training (d) Test

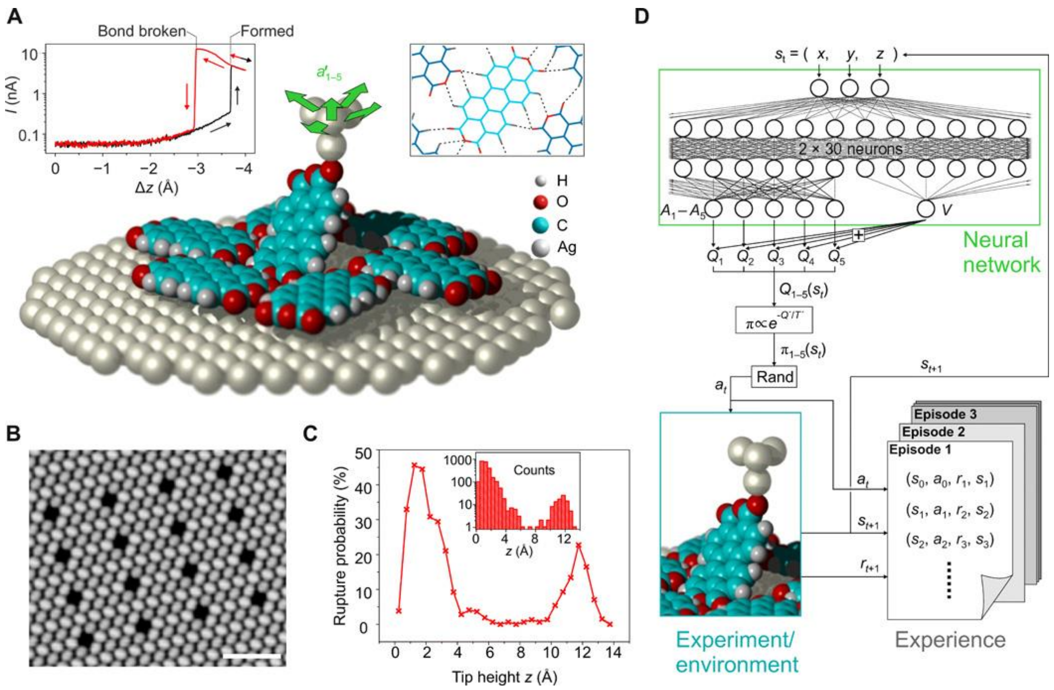
Web navigation (e.g., book a flight)

Reinforcement Learning Successes



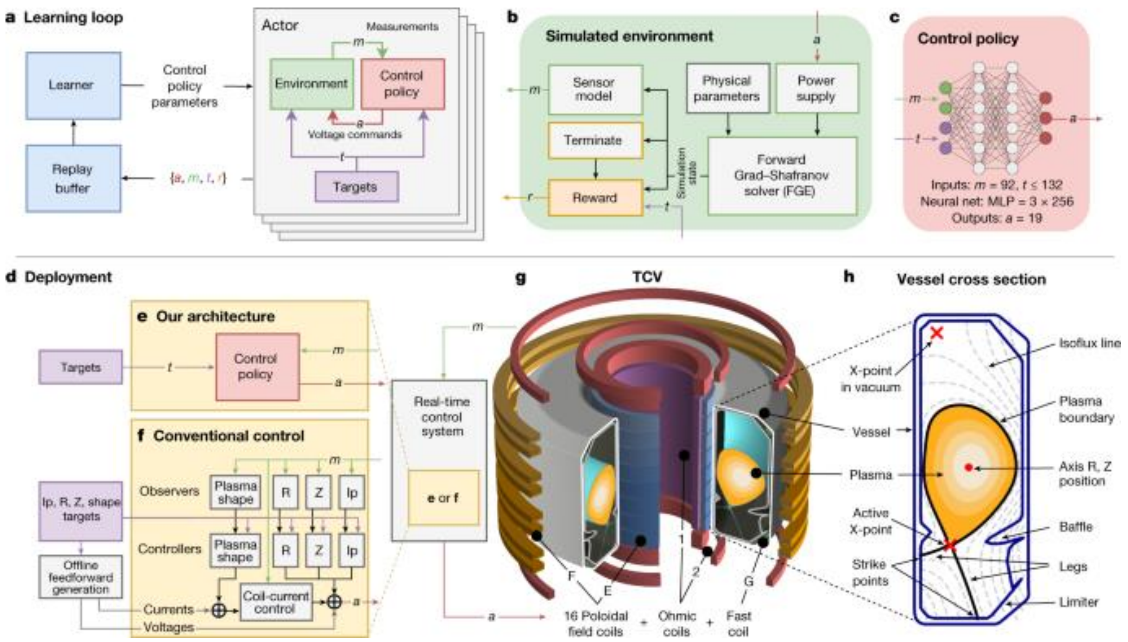
Robotics (e.g., Rubik's cube manipulation)

Reinforcement Learning Successes



Steering microscope to separate molecules

https://www.science.org/doi/10.1126/sciadv.abb6987



Controlling magnetic fields to stabilize plasma (in simulation)

Degrave et al 2022, Magnetic control of tokamak plasmas through deep reinforcement learning

Reinforcement Learning Successes

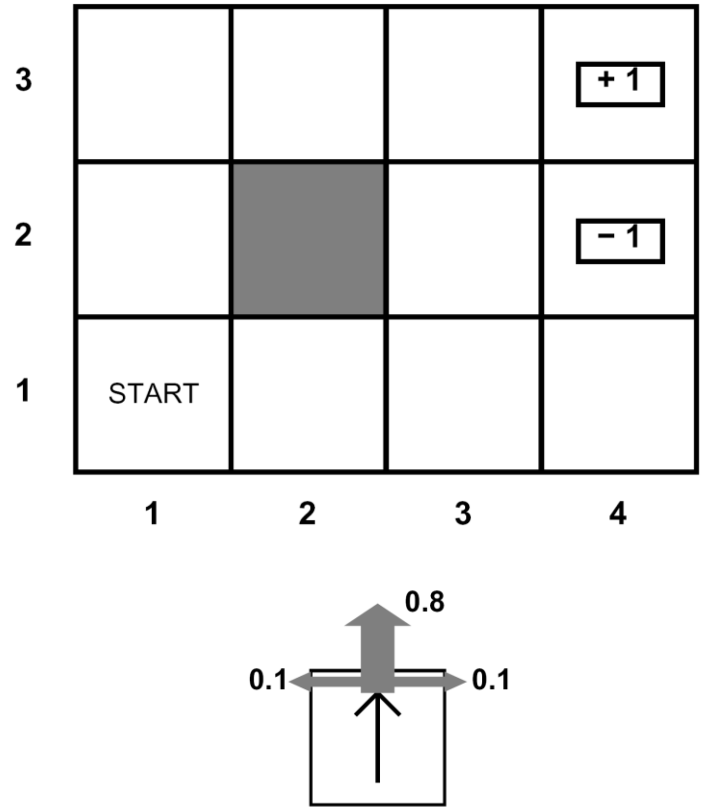
- **Power grids:** Reinforcement learning for demand response
  - A review of algorithms and modeling techniques, J. Vázquez-Canteli, Z. Nagy
- **Recommender systems**
  - https://github.com/google-research/recsim
- **Many potential applications**
  - https://arxiv.org/abs/1904.12901

Reinforcement Learning Problem

- At a high level, we need to specify the following:
  - **State space:** What are the observations the agent may encounter?
  - **Action space:** What are the actions the agent can take?
  - **Transitions/dynamics:** How the state is updated when taking an action
  - **Rewards:** What rewards the agent receives for taking an action in a state
- For most of today, assume state and action spaces are finite

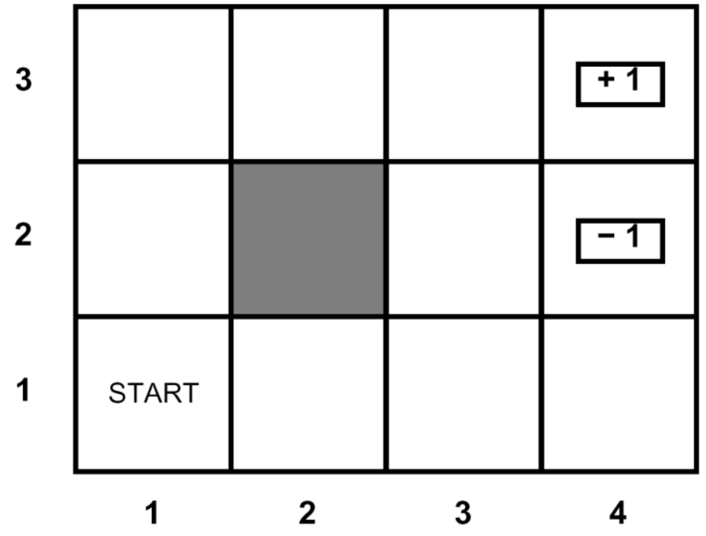
Toy Example

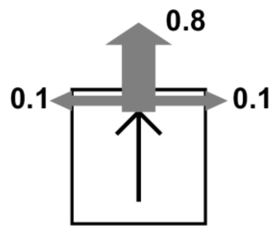
- Grid map with solid/open cells
- **State:** An open grid cell
- **Actions:** Move North, East, South, West



Toy Example

- **Dynamics**
  - Move in chosen direction, but not deterministically!
  - Succeeds 80% of the time
  - 10% of the time, end up $90°$ off
  - 10% of the time, end up $-90°$ off
  - The agent stays put if it tries to move into a solid cell or outside the world
  - At terminal states, any action ends **episode** (or **rollout**)

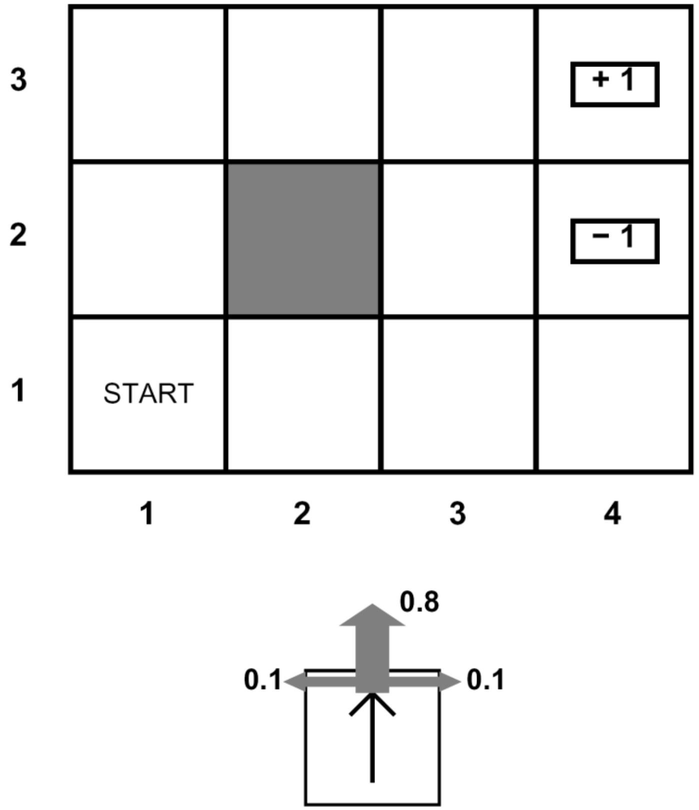




Based on slide by Dan Klein

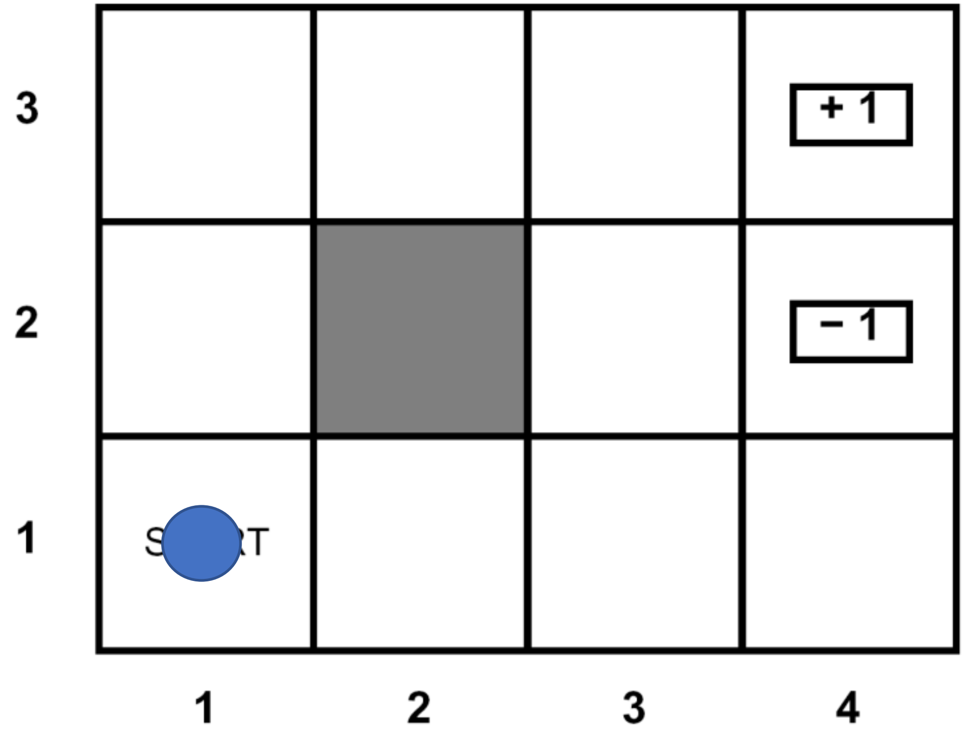
Toy Example

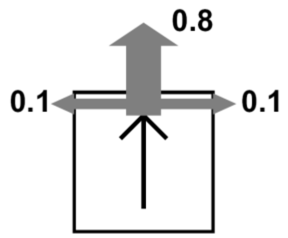
- **Rewards**
  - At terminal state, agent receives the specified reward
  - For each timestep outside terminal states , the agent pays a small cost, e.g., a “reward” of -0.03



Based on slide by Dan Klein

Example Episode (Random Policy)

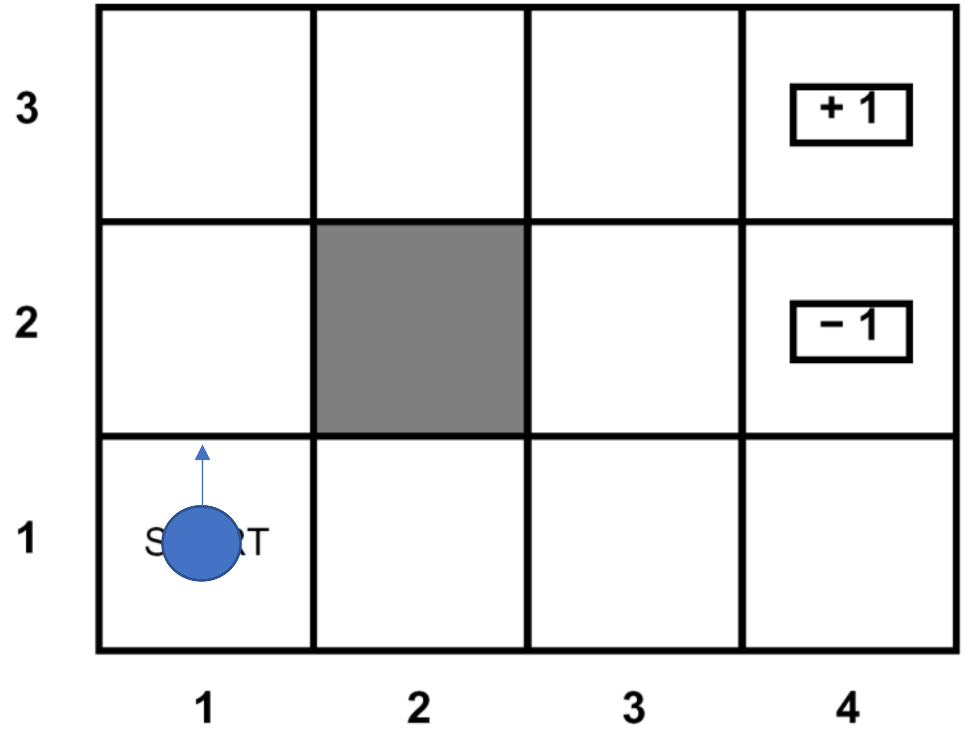


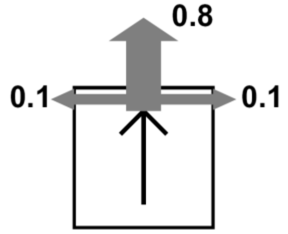


Based on slide by Dan Klein

Example Episode (Random Policy)

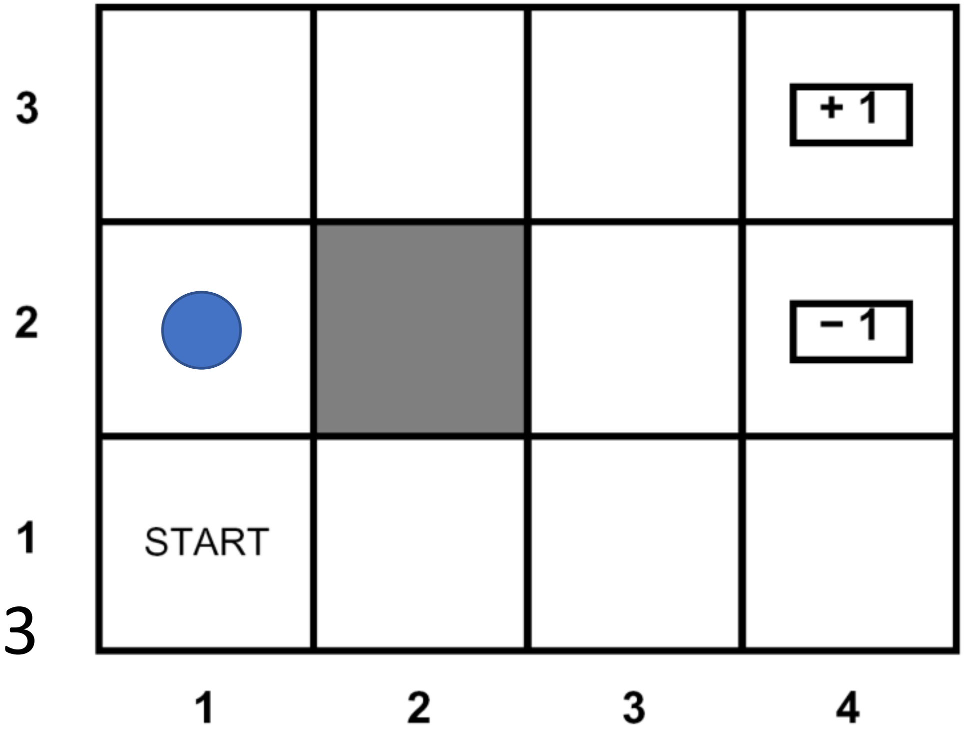
Action= "N"





Based on slide by Dan Klein

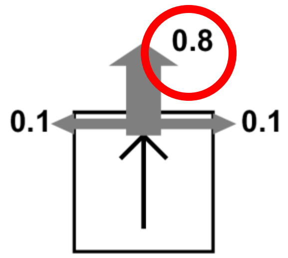
Example Episode (Random Policy)



Action= “N”

Result = “N”

Reward = -0.03



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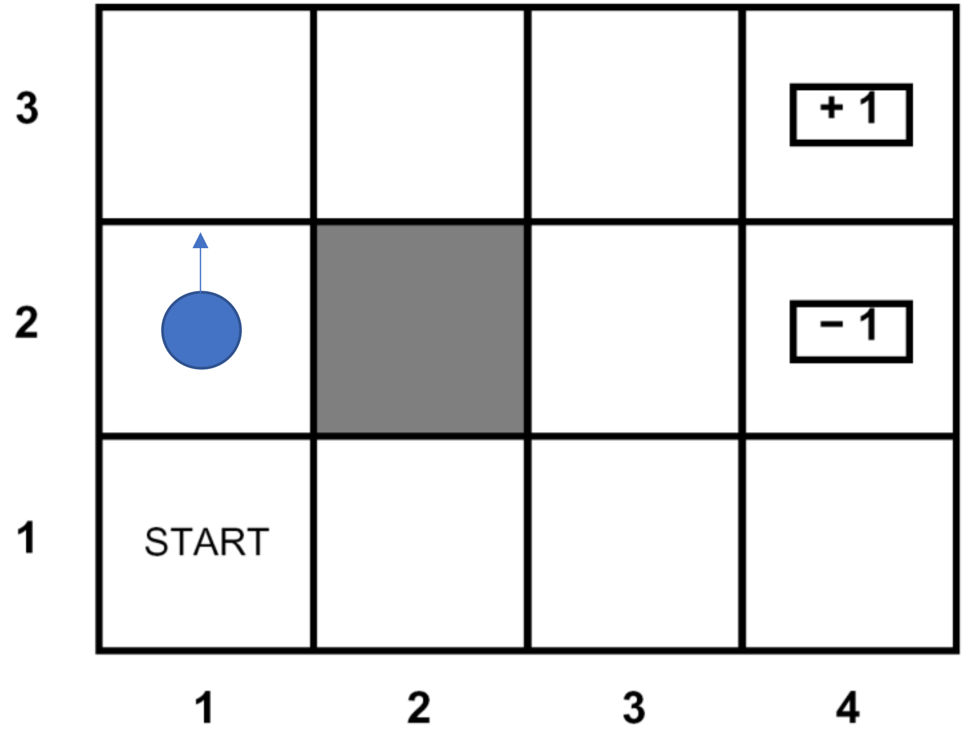
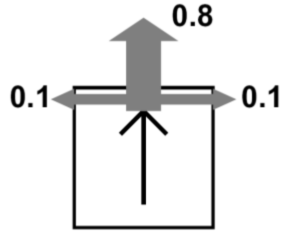
Example Episode (Random Policy)

Action= “N”



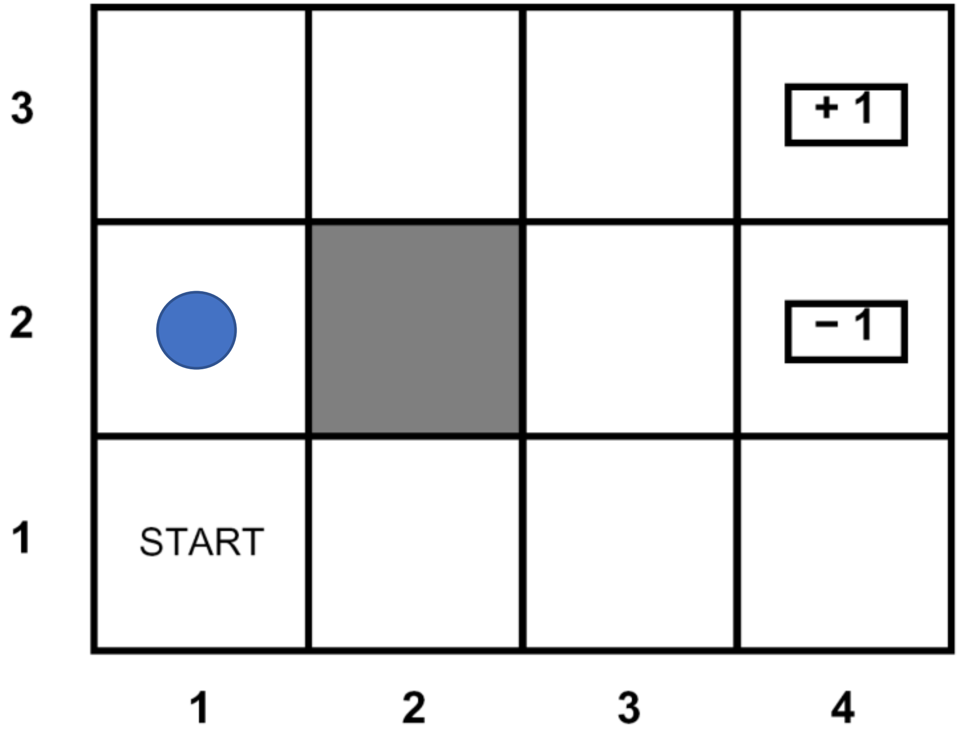


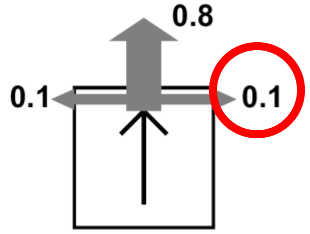
Based on slide by Dan Klein

Example Episode (Random Policy)

(stays still because blocked)

Action= “N”
Result=“E”
Reward = -0.03





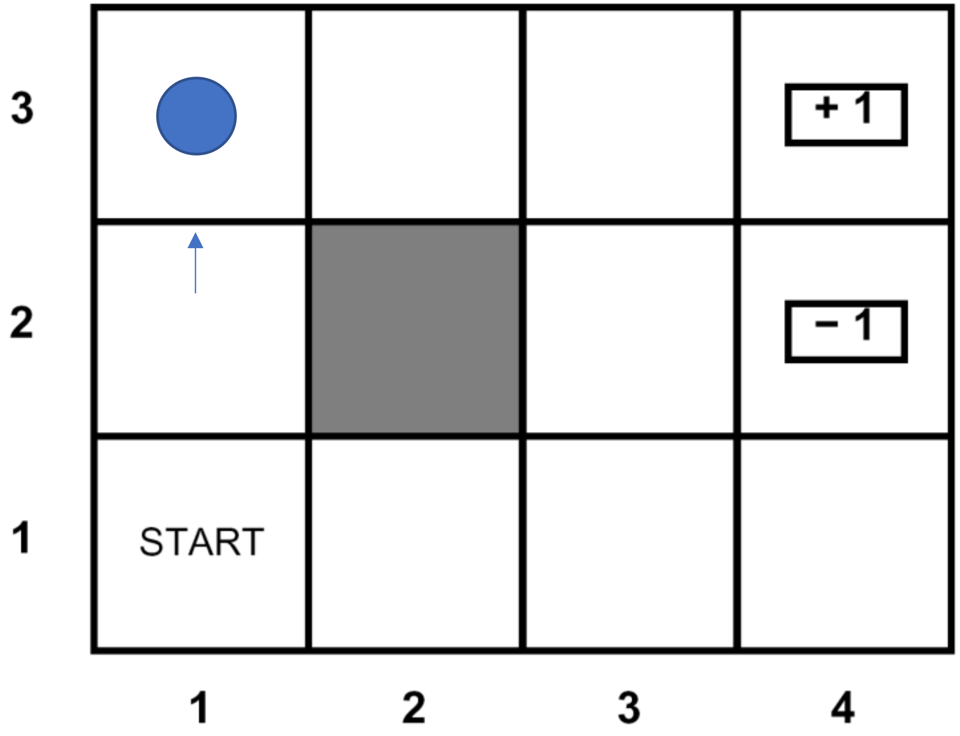
Based on slide by Dan Klein

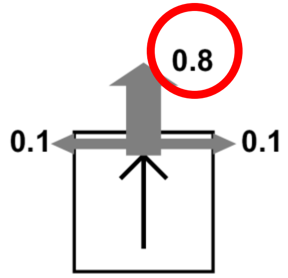
Example Episode (Random Policy)

Action= “N”

Result=“N”

Reward = -0.03





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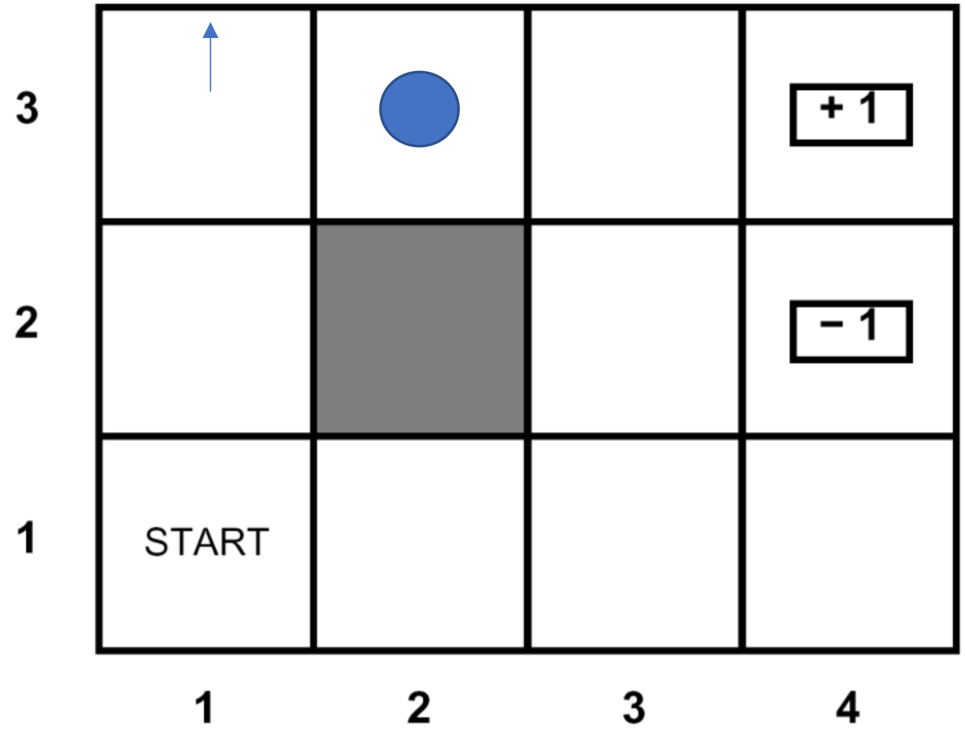
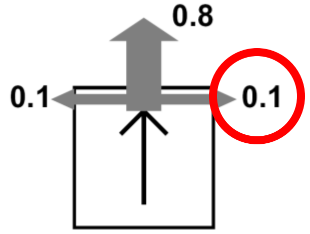
Example Episode (Random Policy)

Action= “N”

Result=“E”

Reward = -0.03





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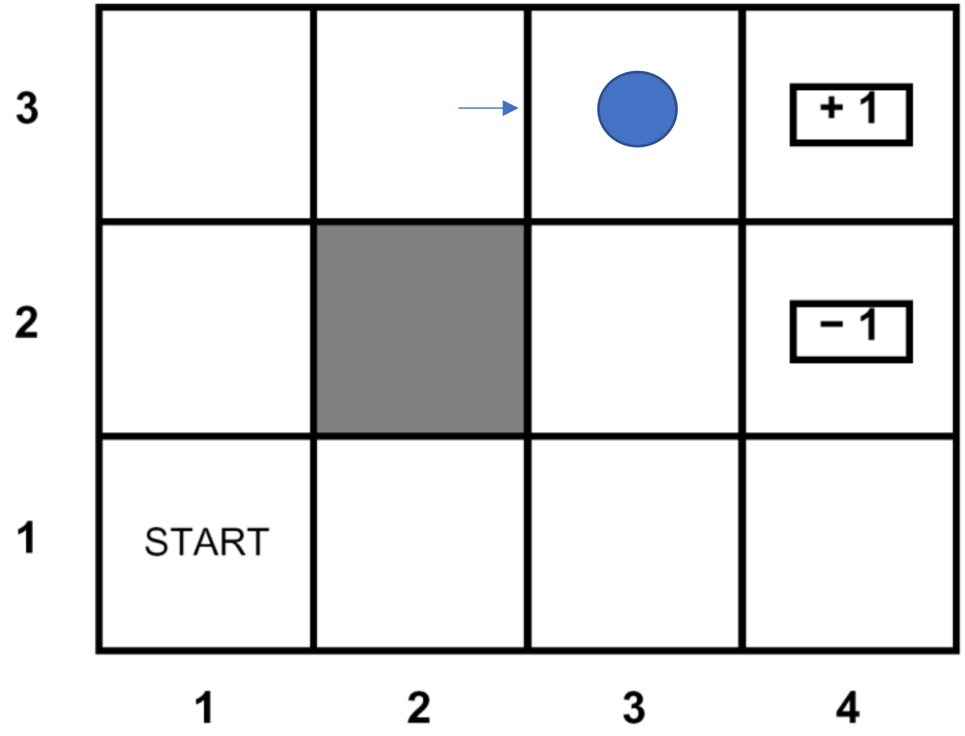
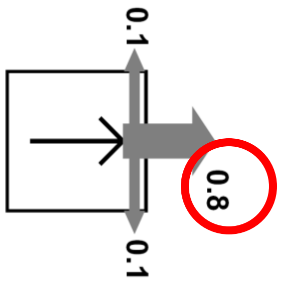
Example Episode (Random Policy)

Action= “E”

Result=“E”

Reward = -0.03





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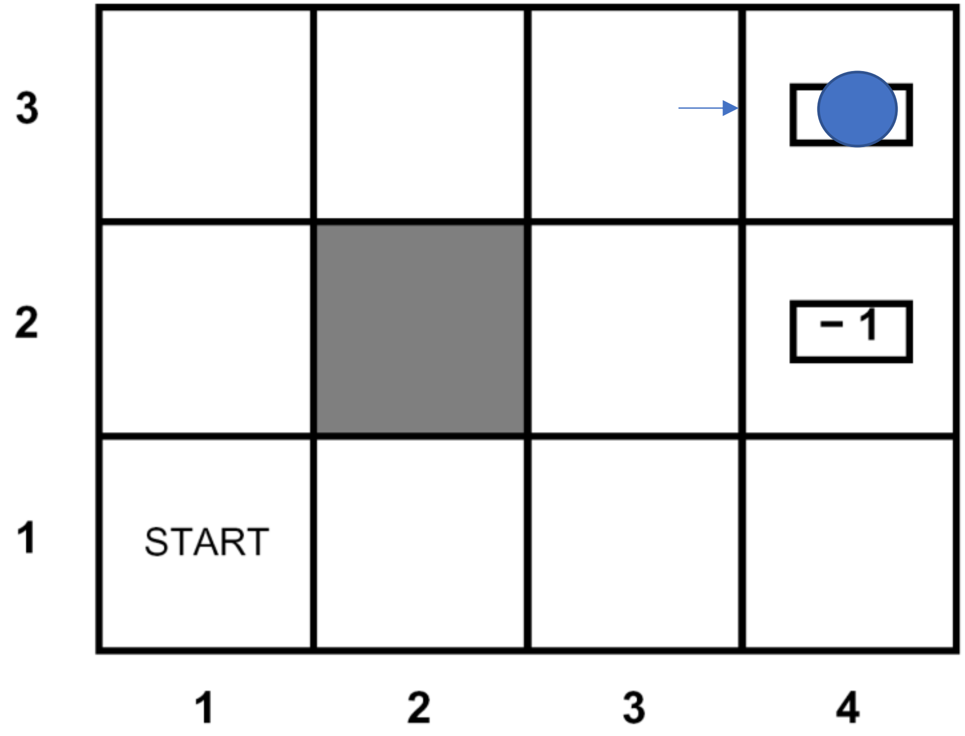
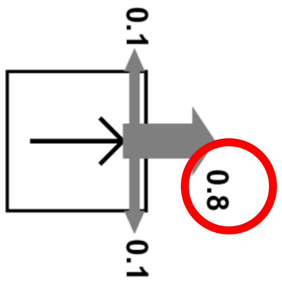
Example Episode (Random Policy)

Action= “E”

Result=“E”

Reward = -0.03





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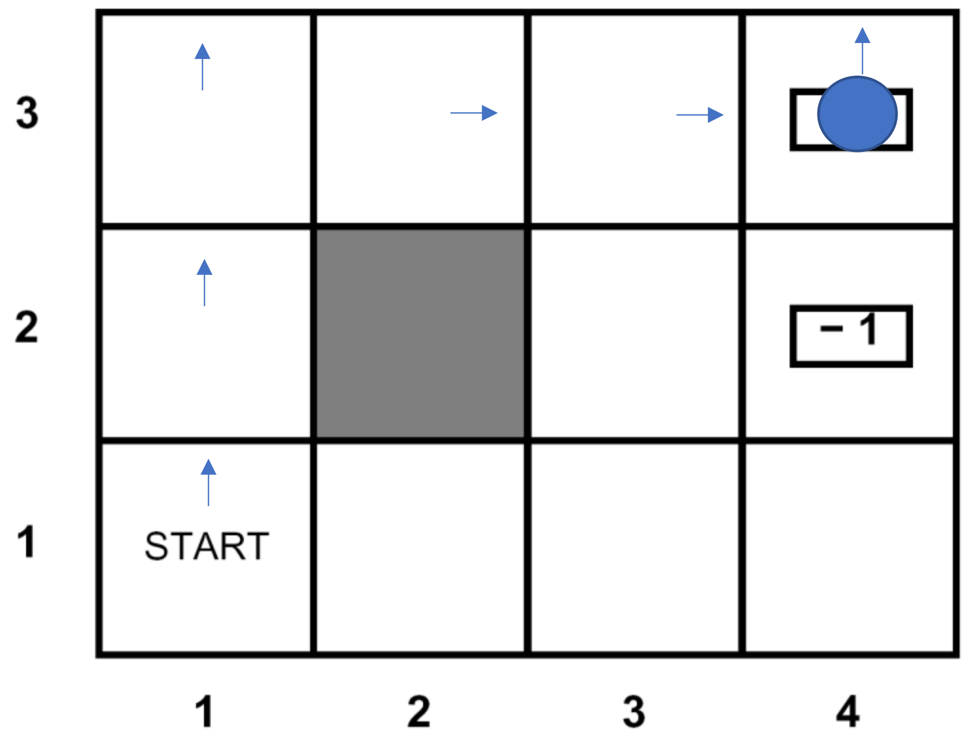
Example Episode (Random Policy)

Action= “N”

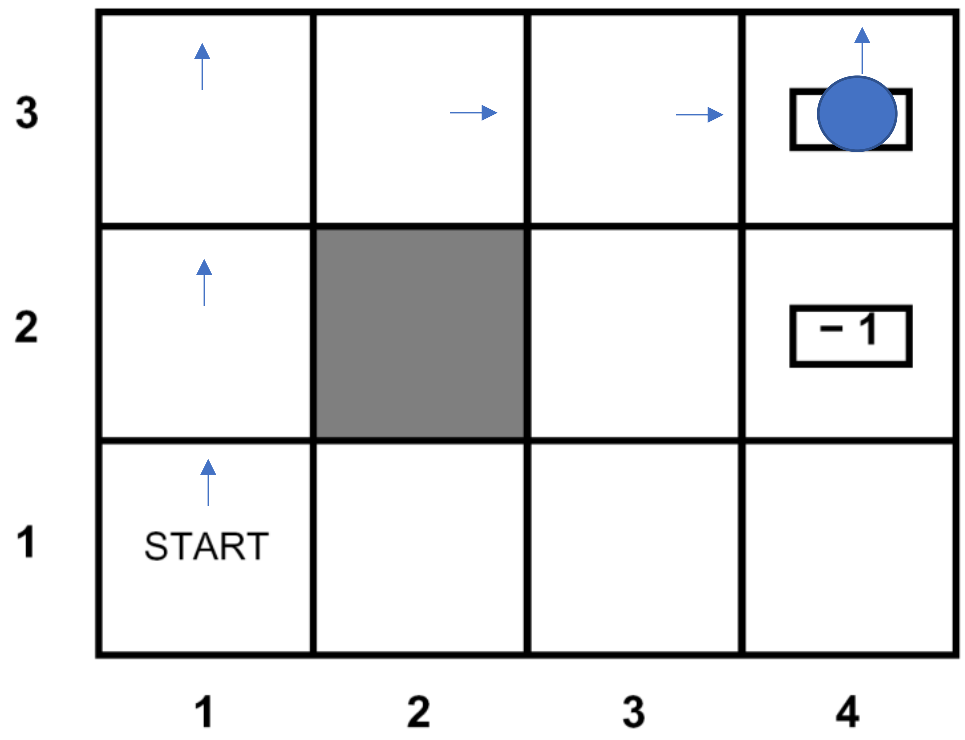
Result=“**<u>the end</u>**”

Reward = +1



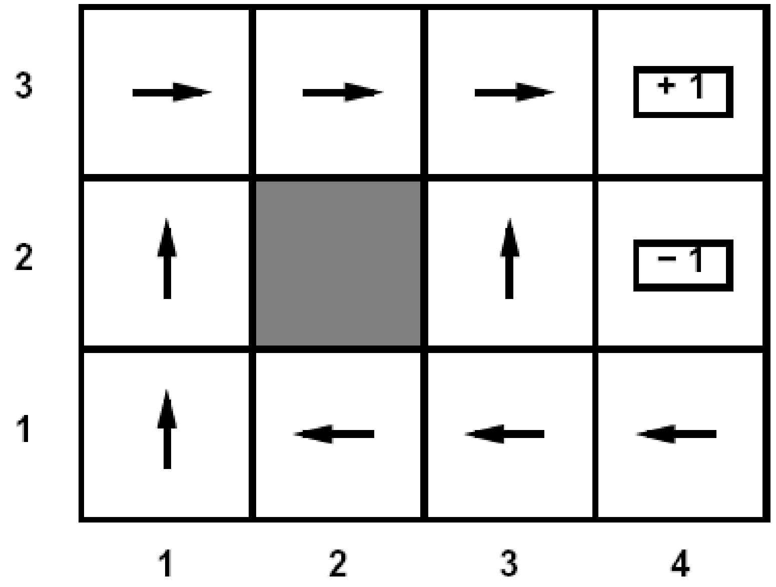
Example Episode (Random Policy)

- Our random trajectory happened to end in the right place!
- Optimal policy? **No!**
  - Only succeeded by random chance



Optimal Policy

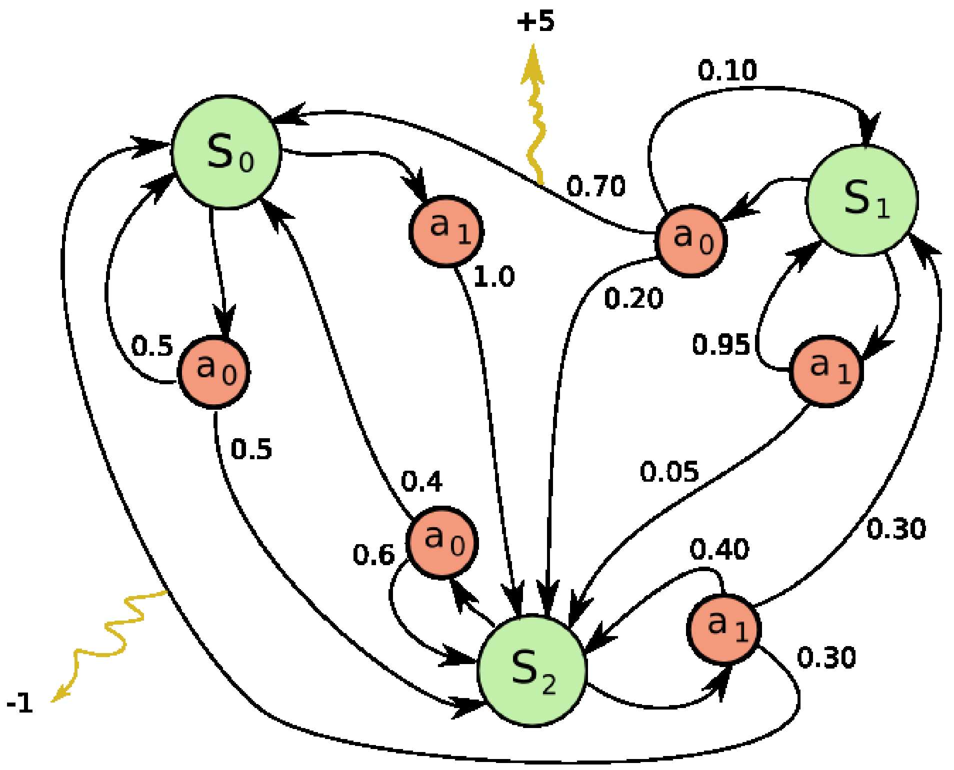
- **Optimal policy:** Following π^* maximizes total reward received
  - **Discounted:** Future rewards are downweighted
  - **In expectation:** On average across randomness of environment and actions



Based on slide by Dan Klein

Markov Decision Process (MDP)

- An MDP (S, A, P, R, γ) is defined by:
  - Set of states $s \in S$
  - Set of actions $a \in \ A$
  - Transition function $P(\,s' \mid s, a\,)$ (also called "dynamics" or the "model")
  - Reward function $R(s, a, s')$
  - Discount factor $\gamma < 1$
- Also assume an initial state distribution $D(s)$
  - Often omitted since optimal policy does not depend on D



[Image: https://towardsdatascience.com/reinforcement-learning-demystified-markov-decision-processes-part-1-bf00dda41690](https://towardsdatascience.com/reinforcement-learning-demystified-markov-decision-processes-part-1-bf00dda41690)

Markov Decision Process (MDP)

- **Goal:** Maximize **cumulative expected discounted reward:**

$$\pi^* = \max_{\pi} J(\pi) \quad \text{where} \quad J(\pi) = \mathbb{E}_{\zeta}\left[\sum_{t=0}^{\infty} \gamma^t \cdot r_t\right]$$

- Expectation over **episodes** $\zeta = (s_0, a_0, r_0, s_1, \dots)$, where
  - $s_0 \sim D$
  - $a_t = \pi(s_t)$
  - $s_{t+1} \sim P(\cdot \mid s_t, a_t)$
  - $r_t = R(s_t, a_t, s_{t+1})$

Markov Decision Process (MDP)

- **Planning:** Given P and R, compute the optimal policy π^*
  - Purely an optimization problem! No learning

- **Reinforcement learning:** Compute the optimal policy π^* without prior knowledge of P and R

Policy Value Function

- **Policy Value Function:** Expected reward if we start in s and use π:

$$V^{\pi}(s) = \mathbb{E}\left(\sum_{t=0}^{\infty} \gamma^{t} \cdot r_t \mid s_0 = s\right)$$

- **Bellman equation:**

$$\underbrace{V^{\pi}(s)}_{\text{current value}} = \sum_{s' \in S} \underbrace{P(\, s' \mid s, \pi(s)\,)}_{\substack{\text{expectation} \\ \text{over next state}}} \cdot \underbrace{\left(R(s, \pi(s), s') + \gamma \cdot V^{\pi}(s')\right)}_{\substack{\text{current reward +} \\ \text{discounted future reward}}}$$

Optimal Value Function

- **Optimal value function:** Expected reward if we start in s and use π^*:

$$V^*(s) = \mathbb{E}\left(\sum_{t=0}^{\infty} \gamma^t \cdot r_t \mid s_0 = s\right)$$

- **Bellman equation:**

Optimal policy selects action that maximizes future expected reward from state s

$$\underbrace{V^*(s)}_{\text{current value}} = \max_{a \in A} \sum_{s' \in S} \underbrace{P(s' \mid s, a)}_{\substack{\text{expectation} \\ \text{over next state}}} \cdot \underbrace{\left(R(s, a, s') + \gamma \cdot V^*(s')\right)}_{\substack{\text{current reward +} \\ \text{discounted future reward}}}$$

Optimal Value Function

- **Bellman equation:**

$$V^*(s) = \max_{a \in A} \sum_{s' \in S} P(\, s' \mid s, a\,) \cdot \left(R(s, a, s') + \gamma \cdot V^*(s')\right)$$

- Do not need to know the optimal policy π^*!

- **Strategy:** Compute V^* and then use it to compute π^*
  - **Caveat:** Latter step requires knowing P

Policy Action-Value Function

- **Policy Action-Value Function (or Q function):** Expected reward if we start in s, take action a, and then use π thereafter:

$$Q^{\pi}(s,a) = \mathbb{E}\left(\sum_{t=0}^{\infty} \gamma^{t} \cdot r_t \mid s_0 = s, a_0 = a\right)$$

- **Bellman equation:**

$$Q^{\pi}(s,a) = \sum_{s' \in S} P(s' \mid s, a) \cdot \left(R(s,a,s') + \gamma \cdot Q^{\pi}\big(s', \pi(s')\big)\right)$$

Optimal Action-Value Function

- **Optimal Action-Value Function (or Q function):** Expected reward if we start in s, take action a, and then act optimally thereafter:

$$Q^*(s,a) = \mathbb{E}\left(\sum_{t=0}^{\infty} \gamma^t \cdot r_t \mid s_0 = s, a_0 = a\right)$$

- **Bellman equation:**

$$Q^*(s,a) = \sum_{s' \in S} P(s' \mid s,a) \cdot \left(R(s,a,s') + \gamma \cdot \max_{a' \in A} Q^*(s',a')\right)$$

Relationship

- We have

$$V^{\pi}(s) = Q^{\pi}\big(s, \pi(s)\big)$$

- Similarly, we have

$$V^{*}(s) = \max_{a} Q^{*}(s, a)$$

Q Iteration

- We have

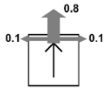
$$\pi^*(s) = \max_{a \in A} Q^*(s, a)$$

- **Strategy:** Compute Q^* and then use it to compute π^*

Q Iteration

- Initialize $Q_1(s, a) \leftarrow 0$ for all s, a
- For $i \in \{1,2, \dots\}$ until convergence:

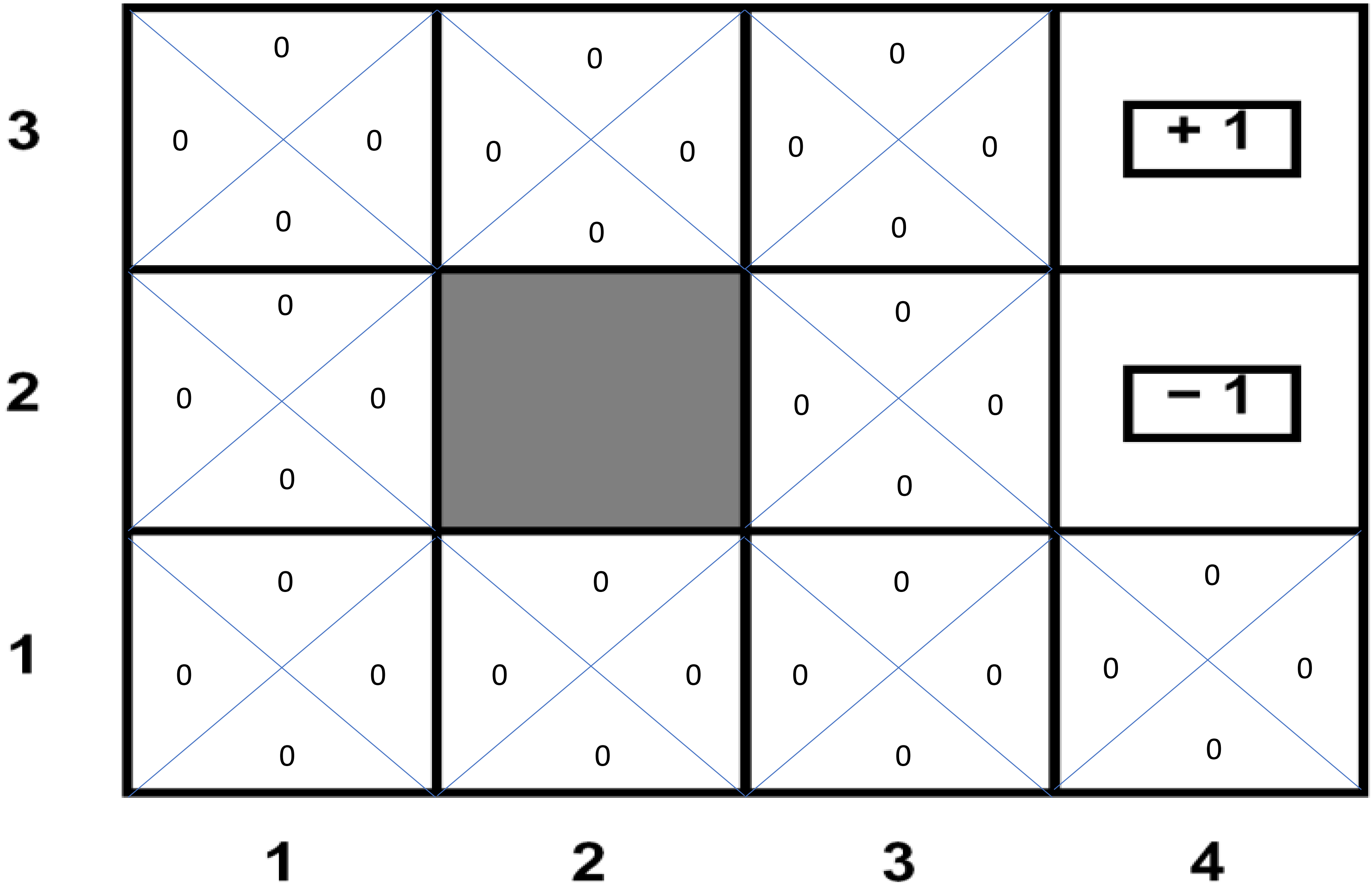
$$Q_{i+1}(s, a) \leftarrow \sum_{s' \in S} P(s' \mid s, a) \cdot \left(R(s, a, s') + \gamma \cdot \max_{a' \in A} Q_i(s', a')\right)$$

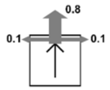


Living cost 0

0.9

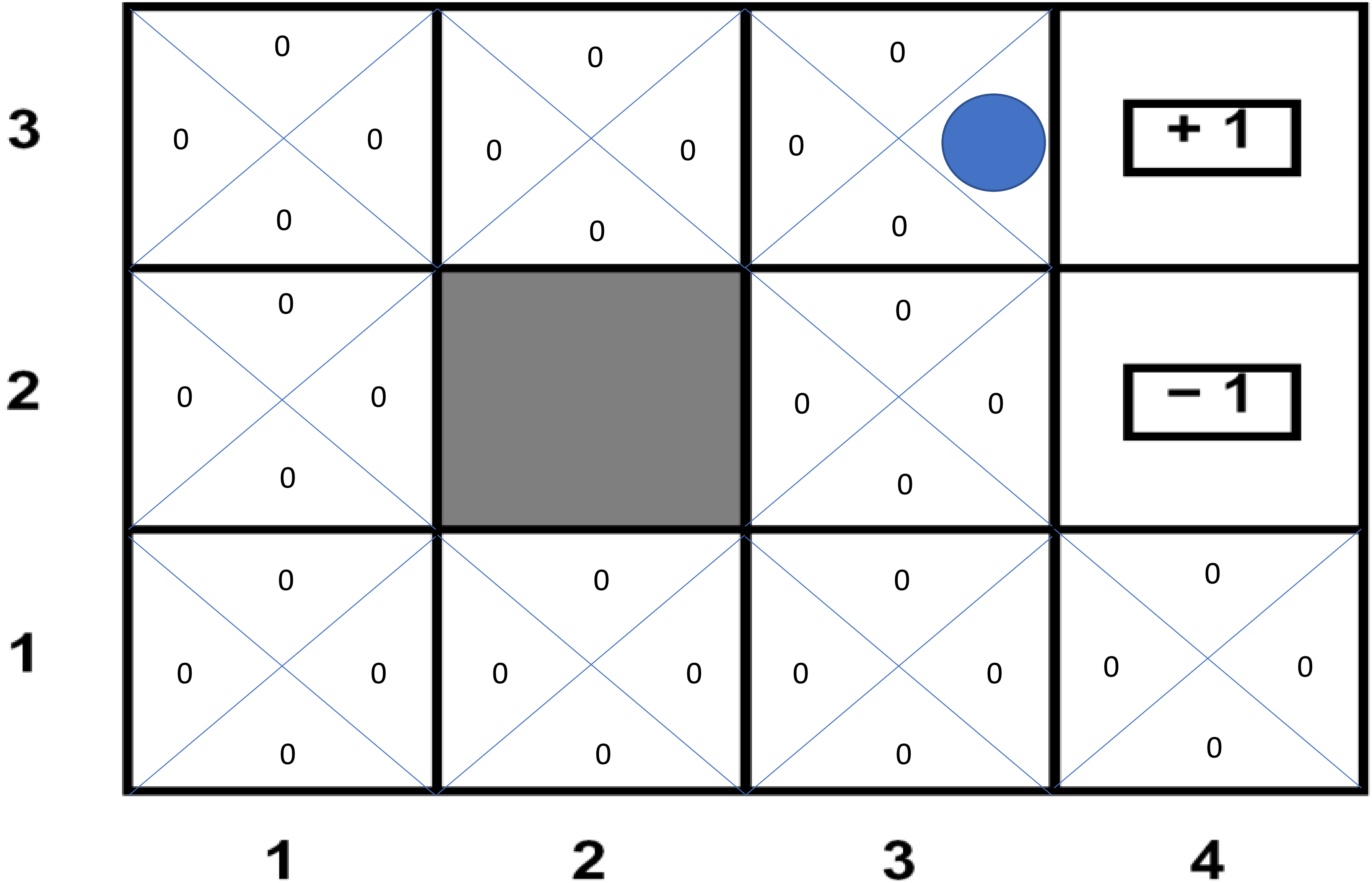
$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[R(s,a,s') + \gamma \boxed{\max_{a'} Q_i(s',a')}\right]$$



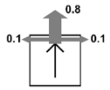


0 0.9

$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[R(s,a,s') + \gamma \boxed{\max_{a'} Q_i(s',a')}\right]$$



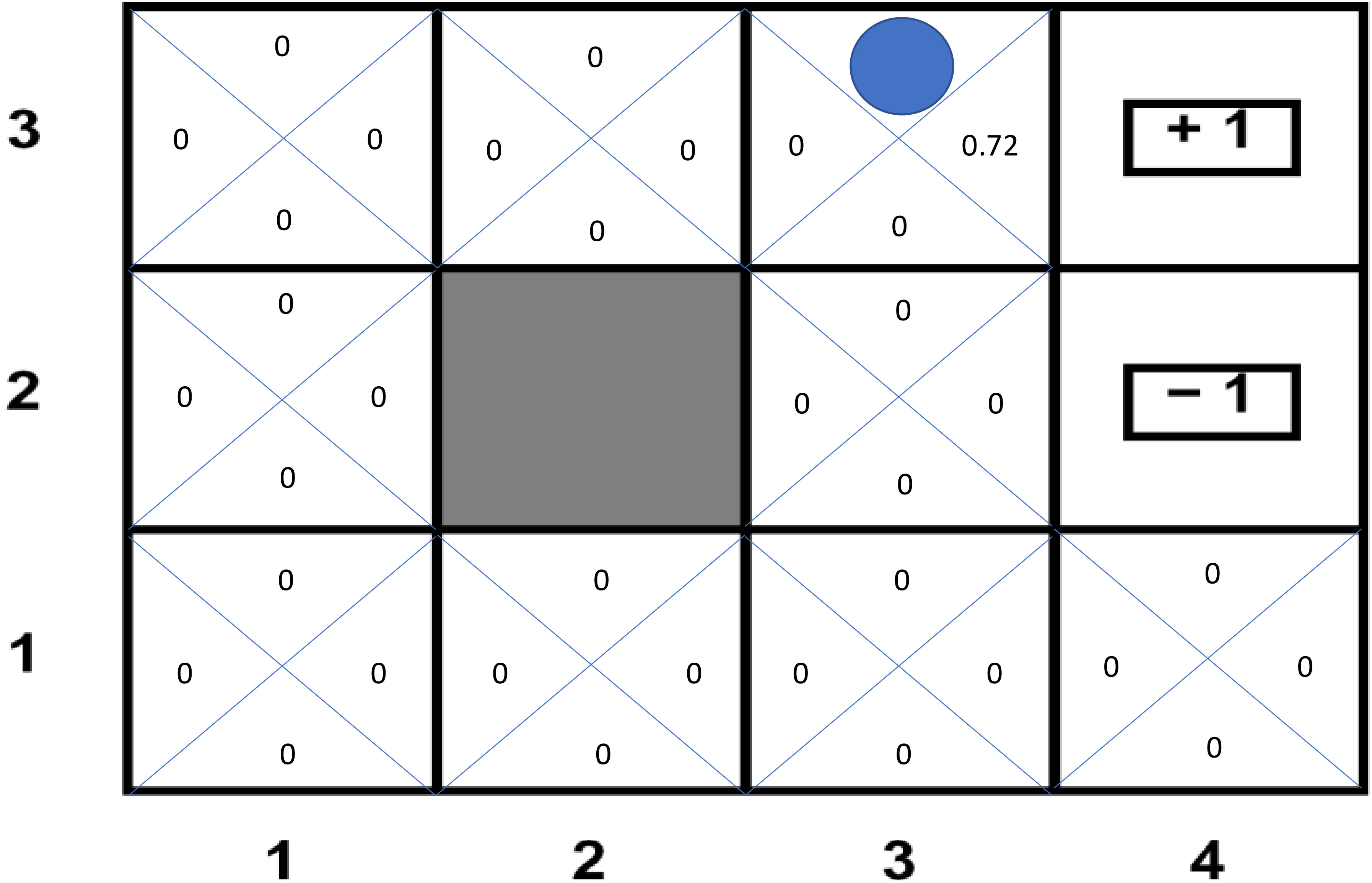
0.8x[0+0.9x1]
+ 0.1x[0 + 0]
+0.1x[0+0]
=0.72



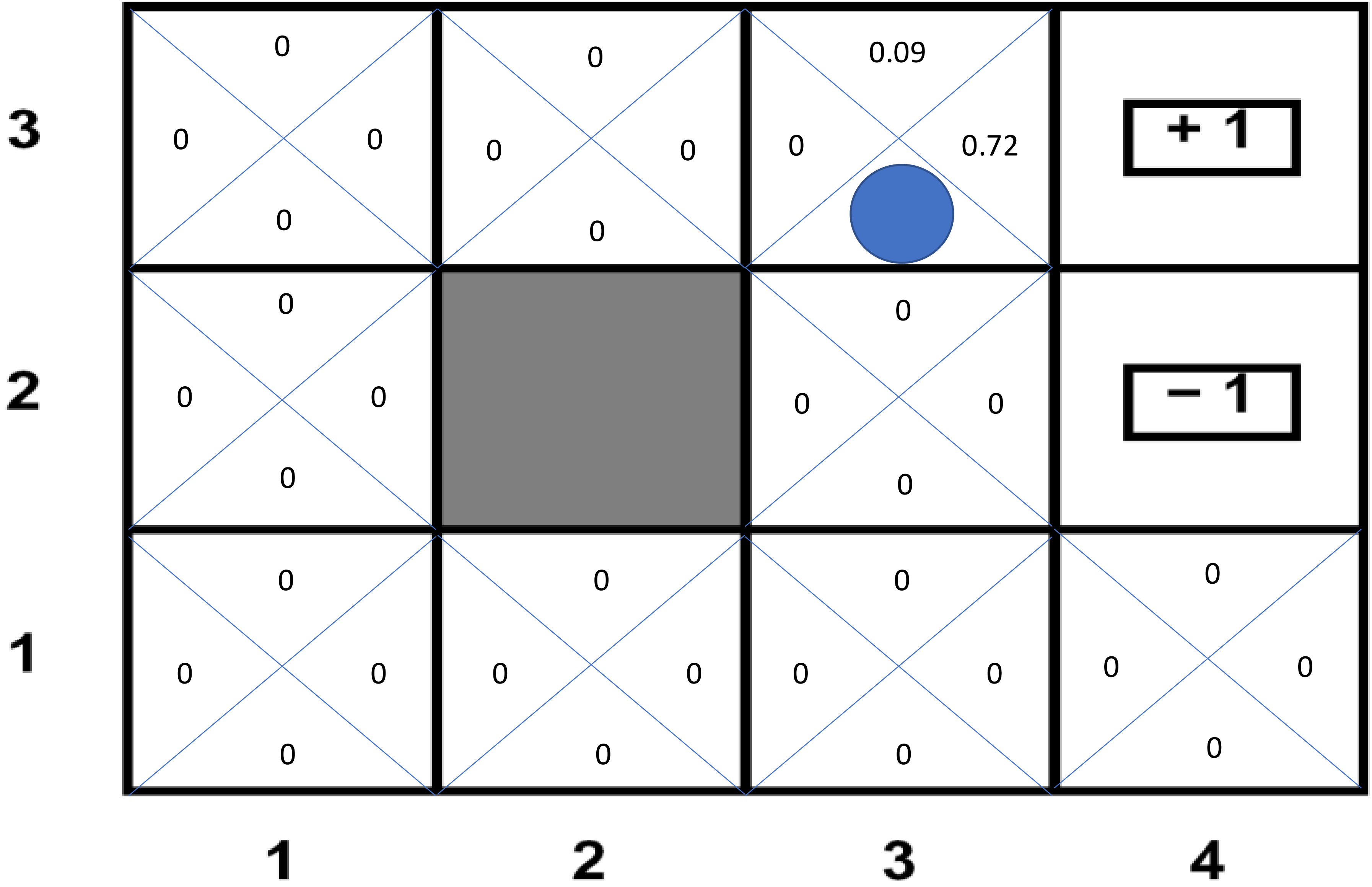
0   0.9

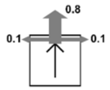
$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[R(s,a,s') + \gamma \boxed{\max_{a'} Q_i(s',a')}\right]$$

0.8x[0+0]
+ 0.1x[0+0.9x1]
+0.1x[0+0]
=0.09

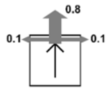


$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[\overbrace{R(s,a,s')}^{0} + \gamma \overbrace{\boxed{\max_{a'} Q_i(s',a')}}^{0.9}\right]$$

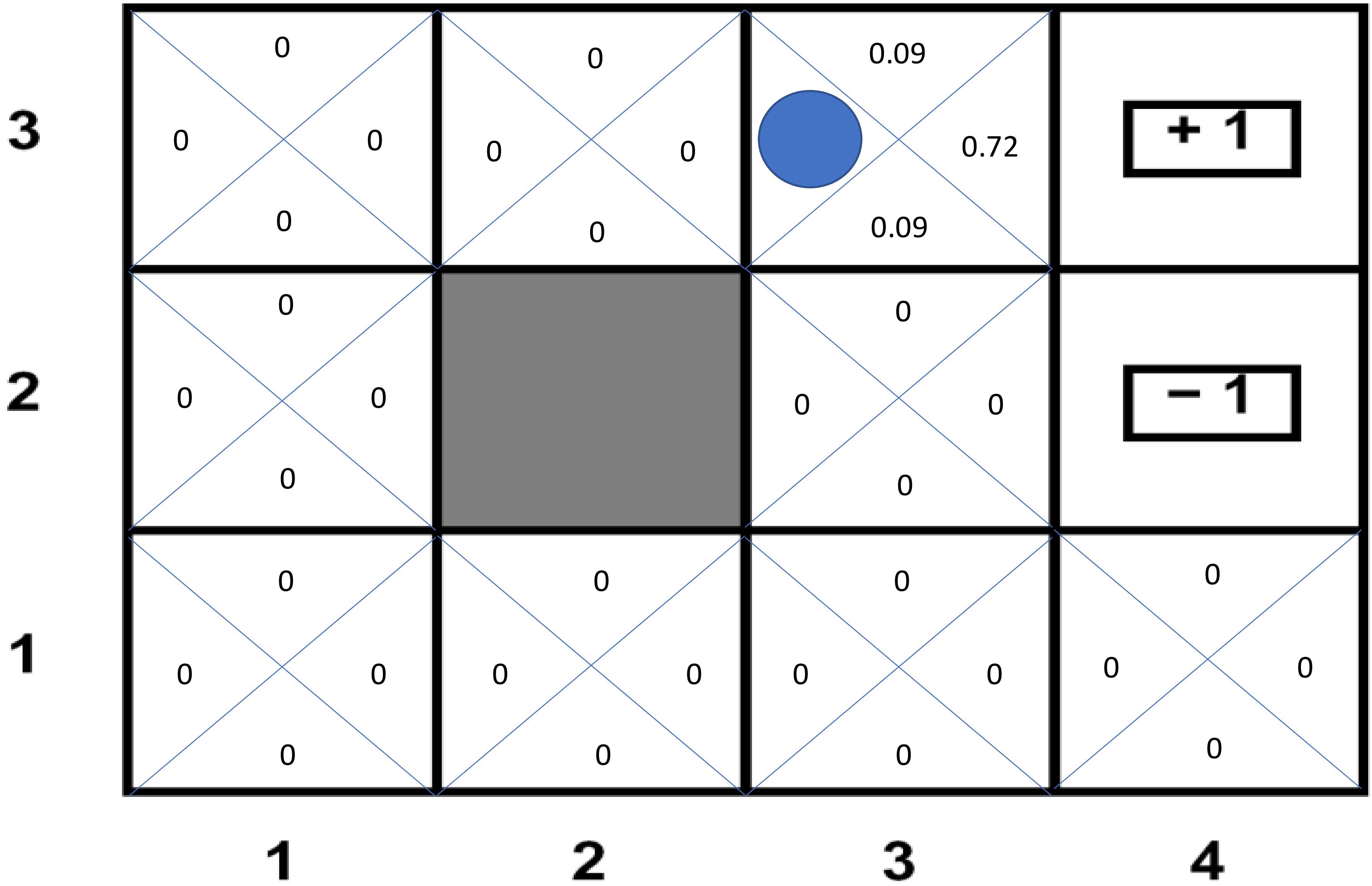




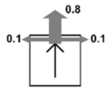
0.8x[0+0]
+ 0.1x[0+0.9x1]
+0.1x[0+0]
=0.09



$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[\overset{0}{R(s,a,s')} + \gamma \overset{0.9}{\boxed{\max_{a'} Q_i(s',a')}}\right]$$

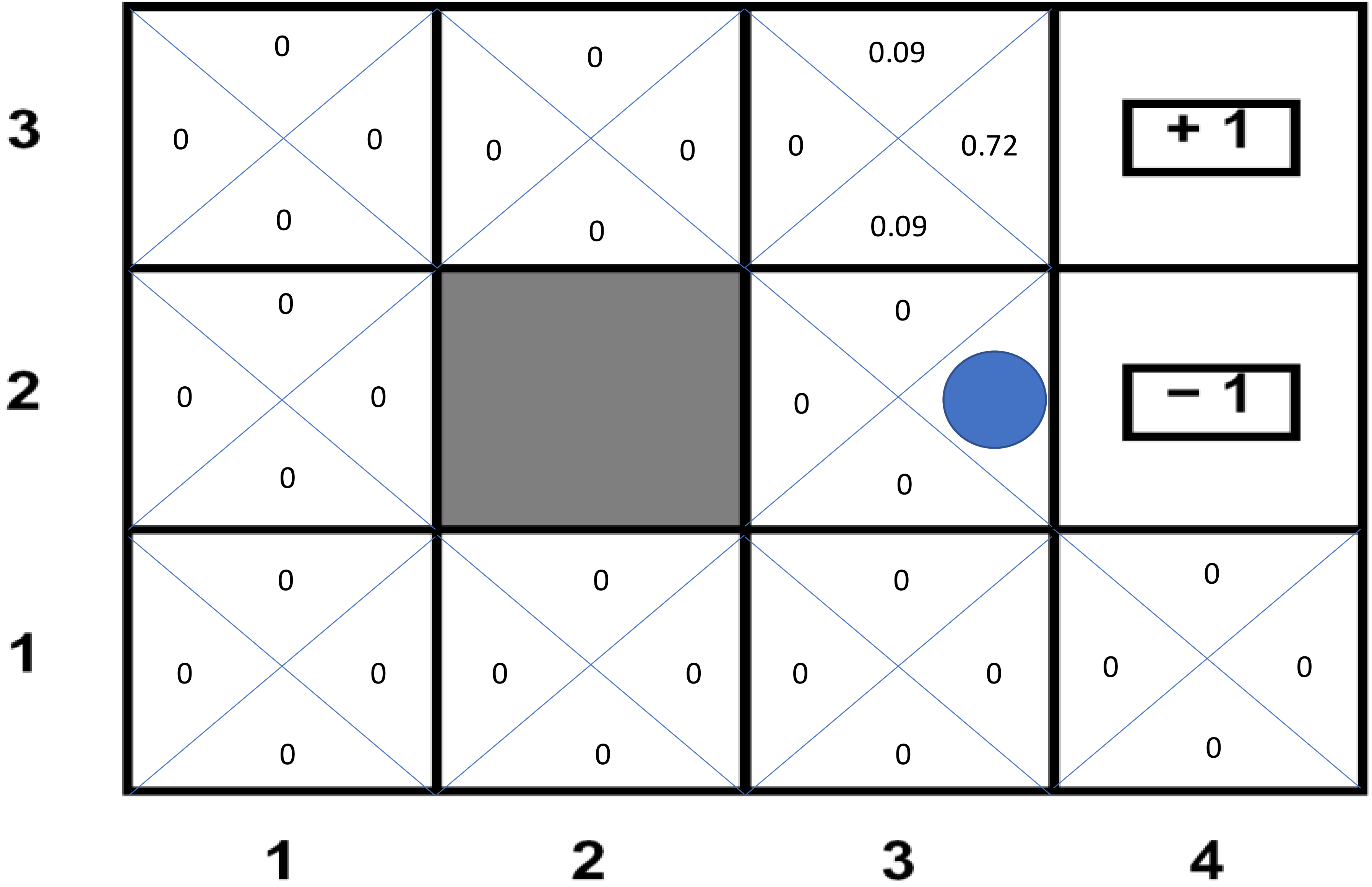


0.8x[0+0]
+ 0.1x[0+0]
+0.1x[0+0]
=0

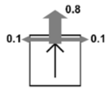


0   0.9

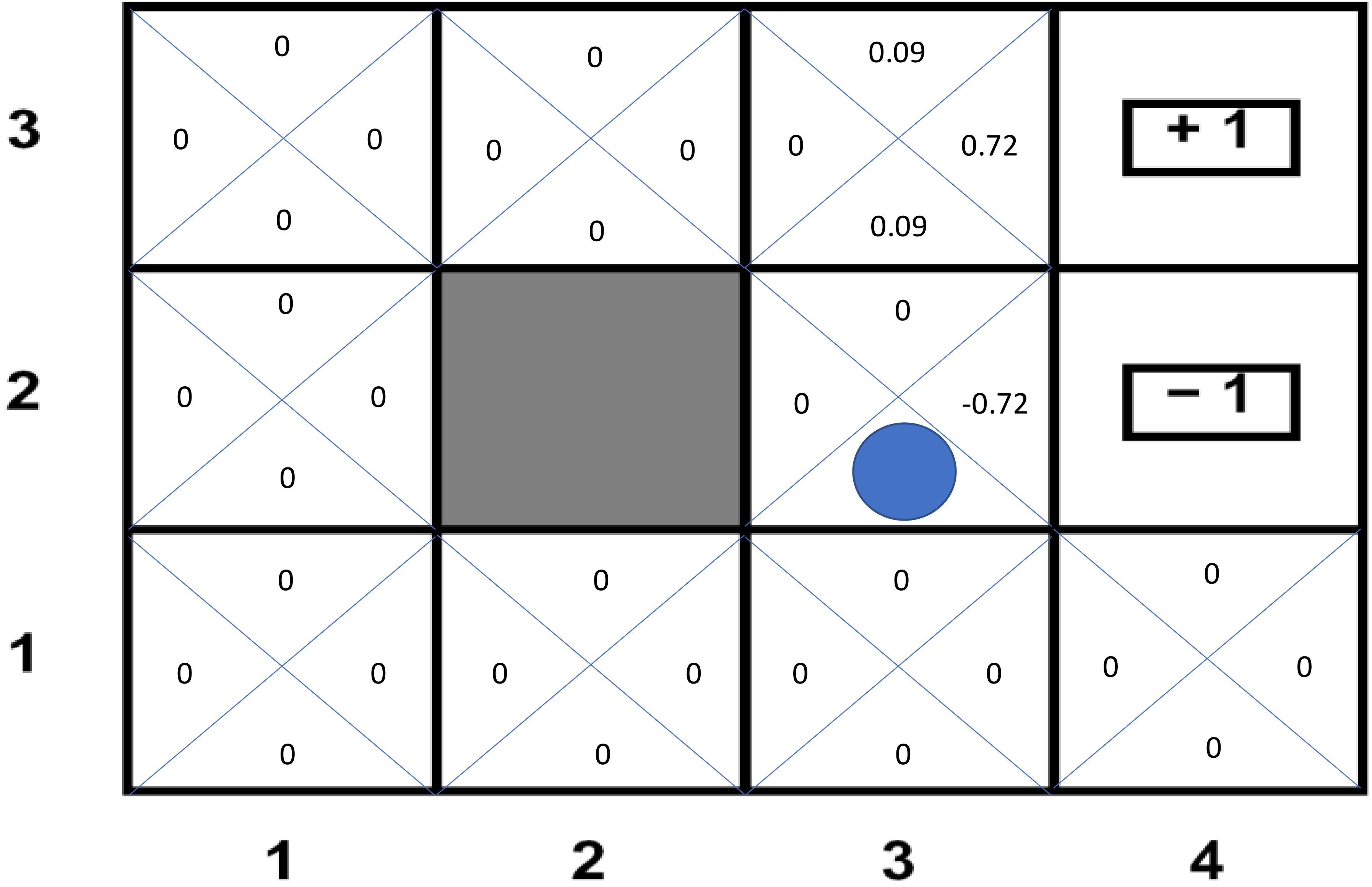
$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[R(s,a,s') + \gamma \boxed{\max_{a'} Q_i(s',a')}\right]$$



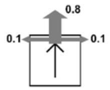
0.8x[0+0.9x-1]
+ 0.1x[0+0]
+0.1x[0+0]
=-0.72



$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[\overset{0}{R(s,a,s')} + \overset{0.9}{\gamma} \boxed{\max_{a'} Q_i(s',a')}\right]$$

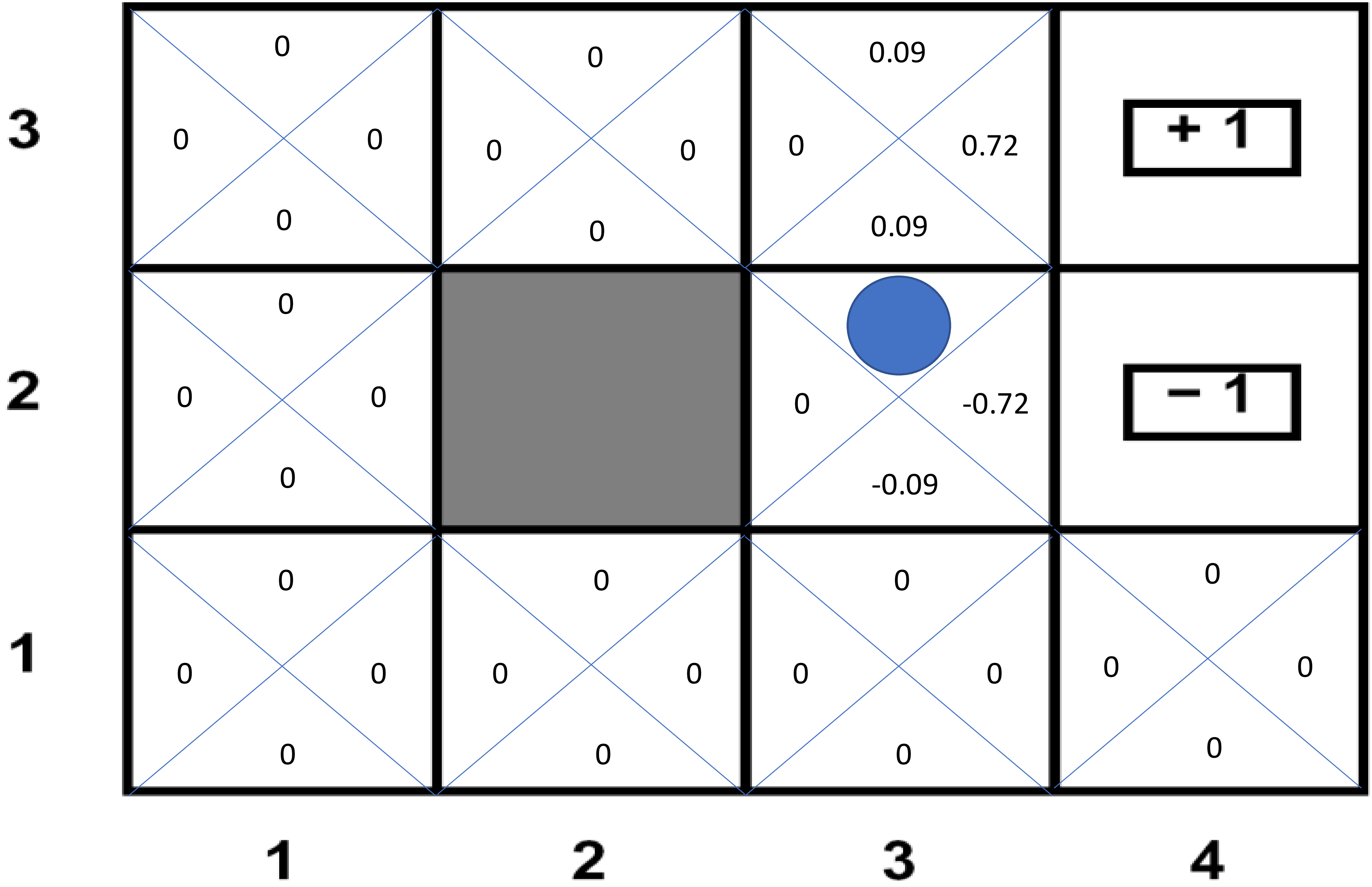


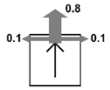
0.8x[0+0]
+ 0.1x[0+0]
+0.1x[0+0.9x-1]
=-0.09



$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[\overset{0}{R(s,a,s')} + \overset{0.9}{\gamma}\, \boxed{\max_{a'} Q_i(s',a')}\right]$$

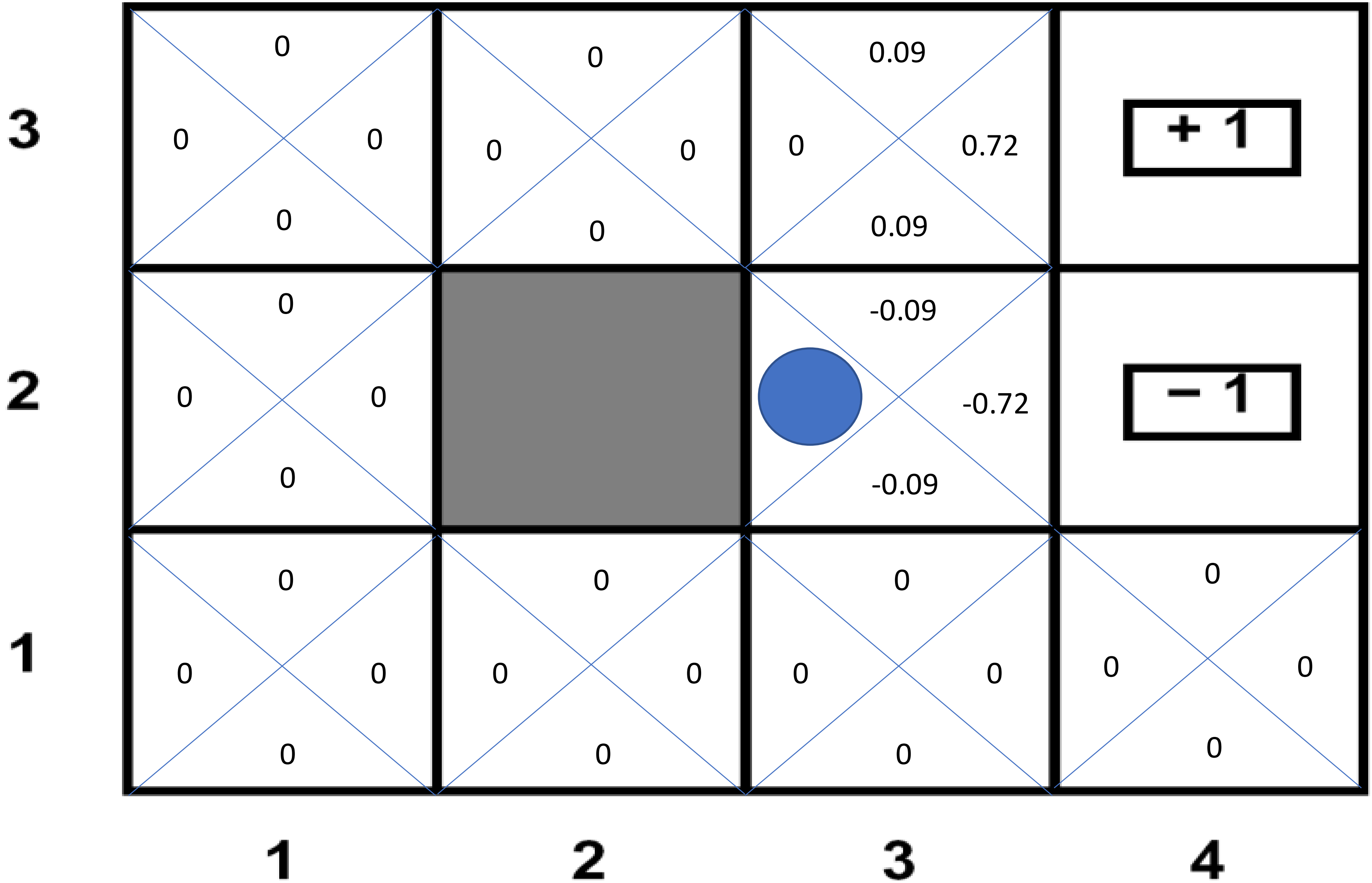
0.8x[0+0]
+ 0.1x[0+0.9x-1]
+0.1x[0+0]
=-0.09



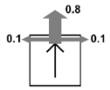


0 0.9

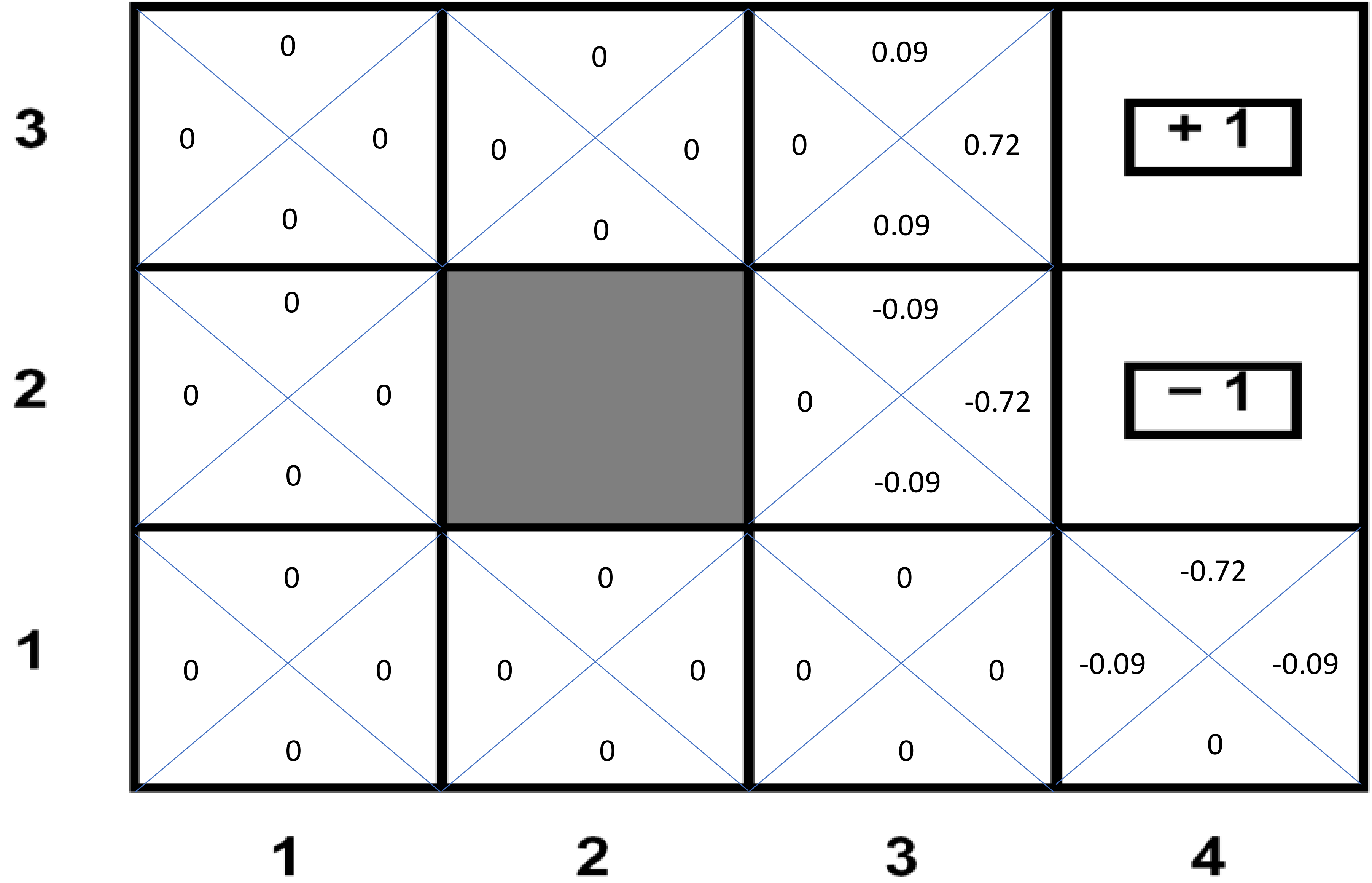
$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[R(s,a,s') + \gamma \boxed{\max_{a'} Q_i(s',a')}\right]$$



0.8x[0+0]
+ 0.1x[0+0]
+0.1x[0+0]
=0

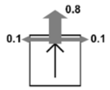


$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[\overset{0}{R(s,a,s')} + \gamma \overset{0.9}{\boxed{\max_{a'} Q_i(s',a')}}\right]$$

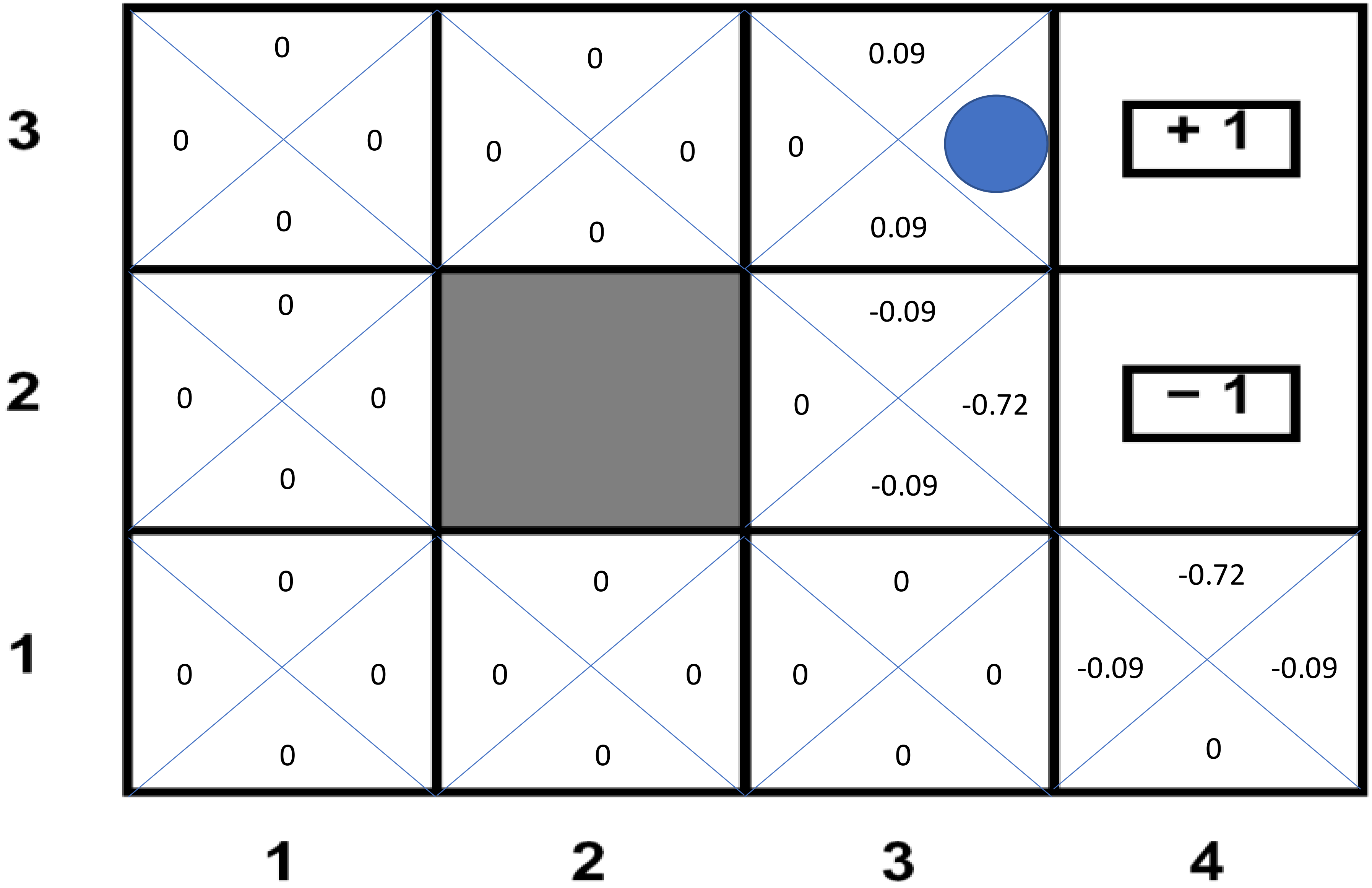


Now we have

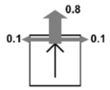
$Q_1(s,a)$ for all (s,a)



$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[\overset{0}{R(s,a,s')} + \gamma\, \overset{0.9}{\boxed{\max_{a'} Q_i(s',a')}}\right]$$

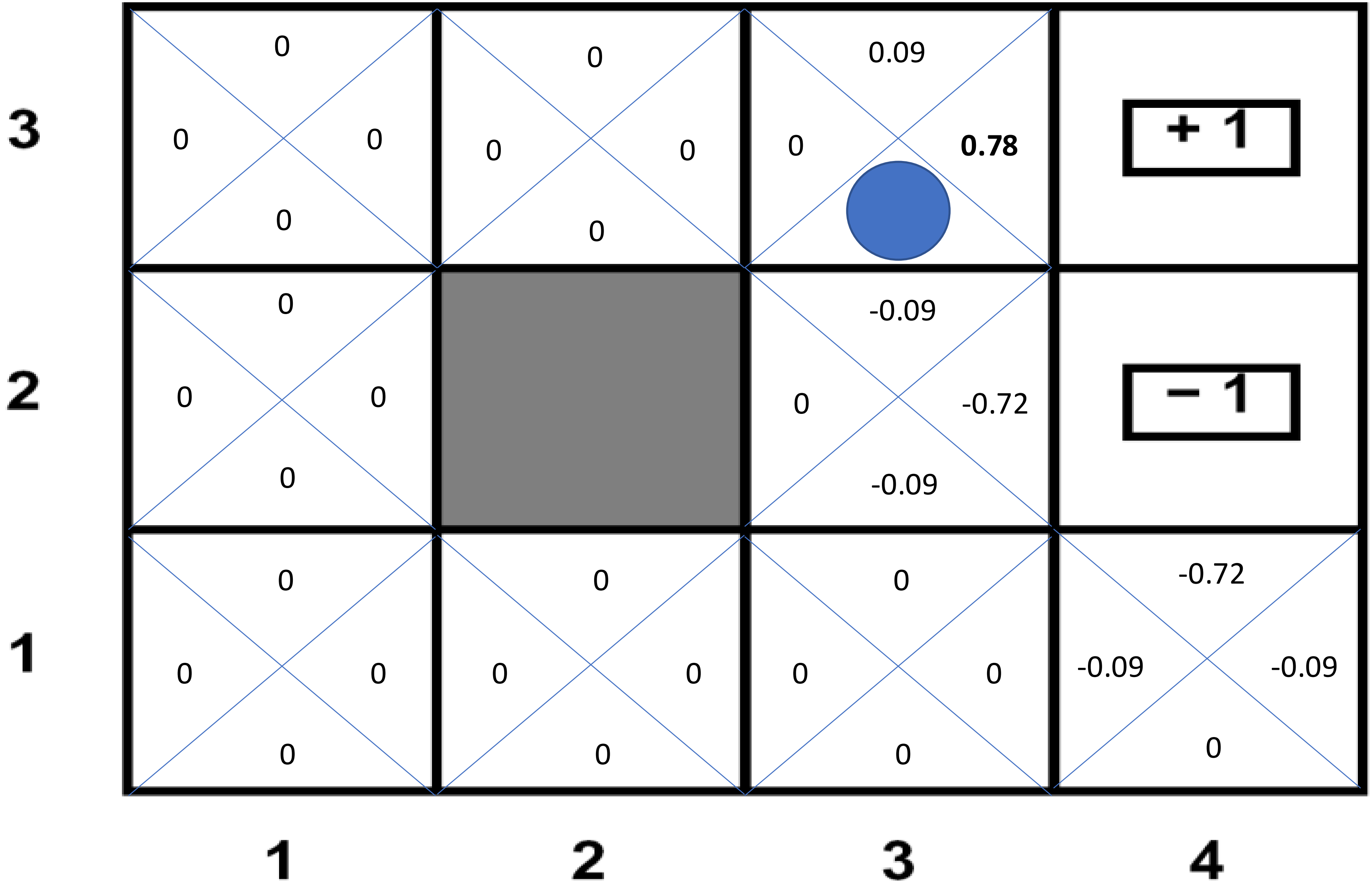


0.8x[0+0.9x1]
+ 0.1x[0+0.9x0.72]
+0.1x[0+0]
=0.7848



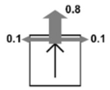
0 0.9

$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[R(s,a,s') + \gamma \boxed{\max_{a'} Q_i(s',a')}\right]$$

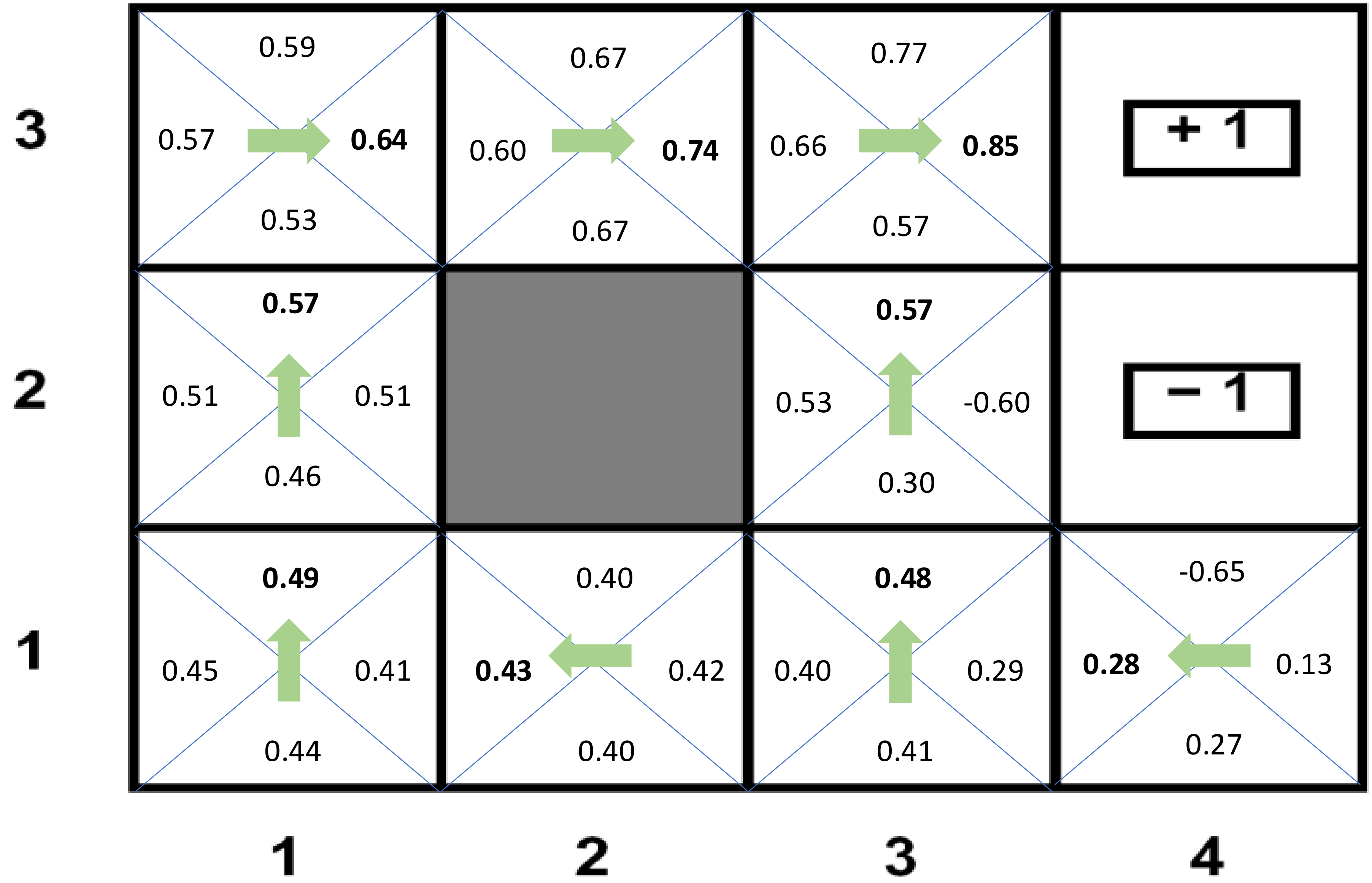


0.8x[0+0]
+ 0.1x[0+0.9x1]
+0.1x[0+0]
=0.09

After 1000 iterations:



$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[\overset{0}{R(s,a,s')} + \overset{0.9}{\gamma}\, \boxed{\max_{a'} Q_i(s',a')}\right]$$



Q Iteration

- Information propagates outward from terminal states
- Eventually all state-action pairs converge to correct Q-value estimates

Aside: Value Iteration

- Analogous to Q-Policy iteration but for computing the value function

- Initialize $V_1(s) \leftarrow 0$ for all s
- For $i \in \{1,2,\ldots\}$ until convergence:

$$V_{i+1}(s) \leftarrow \max_{a \in A} \sum_{s' \in S} P(\, s' \mid s, a\,) \cdot \left(R(s, a, s') + \gamma \cdot V_i(s')\right)$$

$$V_{i+1}(s) \leftarrow \max_{a \in A} \sum_{s' \in S} P(s'|s,a)[\overset{0}{R(s,a,s')} + \overset{0.9}{\gamma} V_i(s')]$$

Example MDP

3				+1
2				− 1
1				
	1	2	3	4

V_0

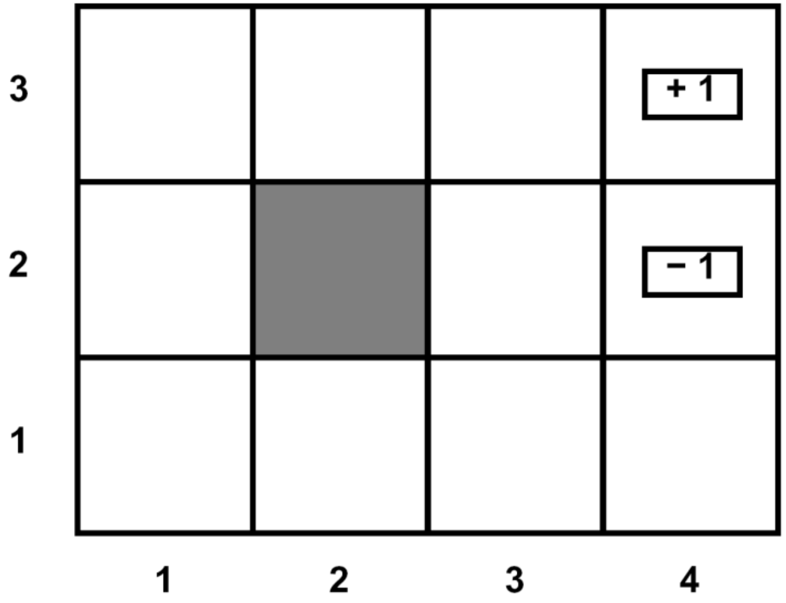
3	0	0	0	0
2	0		0	0
1	0	0	0	0
	1	2	3	4

V_1

3	0	0	0	+1
2	0		0	-1
1	0	0	0	0
	1	2	3	4

$$V_{i+1}(s) \leftarrow \max_{a \in A} \sum_{s' \in S} P(s'|s,a)[\overset{0}{R(s,a,s')} + \overset{0.9}{\gamma} V_i(s')]$$

Example MDP



V_1

	1	2	3	4
3	0	0	0	+1
2	0		0	-1
1	0	0	0	0

V_2

	1	2	3	4
3	0	0	0.72	+1
2	0		0	-1
1	0	0	0	0

$V_2(\langle 4,3 \rangle) \leftarrow 1$

$V_2(\langle 4,2 \rangle) \leftarrow -1$

$$V_{i+1}(s) \leftarrow \max_{a \in A} \sum_{s' \in S} P(s'|s,a)[\overset{0}{R(s,a,s')} + \overset{0.9}{\gamma} V_i(s')]$$

Example MDP

3				+1
2				− 1
1				
	1	2	3	4

V_2

3	0	0	0.72	+1
2	0		0	-1
1	0	0	0	0
	1	2	3	4

V_3

3	0	0.52	0.78	+1
2	0		0.43	-1
1	0	0	0	0
	1	2	3	4

Reinforcement Learning

- Q iteration can be used to compute the optimal Q function when P and R are **known**

- How can we adapt it to the setting where these are unknown?
  - **Observation:** Every time you take action a from state s, you obtain one sample $s' \sim P(\cdot | s, a)$ and observe $R(s, a, s')$
  - Use single sample instead of full P

Q Learning

- Can we learn π^* without explicitly learning P and R?

$$Q_{i+1}(s,a) \leftarrow \sum_{s' \in S} P(\, s' \mid s, a\,) \cdot \left(R(s,a,s') + \gamma \cdot \max_{a' \in A} Q_i(s',a') \right)$$

Q Learning

- Can we learn π^* without explicitly learning P and R?

$$Q_{i+1}(s,a) \leftarrow \mathbb{E}_{s' \sim P(\cdot|s,a)}\left[R(s,a,s') + \gamma \cdot \max_{a' \in A} Q_i(s',a')\right]$$

Q Learning

- **Q Learning update:**

$$Q_{i+1}(s,a) \leftarrow R(s,a,s') + \gamma \cdot \max_{a' \in A} Q_i(s',a')$$

- **Q Iteration:** Update for all (s,a,s') at each step

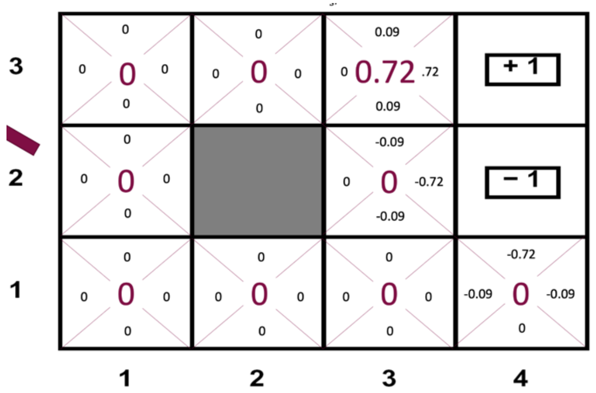
- **Q Learning:** Update just for current (s,a), and approximate with the state s' we actually reached (i.e., a single sample $s' \sim P(\cdot \mid s,a)$)

Q Learning

- **Problem:** Forget everything we learned before (i.e., $Q_i(s, a)$)
- **Solution:** Incremental update:

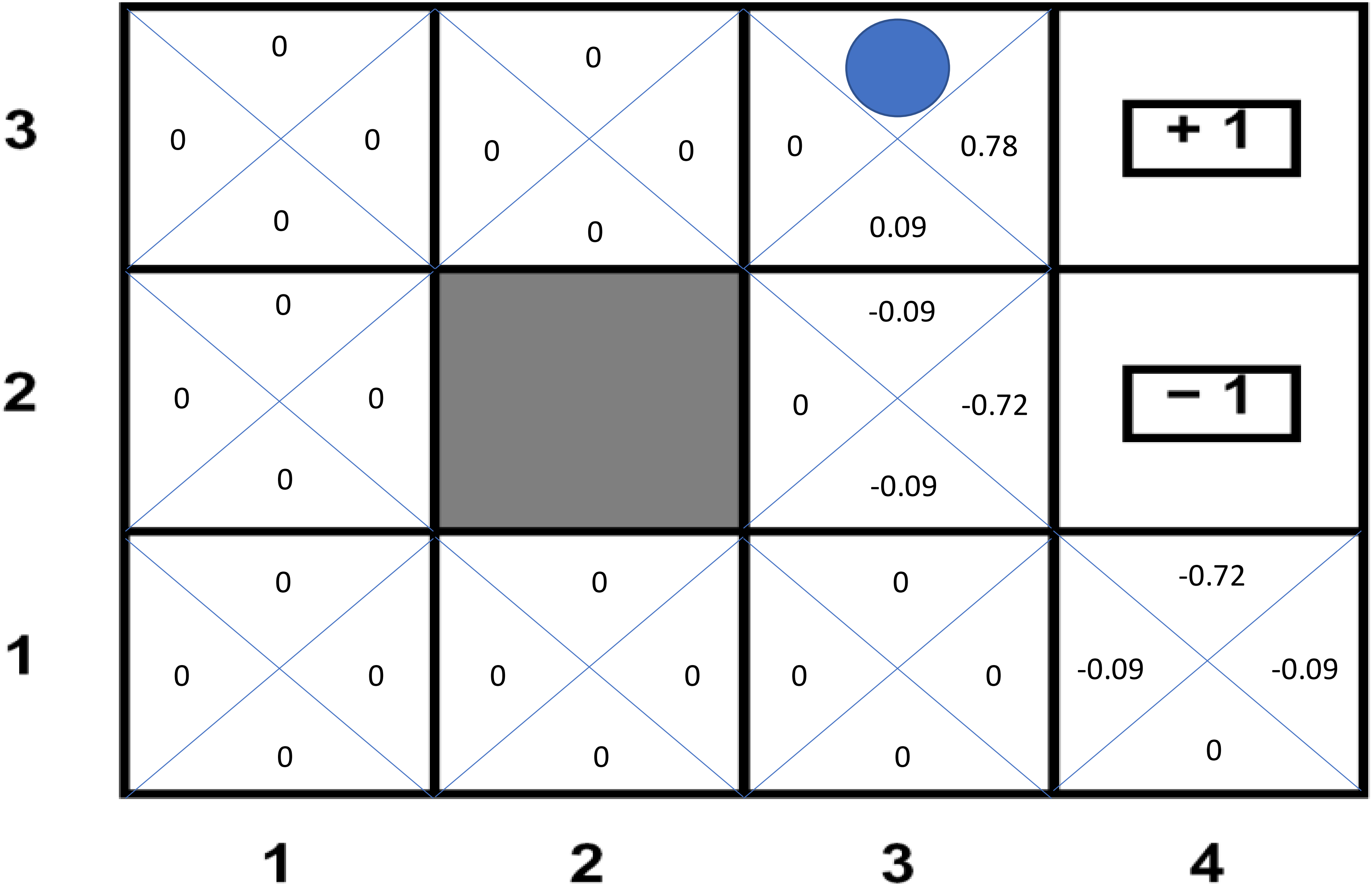
$$Q_{i+1}(s, a) \leftarrow (1 - \alpha) \cdot Q_i(s, a) + \alpha \cdot \left(R(s, a, s') + \gamma \cdot \max_{a' \in A} Q_i(s', a')\right)$$

$$Q(s,a) \leftarrow Q(s,a) + \overset{0.1}{\alpha}\left(R(s,a,s') + \overset{0.9}{\gamma} \max_{a'} Q(s',a') - Q(s,a)\right)$$



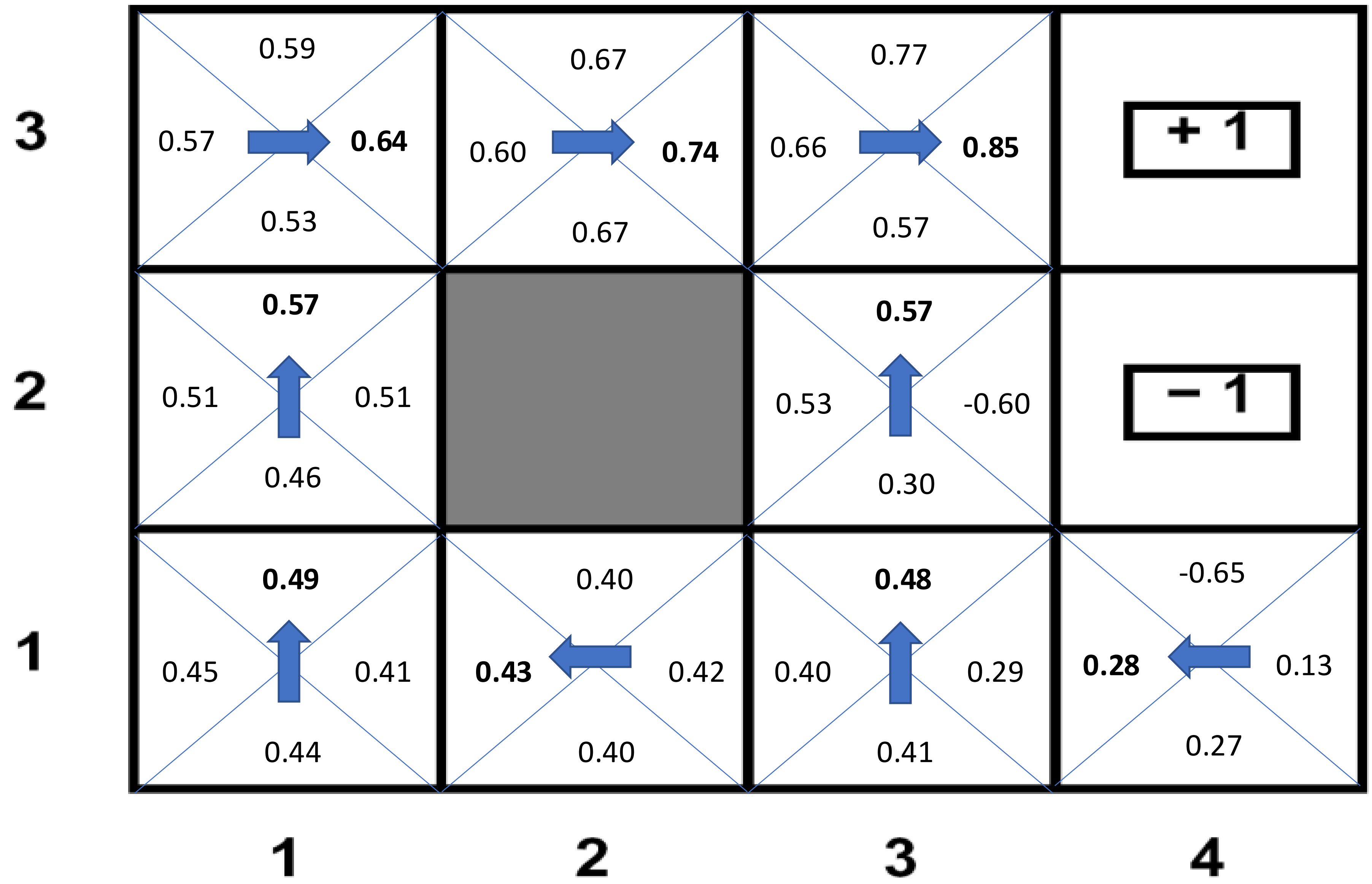
Sample $R + \gamma \max Q$ =
0+0.9x0.78 = 0.702

New Q =
0.09+0.1X(0.702-0.09)
= 0.1512



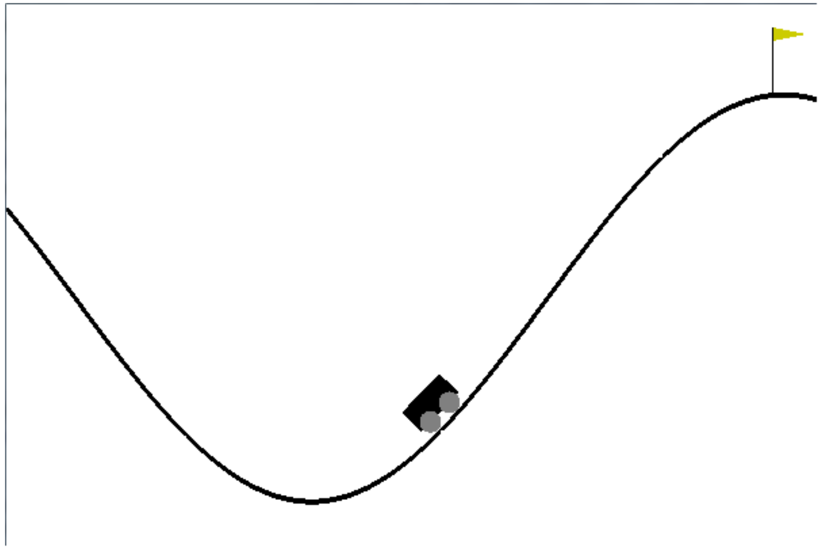
After 100,000 actions:

$$Q(s,a) \leftarrow Q(s,a) + \alpha\left(R(s,a,s') + \gamma \max_{a'} Q(s',a') - Q(s,a)\right)$$



Policy for Gathering Data

- **Strategy 1:** Randomly explore all (s, a) pairs
  - Not obvious how to do so!
  - E.g., if we act randomly, it may take a very long time to explore states that are difficult to reach
- **Strategy 2:** Use current best policy
  - Can get stuck in local minima
  - E.g., we may never discover a shortcut if it sticks to a known route to the goal



Policy for Gathering Data

- **ϵ-greedy:**
  - Play current best with probability $1 - \epsilon$ and randomly with probability ϵ
  - Can reduce ϵ over time
  - Works okay, but exploration is undirected
- **Visitation counts:**
  - Maintain a count $N(s, a)$ of number of times we tried action a in state s
  - Choose $a^* = \arg\max_{a \in A} \left\{ Q(s, a) + \frac{1}{N(s,a)} \right\}$, i.e., inflate less visited states

Summary

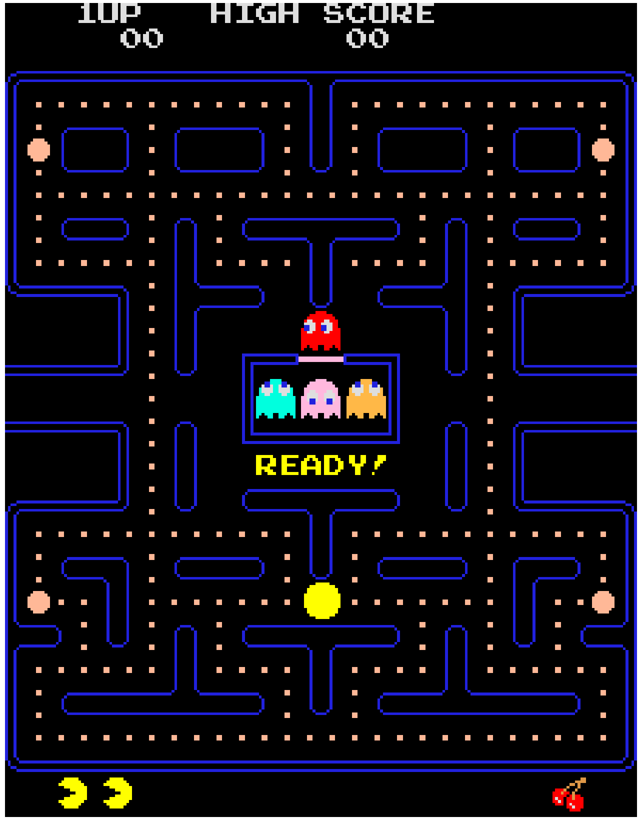
- **Q iteration:** Compute optimal Q function when the transitions and rewards are known
- **Q learning:** Compute optimal Q function when the transitions and rewards are unknown
- **Extensions**
  - Various strategies for exploring the state space during learning
  - Handling large or continuous state spaces

Curse of Dimensionality

- How large is the state space?
  - **Gridworld:** One for each of the n cells
  - **Pacman:** State is $(\mathrm{player}, \mathrm{ghost}_1, \dots, \mathrm{ghost}_k)$, so there are n^k states!

- **Problem:** Learning in one state does not tell us anything about the other states!

- Many states → learn very slowly



State-Action Features

- Can we learn **across** state-action pairs?

- Yes, use features!
  - $\phi(s,a) \in \mathbb{R}^d$
  - Then, learn to predict $Q^*(s,a) \approx Q_\theta(s,a) = f_\theta\big(\phi(s,a)\big)$
  - Enables generalization to similar states