2

System Models : Transformations on Vector Spaces

The fundamental purpose in modeling a system is to develop a mechanism for predicting the condition or change in condition of the system. In the abstract model $\mathbf{Tx} = \mathbf{y}$ of (1.1), $\mathbf{T}$ represents (or is a model of) the system, whereas $\mathbf{x}$ and $\mathbf{y}$ have to do with the condition of the system. We explore first some familiar models for the condition or changes in condition of systems. These examples lead us to use a generalization of the usual notion of a vector as a model for the condition of a system. We then develop the concept of a transformation of vectors as a model of the system itself. The rest of the chapter is devoted to examination of the most commonly used models-linear models-and their matrix representations.

2.1 The Condition of a System

The physical condition (or change in condition) of many simple systems has been found to possess a magnitude and a direction in our physical three-dimensional space. It is natural, therefore, that a mathematical concept of condition (or change in condition) has developed over time which has these two properties; this concept is the vector. Probably the most obvious example of the use of this concept is the use of arrows in a two-dimensional plane to represent changes in the position of an object on the two-dimensional surface of the earth (see Figure 2.1). Using the usual techniques of analytic geometry, we can represent each such arrow by a pair of numbers that indicates the components of that arrow along each of a pair of coordinate axes. Thus pairs of numbers serve as an equivalent model for changes in position.

An ordinary road map is another model for the two-dimensional surface of the earth. It is equivalent to the arrow diagram; points on the map are

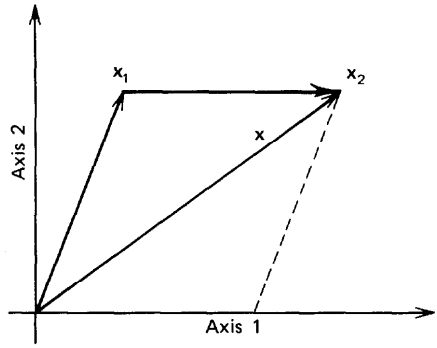

Figure 2.1. An "arrow vector" diagram.

equivalent to the arrow tips of Figure 2.1. The only significant difference between these two models is that the map emphasizes the position (or condition) of an object on the earth, whereas the arrow diagram stresses the changes in position and the manner in which intermediate changes in position add to yield a total change in position. We can also interpret a position on the map as a change from some reference position. The manner in which we combine arrows or changes in position (the parallelogram rule) is the most significant characteristic of either model. Consequently we focus on the arrow model which emphasizes the combination process.

Reference arrows (coordinate axes) are used to tie the arrow model to the physical world. By means of a reference position and a pair of reference "position changes" on the surface of the earth, we relate the positions and changes in position on the earth to positions and arrows in the arrow diagram. However, there are no inherent reference axes on either the physical earth or the two-dimensional plane of arrows.

The same vector model that we use to represent changes in position can be used to represent the forces acting at a point on a physical object. The reason we can use the same model is that the magnitudes and directions of forces also combine according to the parallelogram rule. The physical natures of the reference vectors are different in these three situations: in one case they are changes in position on the earth, in another they are arrows, in the third, forces. Yet once reference vectors are chosen in each, all three situations become in some sense equivalent; corresponding to each vector in one situation is a vector in the other two; corresponding to each sum of vectors in one is a corresponding sum in the other two. We use the set of arrows as a model for the other two situations because it is the most convenient of the three to work with.

The set of complex numbers is one more example of a set of objects which is equivalent to the set of arrows. We usually choose as references in

the set of complex numbers the two numbers 1 and i. Based on these reference numbers and two reference arrows, we interpret every arrow as a complex number. Here we have one set of mathematical (or geometrical) objects serving as a model for another set of mathematical objects.

Consider now a physical system which is more complicated than the two physical systems discussed above. Imagine a flat metal sheet exposed to the sun and partly submerged in a stream. (The sheet is representative of any object subject to heat sources and coolants.) The thermal condition of the sheet is described by the temperature distribution over the surface of the sheet. A change in the cloud cover in the sky will change the pattern in which the sun falls on the sheet. As a result, the temperature distribution will change. Assuming the temperature distribution reaches a new steady state, the new distribution equals the old distribution plus the change in the distribution. We model this situation as follows. Let (s,t) denote a position in some two-dimensional coordinate system on the surface of the sheet. Let $\mathbf{f}(s,$ be the temperature at the point $(s,$ measured in degrees centigrade, for all points (s,t) on the sheet. We model a change in the thermal condition of the sheet by

$$\mathbf{f}_{\text{new}}(s,t) = \mathbf{f}_{\text{old}}(s,t) + \mathbf{f}_{\text{change}}(s,t) \tag{2.1}$$

for all $(s,$ on the sheet. In effect, (2.1) defines $\mathbf{f}_{\text{change}}$. However, we hope to use a model of the system to *predict* $\mathbf{f}_{\text{change}}$. Then (2.1) will determine $\mathbf{f}_{\text{new}}$. Equation (2.1) is a "distributed" equivalent of the arrow diagram in Figure 2.1; each of these models illustrates the manner in which changes in condition combine to yield a net condition of the system in question. Once again, references have been chosen in both the physical system and the model (mathematical system) in order to equate the two systems; choosing physical units of measurement (degrees centigrade) amounts to fixing the relationship between the physical and mathematical systems.

The most significant difference between a system' modeled by Figure 2.1 and a system modeled by (2.1) consists in the nature of the conditions in each system. In one case we have a quantity with magnitude and direction (e.g., force); in the other, a quantity without magnitude and direction—a quantity that is distributed over a two-dimensional region. Yet there are important similarities between the two systems. The changes in condition of the system are under scrutiny; also, several changes in condition combine by simple rules to yield a total or net condition.

Vector Spaces

By expressing various types of problems in a common framework, we learn to use concepts derived from one type of problem in understanding other types of problems. In particular, we are able to draw useful analogies

between algebraic equations and differential equations by expressing both types of equations as "vector" equations. Therefore, we now generalize the common notion of a vector to include all the examples discussed in the previous section.

Definition. A **linear space** (or **vector space)** $\mathcal{V}$ is a set of elements $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots,$ called vectors, together with definitions of *vector addition* and *scalar multiplication.*

a. The definition of vector addition is such that:
   1. To every pair, $\mathbf{x}$ and $\mathbf{y}$, of vectors in $\mathcal{V}$ there corresponds a unique vector $\mathbf{x}+\mathbf{y}$ in $\mathcal{V}$, called the **sum** of $\mathbf{x}$ and $\mathbf{y}$.
   2. $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$.
   3. $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$.
   4. There is a unique vector $\boldsymbol{\theta}$ in $\mathcal{V}$, called the **zero vector** (or origin), such that $\mathbf{x}+\boldsymbol{\theta}=\mathbf{x}$ for all $\mathbf{x}$ in $\mathcal{V}$.
   5. Corresponding to each $\mathbf{x}$ in $\mathcal{V}$ there is a unique vector "$-\mathbf{x}$" in $\mathcal{V}$ such that $\mathbf{x}+(-\mathbf{x})=\boldsymbol{\theta}$.

b. The definition of scalar multiplication is such that:
   1. To every vector $\mathbf{x}$ in $\mathcal{V}$ and every scalar a there corresponds a unique vector $a\mathbf{x}$ in $\mathcal{V}$, called the **scalar multiple** of $\mathbf{x}$.*
   2. $a(b\mathbf{x})=(ab)\mathbf{x}$.
   3. $1(\mathbf{x})=\mathbf{x}$ (where 1 is the unit scalar).
   4. $a(\mathbf{x}+\mathbf{y})=a\mathbf{x}+a\mathbf{y}$.
   5. $(a+b)\mathbf{x}=a\mathbf{x}+b\mathbf{x}$.

Notice that a vector space includes not only a set of elements (vectors) but also "valid" definitions of vector addition and scalar multiplication. Also inherent in the definition is the fact that the vector space $\mathcal{V}$ contains all "combinations" of its own vectors: if $\mathbf{x}$ and $\mathbf{y}$ are in $\mathcal{V}$, then $a\mathbf{x}+b\mathbf{y}$ is also in $\mathcal{V}$. The rules of algebra are so much a part of us that some of the requirements may at first appear above definition; however, they are necessary. A few more vector space properties which may be deduced from the above definition are as follows:

1. $0\mathbf{x}=\boldsymbol{\theta}$ (where "0" is the zero scalar).
2. $a\boldsymbol{\theta}=\boldsymbol{\theta}$.
3. $(-1)\mathbf{x}=-\mathbf{x}$.

Example 1. The Real 3-tuple Space $\mathcal{R}^3$. The space $\mathcal{R}^3$ consists in the set of all

*The scalars are any set of elements which obey the usual rules of algebra. A set of elements which obeys these rules constitutes a field (see Hoffman and Kunze [2.6]). We usually use as scalars either the real numbers or the complex numbers. There are other useful fields, however (P&C 2.4).

real 3-tuples (all ordered sequences of three real numbers), $\mathbf{x} = (\xi_1, \mathbf{y} = (\eta_1, \eta_2, \eta_3)$, with the following definitions of addition and scalar multiplication:

$$\begin{aligned} \mathbf{x} + \mathbf{y} &\triangleq (\xi_1 + \eta_1, \xi_2 + \eta_2, \xi_3 + \eta_3) \\ a\mathbf{x} &\triangleq (a\xi_1, a\xi_2, a\xi_3) \end{aligned} \tag{2.2}$$

It is clear that the zero vector for this 3-tuple space, $\boldsymbol{\theta} = (0,0,0)$, satisfies $\mathbf{x} + \boldsymbol{\theta} = \mathbf{x}$. We show that $\boldsymbol{\theta}$ is unique by assuming another vector $\mathbf{y}$ also satisfies $\mathbf{x} + \mathbf{y} = \mathbf{x}$; that is,

$$(\xi_1 + \eta_1, \xi_2 + \eta_2, \xi_3 + \eta_3) = (\xi_1, \xi_2, \xi_3)$$

or $\xi_i + \eta_i = \xi_i$. The properties of scalars then require $\eta_i = 0$ (or $\mathbf{y} = \boldsymbol{\theta}$). It is easy to prove that $\mathfrak{R}^3$, as defined above, satisfies the other requirements for a linear space. In each instance, questions about vectors are reduced to questions about scalars.

We emphasize that the definition of $\mathfrak{R}^3$ says nothing about coordinates. Coordinates are multipliers for reference vectors (reference arrows, for instance). The 3-tuples are vectors in their own right. However, there is a commonly used correspondence between $\mathfrak{R}^3$ and the set of vectors (arrows) in the usual three-dimensional space which makes it difficult not to think of the 3-tuples as coordinates. The two sets of vectors are certainly equivalent. We will, in fact, use this natural correspondence to help illustrate vector concepts graphically.

Example 2. The Two-Dimensional Space of Points (or Arrows). This space consists in the set of all points in a plane. Addition is defined by the parallelogram rule using a fixed reference point (see Figure 2.2). Scalar multiplication is defined as "length" multiplication using the reference point. The zero vector is obviously the

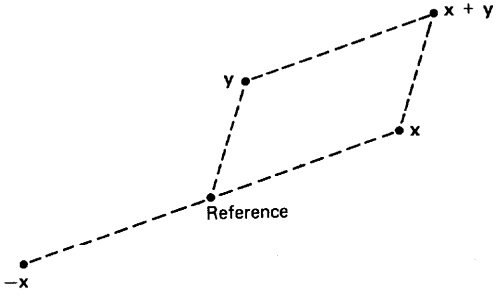

Figure 2.2. The two-dimensional space of points.

reference point. Each of the requirements can be verified by geometrical arguments.

An equivalent (but not identical) space is one where the vectors are not the points, but rather, arrows to the points from the reference point. We distinguish only the magnitude and direction of each arrow; *two parallel arrows of the same length are considered identical.*

Both the arrow space and the point space are easily visualized: we often use the arrow space in two or three dimensions to demonstrate concepts graphically. Although the arrow space contains no *inherent* reference arrows, we sometimes *specify* reference arrows in order to equate the arrows to vectors in $\mathcal{R}^3$. Because of the equivalence between vectors in $\mathcal{R}^3$ and vectors in the three-dimensional space of points, we occasionally refer to vectors in $\mathcal{R}^3$ and in other spaces as *points.*

Example 3. The Space of Column Vectors $\mathcal{M}^{3\times 1}$**.** The space $\mathcal{M}^{3\times 1}$ consists in the set of all real 3x1 column matrices (or column vectors), denoted by

$$\mathbf{x}=\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} \qquad \mathbf{y}=\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}$$

with the following definitions of addition and scalar multiplication:

$$\mathbf{x}+\mathbf{y}\triangleq\begin{pmatrix}\xi_1+\eta_1\\ \xi_2+\eta_2\\ \xi_3+\eta_3\end{pmatrix} \qquad a\mathbf{x}\triangleq\begin{pmatrix}a\xi_1\\ a\xi_2\\ a\xi_3\end{pmatrix} \tag{2.3}$$

In order to save space in writing, we occasionally write vectors from $\mathcal{M}^{3\times 1}$ in the transposed form $\mathbf{x} = (\xi_1\ \xi_2\ \xi_3)^T$. The equivalence between $\mathcal{M}^{3\times 1}$ and $\mathcal{R}^3$ is obvious. The only difference between the two vector spaces is in the nature of their vectors. Vectors in $\mathcal{M}^{3\times 1}$ can be multiplied by $m \times 3$ matrices (as in Section 1.5), whereas vectors in CR^3 cannot.

Example 4. The Space of Real Square-Summable Sequences, l_2**.** The space l_2 consists in the set of all infinite sequences of real numbers, $\mathbf{x} = (\xi_1, \xi_2, \xi_3, \ldots)$, $\mathbf{y} = (\eta_1, \eta_2, \eta_3, \ldots)$ which are square summable; that is, for which $\sum_{i=1}^{\infty}\xi_i^2 < \infty$. Addition and scalar multiplication in l_2 are defined by

$$\begin{aligned}\mathbf{x}+\mathbf{y}&\triangleq(\xi_1+\eta_1, \xi_2+\eta_2, \xi_3+\eta_3, \ldots)\\ a\mathbf{x}&\triangleq(a\xi_1, a\xi_2, a\xi_3, \ldots)\end{aligned} \tag{2.4}$$

Most of the properties required by the definition of a linear space are easily verified for l_2; for instance, the zero vector is obviously $\boldsymbol{\theta}$ = (0,0, . . .). However, there is one subtle difference between l_2 and the space $\mathcal{R}^3$ of Example 1. Because

the sequences in l_2 are infinite, it is not obvious that if $\mathbf{x}$ and $\mathbf{y}$ are in l_2, $\mathbf{x}+\mathbf{y}$ is also in l_2. It can be shown that

$$\sqrt{\Sigma_{i=1}^{\infty}(\xi_i+\eta_i)^2} \leqslant \sqrt{\Sigma_{i=1}^{\infty}\xi_i^2} + \sqrt{\Sigma_{i=1}^{\infty}\eta_i^2}$$

[This fact is known as the triangle inequality (P&C 5.4)]. Therefore,

$$\sum_{i=1}^{\infty}(\xi_i+\eta_i)^2 < \infty$$

and $\mathbf{x}+\mathbf{y}$ is square-summable. The requirement of square summability is a definite restriction on the elements of l_2; the simple sequence $(1, 1, 1, \ldots)$, for instance, is not in l_2.

The definition of $\mathcal{R}^3$ extends easily to $\mathcal{R}^n$, the space of n-tuples of real numbers (where n is a positive integer). The space $\mathcal{M}^{n\times 1}$ is a similar extension of $\mathcal{M}^{3\times 1}$. Mathematically these "n-dimensional" spaces are no more complicated than their three-dimensional counterparts. Yet we are not able to draw arrow-space equivalents because our physical world is three-dimensional. Visualization of an abstract vector space is most easily accomplished by thinking in terms of its three-dimensional counterpart.

The spaces $\mathcal{R}^n$, $\mathcal{M}^{n\times 1}$, and l_2 can also be redefined using complex numbers, rather than real numbers, for scalars. We denote by $\mathcal{R}_c^n$ the complex n-tuple space. We use the symbol $\mathcal{M}_c^{n\times 1}$ for the space of complex $n \times 1$ column vectors. Let l_2^c represent the space of complex square-summable sequences. (We need a slightly different definition of square summability for the space l_2^c: $\Sigma_{i=1}^{\infty}|\xi_i|^2 < \infty$). In most vector space definitions, either set of scalars can be used. A notable exception to interchangeability of scalars is the arrow space in two or three dimensions. The primary value of the arrow space is in graphical illustration. We have already discussed the equivalence of the set of complex scalars to the two-dimensional space of arrows. Therefore, substituting complex scalars in the real two-dimensional arrow space would require four-dimensional graphical illustration.

We eventually find it useful to combine simple vector spaces to form more complicated spaces.

Definition. Suppose $\mathcal{V}$ and $\mathcal{W}$ are vector spaces. We define the **Cartesian product** $\mathcal{V}\times\mathcal{W}$ of the spaces $\mathcal{V}$ and $\mathcal{W}$ to be the set of pairs of vectors $\mathbf{z} \triangleq (\mathbf{x},\mathbf{y})$, with $\mathbf{x}$ in $\mathcal{V}$ and $\mathbf{y}$ in $\mathcal{W}$. We define addition and scalar multiplication of vectors in $\mathcal{V}\times\mathcal{W}$ in terms of the corresponding operations in $\mathcal{V}$ and in $\mathcal{W}$: if $\mathbf{z}_1 = (\mathbf{x}_1,\mathbf{y}_1)$ and $\mathbf{z}_2 = (\mathbf{x}_2,\mathbf{y}_2)$, then

$$\mathbf{z}_1+\mathbf{z}_2 \triangleq (\mathbf{x}_1+\mathbf{x}_2, \mathbf{y}_1+\mathbf{y}_2)$$

$$a\mathbf{z}_1 \triangleq (a\mathbf{x}_1, a\mathbf{y}_1)$$

Example 5. A Cartesian Product. Let $x = (\xi_1, \xi_2)$, a vector in $\mathcal{R}^2$. Let $\mathbf{y} = (\eta_1)$, a vector in $\mathcal{R}^1$. Then $\mathbf{z} \triangleq ((\xi_1, \xi_2), (\eta_1))$ is a typical vector in $\mathcal{R}^2 \times \mathcal{R}^1$. This Cartesian product space is clearly equivalent to $\mathcal{R}^3$. Strictly speaking, however, $\mathbf{z}$ is not in $\mathcal{R}^3$. It is not a 3-tuple, but rather a 2-tuple followed by a 1-tuple. Yet we have no need to distinguish between $\mathcal{R}^3$ and $\mathcal{R}^2 \times \mathcal{R}^1$.

Function Spaces

Each vector in the above examples has discrete elements. It is a small conceptual step from the notion of an infinite sequence of discrete numbers (a vector in l_2) to the usual notion of a function—a "continuum" of numbers. Yet vectors and functions are seldom related in the thinking of engineers. We will find that vectors and functions can be viewed as essentially equivalent objects; functions can be treated as vectors, and vectors can be treated as functions. A **function space** is a linear space whose elements are functions. We usually think of a function as a rule or graph which associates with each scalar in its domain a single scalar value. We do not confuse the graph with particular values of the function. Our notation should also keep this distinction. Let $\mathbf{f}$ denote a **function;** that is, the symbol $\mathbf{f}$ recalls to mind a particular rule or graph. Let $\mathbf{f}(t)$ denote the **value of the function at** t. By $\mathbf{f} = \mathbf{g}$, we mean that the scalars $\mathbf{f}(t)$ and $\mathbf{g}(t)$ are equal for each t of interest.

Example 6. $\mathcal{P}^n$, The Polynomials of Degree Less Than n. The space $\mathcal{P}^n$ consists in all real-valued polynomial functions of degree less than n: $\mathbf{f}(t) = \xi_1 + \xi_2 t + \cdots + \xi_n t^{n-1}$ for all real t. Addition and scalar multiplication of vectors (functions) in $\mathcal{P}^n$ are defined by

$$(\mathbf{f}+\mathbf{g})(t) \triangleq \mathbf{f}(t)+\mathbf{g}(t)$$
$$(a\mathbf{f})(t) \triangleq a(\mathbf{f}(t)) \tag{2.5}$$

for all t. The zero function is $\boldsymbol{\theta}(t) = 0$ for all t. This zero function is unique; if the function $\mathbf{g}$ also satisfied $\mathbf{f} + \mathbf{g} = \mathbf{f}$, then the values of $\mathbf{f}$ and $\mathbf{g}$ would satisfy

$$(\mathbf{f}+\mathbf{g})(t) = \mathbf{f}(t)+\mathbf{g}(t) = \mathbf{f}(t)$$

It would follow that $\mathbf{g}(t) = 0$ for all t, or $\mathbf{g} = \boldsymbol{\theta}$. The other requirements for a vector space are easily verified for $\mathcal{P}^n$.

We emphasize that the vector $\mathbf{f}$ in Example 6 is the entire portrait of the function $\mathbf{f}$. The scalar variable t is a "dummy" variable. The only purpose of this variable is to order the values of the function in precisely the same way that the subscript i orders the elements in the following vector from l_2:

$$\mathbf{x} = (\xi_1, \xi_2, \ldots \xi_i, \ldots)$$

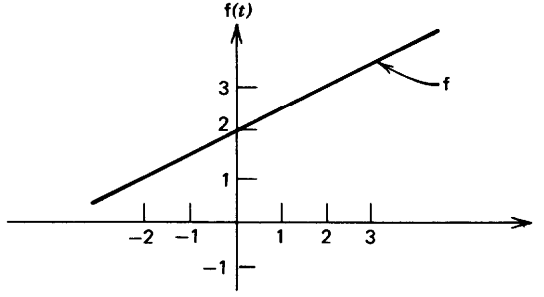

Figure 2.3. A function **f** and its values **f**(*t*).

Figure 2.3 distinguishes graphically between the vector **f** and its value at t for the specific function **f** defined by $\mathbf{f}(t) = 2 + 0.5\,t$. Figure 2.4 distinguishes in a similar manner between an infinite sequence **x** and its ith element.

It is evident that the vector **x** from l_2 is just as much a function as is the polynomial **f** from $\mathcal{P}^n$. In the space of polynomials, the index t is continuous; in the space of infinite sequences the index i is discrete—it takes on only positive integral values. In the latter case, we could as well refer to the ith element ξ_i as the value of **x** at i. In point of fact, most vector spaces can be interpreted as spaces of functions; the terms vector space and function space are somewhat interchangeable. However, it is common practice to use the term function space only for a space in which the index t varies continuously over an interval.

It is unfortunate that the symbol $\mathbf{f}(t)$ is commonly used to represent both a function and the value of that function at t. This blurring of the meaning of symbols is particularly true of the sinusoidal and exponential functions. We will try to be explicit in our distinction between the two concepts. As discussed in the preface, boldface type is used to *emphasize* the interpretation of a function as a vector. However, to avoid overuse of boldface type, it is not used where emphasis on the vector interpretation appears un-

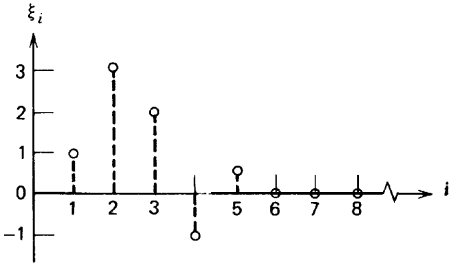

Figure 2.4. The elements ξ_i of an infinite sequence **x**.

necessary; thus the value of a function $\mathbf{f}$ at t may appear either as $\mathbf{f}(t)$ or as $f(t)$. Furthermore, where confusion is unlikely, we sometimes use standard mathematical shorthand; for example, we use $\int_a^b \mathbf{fg}\,dt$ to mean $\int_a^b \mathbf{f}(t)\mathbf{g}(t)\,dt$.

It is difficult to describe or discuss functions in any detail except in terms of their scalar values. In Example 6, for instance, the definitions of addition and scalar multiplication were given in terms of function values. Furthermore, we resorted again to function values to verify that the vector space requirements were met. We will find ourselves continually reducing questions about functions to questions about the scalar values of those functions. Why then do we emphasize the function $\mathbf{f}$ rather than the value f(t)? Because system models act on the whole vector $\mathbf{f}$ rather than on its individual values. As an example, we turn to the one system model we have explored thus far-the matrix equation $\mathbf{Ax} = \mathbf{y}$ which was introduced in Section 1.5. If $\mathbf{A}$ is an $m \times n$ matrix, the vector $\mathbf{x}$ is a column matrix in $\mathfrak{M}^{n\times 1}$; $\mathbf{y}$ is in $\mathfrak{M}^{m\times 1}$. Even though the matrix multiplication requires manipulation of the individual elements (or values) of $\mathbf{x}$, it is impossible to determine *any* element of $\mathbf{y}$ without operating on *all* elements of $\mathbf{x}$. Thus it is natural to think in terms of $\mathbf{A}$ operating on the whole vector $\mathbf{x}$. Similarly, equations involving functions require operations on the whole function (e.g., integration), as we shall see in Section 2.3.

Example 7. The Space $\mathcal{C}(a, b)$ of Continuous Functions. The vectors in $\mathcal{C}$ are those real functions which are defined and continuous on the interval $[a,b]$. Addition and scalar multiplication of functions in $\mathcal{C}(a, b)$ are defined by the standard function space definitions (2.5) for all t in $[a,b]$. It is clear that the sums and scalar multiples of continuous functions are also continuous functions.

Example 8. $\mathcal{L}_2(a,b)$ The Real Square-integrable Functions. The space $\mathcal{L}_2(a,b)$ consists in all real functions which are defined and square integrable on the interval $[a,b]$; that is, functions $\mathbf{f}$ for which*

$$\int_a^b \mathbf{f}^2(t)\,dt < \infty$$

Addition and scalar multiplication of functions in $\mathcal{L}_2(a,b)$ are defined by (2.5) for all t in $[a,$ The space $\mathcal{L}_2(a,b)$ is analogous to l_2. It is not clear that the sum of two square-integrable functions is itself square integrable. As in Example 4, we must rely on P&C 5.4 and the concepts of Chapter 5 to find that

$$\sqrt{\int_a^b [\mathbf{f}(t)+\mathbf{g}(t)]^2\,dt} \leq \sqrt{\int_a^b \mathbf{f}^2(t)\,dt} + \sqrt{\int_a^b \mathbf{g}^2(t)\,dt}$$

*The integral used in the definition of $\mathcal{L}_2(a,b)$ is the Lebesgue integral. For all practical purposes, Lebesgue integration can be considered the same as the usual Riemann integration. Whenever the Riemann integral exists, it yields the same result as the Lebesgue integral. (See Royden [2.l].)

It follows that if $\mathbf{f}$ and $\mathbf{g}$ are square integrable, then $\mathbf{f} + \mathbf{g}$ is square integrable.

Example 9. A Set of Functions. The set of positive real functions [together with the definitions of addition and scalar multiplication in (2.5)] does *not* form a vector space. This set contains a positive valued function $\mathbf{f}$, but not the negative valued function $-\mathbf{f}$; therefore, this set does not include all sums and multiples of its members.

Example 10. ***Functions of a Complex Variable.*** Let $\mathcal{V}$ be the space of all complex functions $\mathbf{w}$ of the complex variable z which are defined and analytic on some region Ω of the complex z plane.* For instance, Ω might be the circle $|z| \leqslant 1$. We define addition and scalar multiplication of functions in $\mathcal{V}$ by

$$\begin{aligned}(\mathbf{w}_1 + \mathbf{w}_2)(z) &\triangleq \mathbf{w}_1(z) + \mathbf{w}_2(z)\\ (a\mathbf{w})(z) &\triangleq a(\mathbf{w}(z))\end{aligned} \tag{2.6}$$

for all z in Ω. In this example, the zero vector $\boldsymbol{\theta}$ is defined by $\boldsymbol{\theta}(z) = 0$ for all z in Ω. (We do not care about the values of the functions $\boldsymbol{\theta}$ and $\mathbf{w}$ outside of Ω.)

Exercise 1. Show that if $\mathbf{w}_1$ and $\mathbf{w}_2$ are in the space $\mathcal{V}$ of Example 10, then $\mathbf{w}_1 + \mathbf{w}_2$ is also in $\mathcal{V}$.

Example 11. A Vector Space of Random Variables † A **random variable** $\mathbf{x}$ is a numerical-valued function whose domain consists in the possible outcomes of an experiment or phenomenon. Associated with the experiment is a probability distribution. Therefore, there is a probability distribution associated with the values of the random variable. For example, the throwing of a single die is an experiment. We define the random variable $\mathbf{x}$ in terms of the possible outcomes σ by

$$\begin{aligned}\mathbf{x}(\sigma) &\triangleq 0 \quad \text{for } \sigma = 2,4,6 \text{ (the die is even)}\\ &\triangleq 1 \quad \text{for } \sigma = 1,3,5 \text{ (the die is odd)}\end{aligned}$$

The probability mass function ω associated with the outcome σ of the experiment is given by

$$\omega(\sigma) = \tfrac{1}{6} \quad \text{for } \sigma = 1,2,3,4,5,6$$

*Express the complex variable z in the form $s + it$, where s and t are real. Let the complex function $\mathbf{w}$ be written as $\mathbf{u} + i\mathbf{v}$, where $\mathbf{u}(z)$ and $\mathbf{v}(z)$ are real. Then $\mathbf{w}$ is analytic in Ω if and only if the partial derivatives of $\mathbf{u}$ and $\mathbf{v}$ are continuous and satisfy the Cauchy-Riemann conditions in Ω:

$$\frac{\partial \mathbf{u}(z)}{\partial s} = \frac{\partial \mathbf{v}(z)}{\partial t}, \quad \frac{\partial \mathbf{v}(z)}{\partial s} = -\frac{\partial \mathbf{u}(z)}{\partial t}$$

For instance, $\mathbf{w}(z) \triangleq z^2$ is analytic in the whole z plane. See Wylie [2.11].

† See Papoulis [2.7], or Cramér and Leadbetter [2.2].

Then the probability mass function $\omega_{\mathbf{x}}$ associated with the values of the random variable $\mathbf{x}$ is

$$\omega_{\mathbf{x}}(x) = \tfrac{1}{2} \quad \text{for } x = 0, 1$$

We can define many other random variables (functions) for the same die-throwing experiment. One other random variable is

$$\mathbf{y}(\sigma) \triangleq 1 \quad \text{for } \sigma = 1 \text{ (the die is 1)}$$

$$\triangleq 0 \quad \text{for } \sigma = 2,3,4,5,6 \text{ (the die is not 1)}$$

where

$$\omega_{\mathbf{y}}(y) = \tfrac{5}{6} \quad \text{for } y = 0$$

$$= \tfrac{1}{6} \quad \text{for } y = 1$$

Two random variables $\mathbf{x}_1$ and $\mathbf{x}_2$ are equal if and only if their values $\mathbf{x}_1(\sigma)$ and $\mathbf{x}_2(\sigma)$ are identical for all possible outcomes σ of the experiment.

A vector **space of random variables** defined on a given experiment consists in a set of functions defined on the possible outcomes of the experiment, together with the following definitions of addition and scalar multiplication*:

$$(a\mathbf{x})(\sigma) \triangleq a(\mathbf{x}(\sigma)) \qquad (\mathbf{x}+\mathbf{y})(\sigma) \triangleq \mathbf{x}(\sigma)+\mathbf{y}(\sigma)$$

for all possible outcomes σ of the experiment. Let $\mathcal{V}$ be the space of all possible random variables defined on the above die-throwing experiment. If $\mathbf{x}$ and $\mathbf{y}$ are the particular vectors described above, then $\mathbf{x}+\mathbf{y}$ is given by

$$(\mathbf{x}+\mathbf{y})(\sigma) \triangleq 2 \quad \text{for } \sigma = 1$$

$$\triangleq 1 \quad \text{for } \sigma = 3,5$$

$$\triangleq 0 \quad \text{for } \sigma = 2,4,6$$

and

$$\omega_{\mathbf{x}+\mathbf{y}}(z) = \tfrac{1}{2} \quad \text{for } z = 0$$

$$= \tfrac{1}{3} \quad \text{for } z = 1$$

$$= \tfrac{1}{6} \quad \text{for } z = 2$$

What is the zero random variable for the vector space $\mathcal{V}$? It is $\boldsymbol{\theta}(\sigma) = 0$ for $\sigma = 1,\ldots,6$.

*We note that the set of functions must be such that it includes all sums and scalar multiples of its members.

2.2 Relations Among Vectors

Combining Vectors

Assuming a vector represents the condition or change in condition of a system, we can use the definitions of addition and scalar multiplication of vectors to find the net result of several successive changes in condition of the system.

Definition. A vector $\mathbf{x}$ is said to be a **linear combination** of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ if it can be expressed as

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n \tag{2.7}$$

for some set of scalars ci, . . . , cn. This concept is illustrated in Figure 2.5 where x= fxi +x2-xX,.

A vector space 'Y is simply a set of elements and a definition of linear combination (addition and scalar multiplication); the space V includes all linear combinations of its own elements. If $\boldsymbol{S}$ is a subset of ?r, the set of all linear combinations of vectors from $\boldsymbol{S}$, using the same definition of linear combination, is also a vector space. We call it a subspace of $\mathcal{V}$. A line or plane through the origin of the three-dimensional arrow space is an example of a subspace.

Definition. A subset $\mathcal{W}$ of a linear space $\mathcal{V}$ is a **linear subspace** (or **linear manifold)** of $\mathcal{V}$ if along with every pair, $\mathbf{x}_1$ and $\mathbf{x}_2$, of vectors in $\mathcal{W}$, every linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is also in $\mathcal{W}$.* We call $\mathcal{W}$ a *proper subspace* if it is smaller than $\mathcal{V}$; that is if $\mathcal{W}$ is not $\mathcal{V}$ itself.

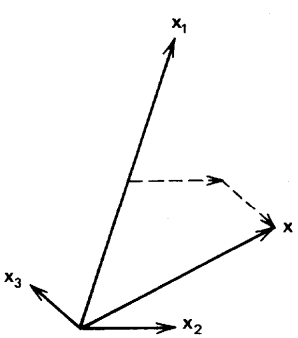

Figure 2.5. A linear combination of arrows.

*In the discussion of infinite-dimensional Hilbert spaces (Section 5.3), we distinguish between a linear subspace and a linear manifold. Linear manifold is the correct term to use in this definition. Yet because a finite-dimensional linear manifold is a linear subspace as well, we emphasize the physically motivated term subspace.

Example 1. A Linear Subspace. The set of vectors from $\mathcal{R}^3$ which are of the form $(c_1, c_2, c_1 + c_2)$ forms a subspace of $\mathcal{R}^3$. It is, in fact, the set of all linear combinations of the two vectors (1, 0, 1) and (0, 1, 1).

Example 2. A Solution Space. The set $\mathcal{W}$ of all solutions to the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is a subspace of $\mathcal{M}^{3\times 1}$. By elimination (Section 1.5), we find that $\mathcal{W}$ contains all vectors of the form $(0\ \xi_2\ -\xi_2)^T$. Clearly, $\mathcal{W}$ consists in all linear combinations of the single vector $(0\ 1\ -1)^T$. This example extends to general matrices. Let A be an $m \times n$ matrix. Let $\mathbf{x}$ be in $\mathcal{M}^{n\times 1}$. Using the rules of matrix multiplication (Appendix 1) it can be shown that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to $\mathbf{Ax} = \boldsymbol{\theta}$, then an arbitrary linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is also a solution. Thus the space of solutions is a subspace of $\mathcal{M}^{n\times 1}$.

Example 3. Subspaces (Linear Manifolds) of Functions. Let $\mathcal{C}^2(\Omega)$ be the space of all real-valued functions which are defined and have continuous second partial derivatives in the two-dimensional region Ω. (This region could be the square $0 \leqslant s \leqslant 1, 0 \leqslant t \leqslant 1$, for instance.) Let Γ denote the boundary of the region Ω. Linear combination in $\mathcal{C}^2(\Omega)$ is defined by

$$\begin{aligned} (\mathbf{f}+\mathbf{g})(s,t) &\triangleq \mathbf{f}(s,t)+\mathbf{g}(s,t) \\ (a\mathbf{f})(s,t) &\triangleq a(\mathbf{f}(s,t)) \end{aligned} \tag{2.8}$$

for all (s, t) in Ω. The functions $\mathbf{f}$ in $\mathcal{C}^2(\Omega)$ which satisfy the homogeneous boundary condition $\mathbf{f}(s, t) = 0$ for (s, t) on Γ constitute a linear manifold of $\mathcal{C}^2(\Omega)$. For if $\mathbf{f}_1$ and $\mathbf{f}_2$ satisfy the boundary condition, then $(c_1\mathbf{f}_1 + c_2\mathbf{f}_2)(s, t) = c_1\mathbf{f}_1(s, t) + c_2\mathbf{f}_2(s, t) = 0$, and the arbitrary linear combination $c_1\mathbf{f}_1 + c_2\mathbf{f}_2$ also satisfies the boundary condition.

The set of solutions to Laplace's equation,

$$\frac{\partial^2 \mathbf{f}(s,t)}{\partial s^2} + \frac{\partial^2 \mathbf{f}(s,t)}{\partial t^2} = 0 \tag{2.9}$$

for all (s, t) in Ω, also forms a linear manifold of $\mathcal{C}^2(\Omega)$. For if $\mathbf{f}_1$ and $\mathbf{f}_2$ both satisfy (2.9), then

$$\frac{\partial^2 [c_1\mathbf{f}_1(s,t) + c_2\mathbf{f}_2(s,t)]}{\partial s^2} + \frac{\partial^2 [c_1\mathbf{f}_1(s,t) + c_2\mathbf{f}_2(s,t)]}{\partial t^2} = 0$$

and the arbitrary linear combination $c_1\mathbf{f}_1 + c_2\mathbf{f}_2$ also satisfies (2.9). Equation (2.9) is phrased in terms of the values of $\mathbf{f}$. Laplace's equation can also be expressed in the

vector notation

$$\nabla^2 \mathbf{f} = \boldsymbol{\theta} \tag{2.10}$$

The domain of definition Ω is implicit in (2.10). The vector $\boldsymbol{\theta}$ is defined by $\boldsymbol{\theta}(s, t) = 0$ for all (s, t) in Ω.

In using vector diagrams to analyze physical problems, we often resolve a vector into a linear combination of component vectors. We usually do this in a unique manner. In Figure 2.5, $\mathbf{x}$ is not a unique linear combination of $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$; $\mathbf{x} = 0\mathbf{x}_1 + 3\mathbf{x}_2 + 2\mathbf{x}_3$ is a second resolution of $\mathbf{x}$; the number of possible resolutions is infinite. In point of fact, $\mathbf{x}$ can be represented as a linear combination of any two of the other vectors; the three vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are redundant as far as representation of $\mathbf{x}$ is concerned.

Definition. The vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are **linearly dependent** (or coplanar) if at least one of them can be written as a linear combination of the others. Otherwise they are **linearly independent.** (We often refer to sets of vectors as simply "dependent" or "independent.")

In Figure 2.5 the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is dependent. Any two of the vectors form an independent set. In any vector space, a set which contains the $\boldsymbol{\theta}$ vector is dependent, for $\boldsymbol{\theta}$ can be written as zero times any other vector in the set. We define the $\boldsymbol{\theta}$ vector by itself as a dependent set.

The following statement is equivalent to the above definition of independence: the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly independent if and only if

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \boldsymbol{\theta} \implies c_1 = \cdots = c_n = 0 \tag{2.11}$$

Equation (2.11) says the "zero combination" is the only combination that equals $\boldsymbol{\theta}$. For if c_i were not 0, we could simply divide by c_i to find $\mathbf{x}_i$ as a linear combination of the other vectors, and the set $\{\mathbf{x}_j\}$ would be dependent. If $c_i = 0$, $\mathbf{x}_i$ cannot be a linear combination of the other vectors. Equation (2.11) is a practical tool for determining independence of vectors.

Exercise 1. Explore graphically and by means of (2.11) the following set of vectors from $\mathcal{R}^3$: $\{\mathbf{x}_1 = (1, 0, 0), \mathbf{x}_2 = (0, 1, 0), \mathbf{x}_3 = (1, 1, 0), \mathbf{x}_4 = (0, 0, 1)\}$.

Example 4. Determining Independence In the space $\mathcal{R}^3$ let $\mathbf{x}_1 = (1, 2, 1)$, $\mathbf{x}_2 = (2, 3, 1)$, and $\mathbf{x}_3 = (4, 7, 3)$. Equation (2.11) becomes

$$\begin{aligned} c_1(1,2,1) + c_2(2,3,1) + c_3(4,7,3) \\ &= (c_1 + 2c_2 + 4c_3,\quad 2c_1 + 3c_2 + 7c_3,\quad c_1 + c_2 + 3c_3) \\ &= (0,0,0) \end{aligned}$$

Each component of this vector equation is a scalar-valued linear algebraic equation. We write the three equations in the matrix form:

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We solve this equation by elimination (Section 1.5) to find $c_1 = -2c_3$ and $c_2 = -c_3$. Any choice for c_3 will yield a particular nonzero linear combination of the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ which equals $\boldsymbol{\theta}$. The set is linearly dependent.

Definition. Let $\mathcal{S} \triangleq \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be a set of vectors from a linear space $\mathcal{V}$. The set of all linear combinations of vectors from $\mathcal{S}$ is called the subspace of $\mathcal{V}$ **spanned** (or generated) by $\mathcal{S}$.* We often refer to this subspace as span($\mathcal{S}$) or span $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$.

Bases and Coordinates

We have introduced the vector space concept in order to provide a common mathematical framework for different types of systems. We can make the similarities between systems more apparent by converting their vector space representations to a standard form. We perform this standardization by introducing coordinate systems. In the example of Figure 2.5, the vectors $\{\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ span a plane; yet any two of them will span the same plane. Two of them are redundant as far as generation of the plane is concerned.

Definition. A **basis** (or coordinate system) for a linear space $\mathcal{V}$ is a linearly independent set of vectors from $\mathcal{V}$ which spans $\mathcal{V}$.

Example 5. The Standard Bases for $\mathcal{R}^n$, $\mathcal{M}^{n \times 1}$, and $\mathcal{P}^n$. It is evident that any three linearly independent vectors in $\mathcal{R}^3$ form a basis for $\mathcal{R}^3$. The n-tuples

$$\begin{aligned} \boldsymbol{\varepsilon}_1 &= (1, 0, \ldots, 0) \\ \boldsymbol{\varepsilon}_2 &= (0, 1, 0, \ldots, 0) \\ &\vdots \\ \boldsymbol{\varepsilon}_n &= (0, \ldots, 0, 1) \end{aligned} \tag{2.12}$$

form a basis for $\mathcal{R}^n$. The set $\mathcal{E} \triangleq \{\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n\}$ is called the **standard basis for $\mathcal{R}^n$.** We use the same notation to represent the standard basis for $\mathcal{M}^{n \times 1}$: $\mathcal{E} \triangleq \{\boldsymbol{\varepsilon}_i\}$, where $\boldsymbol{\varepsilon}_i$ is a column vector of zeros except for a 1 in the ith place. The set $\mathcal{N} \triangleq \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$ defined by $\mathbf{f}_k(t) = t^{k-1}$ forms a basis for $\mathcal{P}^n$; it is analogous to the standard bases for $\mathcal{R}^n$ and $\mathcal{M}^{n \times 1}$.

*The definition of the space spanned by an infinite set of vectors depends on limiting concepts. We delay the definition until Section 5.3.

Example 6. The Zero Vector Space. The set $\{\boldsymbol{\theta}\}$ together with the obvious definitions of addition and scalar multiplication forms a vector space which we denote $\mathcal{O}$. However, the vector $\boldsymbol{\theta}$, by itself, is a dependent set. Therefore $\mathcal{O}$ has no basis.

If $\mathcal{X} \triangleq \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is a basis for the space $\mathcal{V}$, any vector $\mathbf{x}$ in $\mathcal{V}$ can be written uniquely as some linear combination

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n \tag{2.13}$$

of vectors in $\mathcal{X}$. The multipliers c_i are called the **coordinates of x relative to the ordered basis** $\mathcal{X}$. It is easy to show that the coordinates relative to a particular ordered basis are unique: just expand $\mathbf{x}$ as in (2.13) for a second set $\{d_i\}$ of coordinates; then independence of the basis vectors implies $d_i = c_i$.

It is common to write the coordinates of a vector relative to a particular basis as a column matrix. We will denote by $[\mathbf{x}]_{\mathcal{X}}$ the **coordinate matrix** of the vector $\mathbf{x}$ relative to the (ordered) basis $\mathcal{X}$; thus corresponding to (2.13) we have

$$[\mathbf{x}]_{\mathcal{X}} \triangleq \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \tag{2.14}$$

Some bases are more natural or convenient than others. We use the term **natural basis** to mean a basis relative to which we can find coordinates by inspection. The bases of Example 5 are natural bases for $\mathcal{R}^n$, $\mathcal{M}^{n\times 1}$, and $\mathcal{P}^n$. Thus if $\mathbf{f}(t) = \xi_1 + \xi_2 t + \cdots + \xi_n t^{n-1}$, then $[\mathbf{f}]_{\mathcal{R}} = (\xi_1\ \xi_2 \cdots \xi_n)^T$.

Example 7. Coordinates for Vectors in $\mathcal{R}^3$. Let $\mathcal{X} \triangleq \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ be an ordered basis for $\mathcal{R}^3$, where $\mathbf{x}_1 = (1, 2, 3)$, $\mathbf{x}_2 = (2, 3, 2)$, and $\mathbf{x}_3 = (2, 5, 5)$. Let $\mathbf{x} = (1, 1, 1)$. To find $[\mathbf{x}]_{\mathcal{X}}$, we must solve (2.13):

$$\begin{aligned}(1,1,1) &= c_1(1,2,3) + c_2(2,3,2) + c_3(2,5,5). \\ &= (c_1 + 2c_2 + 2c_3,\quad 2c_1 + 3c_2 + 5c_3,\quad 3c_1 + 2c_2 + 5c_3)\end{aligned}$$

We rewrite the vector (3-tuple) equation in the matrix notation:

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 5 \\ 3 & 2 & 5 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{2.15}$$

We solved this equation in Example 1 of Section 1.5. The result is

$$[\mathbf{x}]_{\mathfrak{X}} \triangleq [(1,1,1)]_{\mathfrak{X}} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{3}{5} \\ -\frac{2}{5} \end{pmatrix}$$

The coordinate matrix of Example 7 is merely a simple way of stating that $\mathbf{x} = \frac{3}{5}\mathbf{x}_1 + \frac{3}{5}\mathbf{x}_2 - \frac{2}{5}\mathbf{x}_3$. We choose to write the coordinates of a vector $\mathbf{x}$ as a column matrix because it allows us to carry out in a standard matrix format all manipulations involving the coordinates of $\mathbf{x}$.

In Example 4 of Section 1.5 we solved (2.15) with a general right-hand side; that is, for $\mathbf{x} = (\eta_1, \eta_2, \eta_3)$. That solution allows us to determine quickly the coordinate matrix, relative to the basis $\mathfrak{X}$ of Example 7, for *any* vector $\mathbf{x}$ in $\mathcal{R}^3$, including the case $\mathbf{x} = (0, 0, 0)$. In general, (2.13) includes (2.11); inherent in the process of finding coordinates for an arbitrary vector $\mathbf{x}$ is the process of determining whether $\mathfrak{X}$ is a basis. If $\mathfrak{X}$ is not independent, there will exist nonzero coordinates for $\mathbf{x} = \boldsymbol{\theta}$. If $\mathfrak{X}$ does not span the space, there will be some vector $\mathbf{x}$ for which no coordinates exist (P&C 2.7).

Example 8. Coordinates for Vectors in $\mathcal{P}^3$. Let $\mathcal{F} \triangleq \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be an ordered basis for $\mathcal{P}^3$, where $\mathbf{f}_1(t) = 1 + 2t + 3t^2$, $\mathbf{f}_2(t) = 2 + 3t + 2t^2$, and $\mathbf{f}_3(t) = 2 + 5t + 5t^2$. Let $\mathbf{f}$ be defined by $\mathbf{f}(t) = 1 + t + t^2$. To find $[\mathbf{f}]_{\mathcal{F}}$, we solve (2.13), $\mathbf{f} = c_1\mathbf{f}_1 + c_2\mathbf{f}_2 + c_3\mathbf{f}_3$. To solve this equation, we evaluate both sides at t:

$$\begin{aligned} \mathbf{f}(t) &= (c_1\mathbf{f}_1 + c_2\mathbf{f}_2 + c_3\mathbf{f}_3)(t) \\ &= c_1\mathbf{f}_1(t) + c_2\mathbf{f}_2(t) + c_3\mathbf{f}_3(t) \end{aligned} \tag{2.16}$$

or

$$\begin{aligned} 1 + t + t^2 &= c_1(1 + 2t + 3t^2) + c_2(2 + 3t + 2t^2) + c_3(2 + 5t + 5t^2) \\ &= (c_1 + 2c_2 + 2c_3) + (2c_1 + 3c_2 + 5c_3)t + (3c_1 + 2c_2 + 5c_3)t^2 \end{aligned}$$

Equating coefficients on like powers of t we again obtain (2.15). The coordinate matrix of f is

$$[\mathbf{f}]_{\mathcal{F}} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{3}{5} \\ -\frac{2}{5} \end{pmatrix}$$

In order to solve the vector (function) equation (2.16) we converted it to a set of scalar equations expressed in matrix form. A second method for

converting (2.16) to a matrix equation in the unknowns $\{c_i\}$ is to evaluate the equation at three different values of t. Each such evaluation yields an algebraic equation in $\{c_i\}$. The resulting matrix equation is different from (2.15), but the solution is the same. We now describe a general method, built around a natural basis, for converting (2.13) to a matrix equation. The coordinate matrix of a vector $\mathbf{x}$ relative to the basis $\mathcal{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is $[\mathbf{x}]_{\mathcal{X}} = (c_1 \cdots c_n)^{\mathrm{T}}$, where the coordinates c_i are obtained by solving the vector equation

$$\mathbf{x} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$$

A general method for obtaining an equivalent matrix equation consists in taking coordinates of the vector equation relative to a natural basis $\mathcal{N}$—a basis relative to which coordinates can be obtained by inspection. The vector equation becomes

$$\begin{aligned}[\mathbf{x}]_{\mathcal{N}} &= \left[\sum_{i=1}^{n} c_i\mathbf{x}_i\right]_{\mathcal{N}} \\ &= \sum_{i=1}^{n} c_i[\mathbf{x}_i]_{\mathcal{N}} \\ &= ([\mathbf{x}_1]_{\mathcal{N}} \vdots \cdots \vdots [\mathbf{x}_n]_{\mathcal{N}})[\mathbf{x}]_{\mathcal{X}} \end{aligned} \tag{2.17}$$

We determine $[\mathbf{x}]_{\mathcal{N}}, [\mathbf{x}_1]_{\mathcal{N}}, \ldots, [\mathbf{x}_n]_{\mathcal{N}}$ by inspection. Then we solve (2.17) routinely for $[\mathbf{x}]_{\mathcal{X}}$.

Example 9. Finding Coordinates via a Natural Basis. Let the set $\mathcal{F} \triangleq \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be a basis for $\mathcal{P}^3$, where $\mathbf{f}_1(t) = 1 + 2t + 3t^2$, $\mathbf{f}_2(t) = 2 + 3t + 2t^2$, and $\mathbf{f}_3(t) = 2 + 5t + 5t^2$. We seek $[\mathbf{f}]_{\mathcal{F}}$ for the vector $\mathbf{f}(t) = 1 + t + t^2$. To convert the defining equation for coordinates into a matrix equation, we use the natural basis $\mathcal{N} \triangleq \{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$, where $\mathbf{g}_k(t) = t^{k-1}$. For this problem, (2.17) becomes

$$[\mathbf{f}]_{\mathcal{N}} = \left([\mathbf{f}_1]_{\mathcal{N}} \vdots [\mathbf{f}_2]_{\mathcal{N}} \vdots [\mathbf{f}_3]_{\mathcal{N}}\right)[\mathbf{f}]_{\mathcal{F}}$$

or

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \vdots 2 \vdots 2 \\ 2 \vdots 3 \vdots 5 \\ 3 \vdots 2 \vdots 5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

The solution to this equation is $[\mathbf{f}]_{\mathcal{F}} = (\frac{3}{5}\ \frac{3}{5}\ -\frac{2}{5})^{\mathrm{T}}$. (Compare with Example 8.)

Typically, the solution of (2.17) requires the elimination procedure

$$([\mathbf{x}_1]_{\mathcal{N}} \vdots \cdots \vdots [\mathbf{x}_n]_{\mathcal{N}} \vdots [\mathbf{x}]_{\mathcal{N}}) \rightarrow (\mathbf{I} \vdots [\mathbf{x}]_{\mathcal{X}}) \tag{2.18}$$

If we wish to solve for the coordinates of more than one vector, we still perform the elimination indicated in (2.18), but augment the matrix with all the vectors whose coordinates we desire. Thus if we wish the coordinates for $\mathbf{z}_1$, $\mathbf{z}_2$, and $\mathbf{z}_3$, we perform elimination on

$$([\mathbf{x}_1]_{\mathcal{N}} \vdots \cdots \vdots [\mathbf{x}_n]_{\mathcal{N}} \vdots [\mathbf{z}_1]_{\mathcal{N}} \vdots [\mathbf{z}_2]_{\mathcal{N}} \vdots [\mathbf{z}_3]_{\mathcal{N}})$$

This elimination requires less computation than does the process which goes through inversion of the matrix $([\mathbf{x}_1]_{\mathcal{N}} \vdots \cdots \vdots [\mathbf{x}_n]_{\mathcal{N}})$, regardless of the number of vectors whose coordinates we desire (P&C 1.3).

Example 10. A Basis and Coordinates for a Subspace. Let $\mathcal{W}$ be the subspace of $\mathcal{P}^3$ consisting in all functions $\mathbf{f}$ defined by the rule $\mathbf{f}(t) = \xi_1 + \xi_2 t + (\xi_1 + \xi_2)t^2$ for some ξ_1 and ξ_2. Note that the standard basis functions for $\mathcal{P}^3$ are not contained in $\mathcal{W}$. The functions defined by $\mathbf{g}_1(t) = 1 + t^2$ and $\mathbf{g}_2(t) = t + t^2$ are clearly independent vectors in $\mathcal{W}$. Because there are two "degrees of freedom" in $\mathcal{W}$ (i.e., two parameters ξ_1 and ξ_2 must be given to specify a particular function in $\mathcal{W}$) we expect the set $\mathcal{G} \triangleq \{\mathbf{g}_1, \mathbf{g}_2\}$ to span $\mathcal{W}$ and thus be a basis. We seek the coordinate matrix $[\mathbf{f}]_{\mathcal{G}}$ of an arbitrary vector $\mathbf{f}$ in $\mathcal{W}$. That is, we seek c_1 and c_2 such that

$$\mathbf{f}(t) = c_1\mathbf{g}_1(t) + c_2\mathbf{g}_2(t)$$

The matrix equation (2.17) can be written by inspection using the natural basis $\mathcal{N}$ of Example 9:

$$[\mathbf{f}]_{\mathcal{N}} = \left([\mathbf{g}_1]_{\mathcal{N}} \vdots [\mathbf{g}_2]_{\mathcal{N}}\right)[\mathbf{f}]_{\mathcal{G}}$$

or

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_1 + \xi_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Then $c_i = \xi_i$ and

$$[\mathbf{f}]_{\mathcal{G}} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

Because we were able to solve uniquely for the coordinates, we know that 4 is indeed a basis for $\mathcal{W}$. The subspace $\mathcal{W}$ is equivalent to the subspace of Example 1. Note that the elimination procedure does not agree precisely

with (2.18) because there are only two degrees of freedom among the three coefficients of the arbitrary vector $\mathbf{f}$ in $\mathcal{W}$.

Dimension

The equivalence between the three vector spaces $\mathcal{R}^3$, $\mathcal{P}^3$, and $\mathcal{M}^{3\times 1}$ is apparent from Examples 7 and 8; The subspace $\mathcal{W}$ of Example 10, however, is equivalent to $\mathcal{M}^{2\times 1}$ rather than $\mathcal{M}^{3\times 1}$, even though the elements of $\mathcal{W}$ are polynomials in $\mathcal{P}^3$. The key to the equivalence lies not in the nature of the elements, but rather in the number of "degrees of freedom" in each space (the number of scalars which must be specified in order to specify a vector); more to the point, the key lies in the number of vectors in a basis for each space.

Definition. A vector space is **finite dimensional** if it is spanned by a finite number of vectors. It is intuitively clear that all bases for a finite-dimensional space contain the same number of vectors. The number of vectors in a basis for a finite-dimensional space $\mathcal{V}$ is called the **dimension** of $\mathcal{V}$ and is denoted by $\dim(\mathcal{V})$.

Thus $\mathcal{R}^3$ and $\mathcal{P}^3$ are both three-dimensional spaces. The subspace $\mathcal{W}$ of Example 10 has dimension 2. Knowledge of the dimension of a space (or a subspace) is obtained in the course of determining a basis for the space (subspace). Since the space $\mathcal{O} \triangleq \{\boldsymbol{\theta}\}$ has no basis, we assign it dimension zero.

Example 11. A Basis for a Space of Random Variables. A vector space $\mathcal{V}$ of random variables, defined on the possible outcomes of a single die-throwing experiment, is described in Example 11 of Section 2.1. A natural basis for $\mathcal{V}$ is the set of random variables $\mathcal{X} \triangleq \{\mathbf{x}_i, i = 1,\ldots,6\}$, where

$$\mathbf{x}_i(\sigma) \triangleq 1 \text{ for } \sigma = i \text{ (the die equals } i\text{)}$$

$$\triangleq 0 \text{ for } \sigma \neq i \text{ (the die does not equal } i\text{)}$$

That $\mathcal{X}$ is a basis for $\mathcal{V}$ can be seen from an attempt to determine the coordinates with respect to $\mathcal{X}$ of an arbitrary random variable $\mathbf{z}$ defined on the experiment. If

$$\mathbf{z}(\sigma) \triangleq \begin{array}{ll} c_1 & \text{for } \sigma = 1 \\ \vdots & \\ c_6 & \text{for } \sigma = 6 \end{array}$$

then $[\mathbf{z}]_{\mathcal{X}} = (c_1 \cdots c_6)^{\mathrm{T}}$; a unique representation exists.

The random variables $\{\mathbf{x}_1, \ldots, \mathbf{x}_6\}$ are linearly independent. However, they are not *statistically* independent. **Statistical independence** of two random variables $\mathbf{x}$ and $\mathbf{y}$ means that knowledge of the *value* of one variable, say, $\mathbf{x}$, does not tell us anything about the outcome of the experiment which determines the value of the other variable $\mathbf{y}$, and therefore it tells us nothing about the value of $\mathbf{y}$. The random variables $\{\mathbf{x}_i\}$ are related by the underlying die-throwing experiment. If we know $\mathbf{x}_1 = 0$, for instance, then we know $\sigma \neq 1$ (the die is not equal to 1); the probability mass functions for $\mathbf{x}_2, \ldots, \mathbf{x}_6$ and for all other vectors in $\mathcal{V}$ are modified by the information concerning the value of $\mathbf{x}_1$. The new probability mass functions for $\mathbf{x}$ and $\mathbf{y}$ of Example 11, Section 2.1, given that $\mathbf{x}_1 = 0$, are

$$\omega_{\mathbf{x}}(x; \mathbf{x}_1 = 0) = \tfrac{3}{5} \quad \text{for } x = 0 \qquad \omega_{\mathbf{y}}(y; \mathbf{x}_1 = 0) = 1 \quad \text{for } y = 0$$
$$= \tfrac{2}{5} \quad \text{for } x = 1 \qquad = 0 \quad \text{for } y = 1$$

The space l_2 of square-summable sequences described in Example 4 of Section 2.1 is obviously *infinite dimensional*. A direct extension of the standard basis for $\mathcal{R}^n$ seems likely to be a basis for l_2. It is common knowledge that functions $\mathbf{f}$ in $\mathcal{C}(0, 2\pi)$, the space of functions continuous on $[0, 2\pi]$, can be expanded uniquely in a Fourier series of the form $\mathbf{f}(t) = b_0 + \sum_{k=1}^{\infty} (a_k \sin kt + b_k \cos kt)$. This fact leads us to suspect that the set of functions

$$\mathcal{F} \triangleq \{1, \sin t, \cos t, \sin 2t, \cos 2t, \ldots\} \tag{2.19}$$

forms a basis for $\mathcal{C}(0, 2\pi)$, and that the coordinates of $\mathbf{f}$ relative to this basis are

$$(b_0, a_1, b_1, a_2, b_2, \ldots)$$

This suspicion is correct. The coordinates (or Fourier coefficients) actually constitute a vector in l_2. We show in Example 11 of Section 5.3 that l_2 serves as a convenient standard space of coordinate vectors for infinite-dimensional spaces; in that sense, it plays the same role that $\mathcal{M}^{n \times 1}$ does for n-dimensional spaces. Unfortunately, the concepts of independence, spanning sets, and bases do not extend easily to infinite-dimensional vector spaces. The concept of linear combination applies only to the combination of a finite number of vectors. We cannot add an infinite number of vectors without the concept of a limit; this concept is introduced in Chapter 5. Hence detailed examination of infinite-dimensional function spaces is left for that chapter.

Summary

There is no inherent basis in any space-one basis is as good as another. Yet a space may have one basis which appears more convenient than others. The standard basis for $\mathcal{R}^n$ is an example. By picking units of measurement in a physical system (e.g., volts, feet, degrees centigrade) we tie together the system and the model; our choice of units may automatically determine convenient or standard basis vectors for the vector space of the model (based on, say, 1 V, 1 ft, or 1 ° C).

By choosing a basis for a space, we remove the most distinguishing feature of that space, the nature of its elements, and thus tie each vector in the space to a unique coordinate matrix. Because of this unique connection which a basis establishes between the elements of a particular vector space and the elements of the corresponding space of coordinate matrices, we are able to carry out most vector manipulations in terms of coordinate matrices which represent the vectors. We have selected $\mathcal{M}^{n\times 1}$, rather than $\mathcal{R}^n$, as our standard n-dimensional space because matrix operations are closely tied to computer algorithms for solving linear algebraic equations (Section 1.5). Most vector space manipulations lead eventually to such equations.

Because coordinate matrices are themselves vectors in a vector space ($\mathcal{M}^{n\times 1}$), we must be careful to distinguish vectors from their coordinates. The confusion is typified by the problem of finding the coordinate matrix of a vector $\mathbf{x}$ from $\mathcal{M}^{n\times 1}$ relative to the standard basis for $\mathcal{M}^{n\times 1}$. In this instance $[\mathbf{x}]_{\mathcal{E}} = \mathbf{x}$; the difference between the vector and its coordinate matrix is only conceptual. A vector is simply one of a set of elements, although we may use it to represent the physical condition of some system. The coordinate matrix of the vector, on the other hand, is the unique set of multipliers which specifies the vector as a linear combination of arbitrarily chosen basis vectors.

2.3 System Models

The concept of a vector as a model for the condition or change in condition of a system is explored in Sections 2.1 and 2.2. We usually separate the variables which pertain to the condition of the system into two broad sets: the independent (or input) variables, the values of which are determined outside of the system, and the dependent (or output) variables, whose values are determined by the system together with the independent variables. A model for the system itself consists in expressions of relations among the variables. In this section we identify properties of system models.

Example 1. An Economic System Let $\mathbf{x}$ represent a set of inputs to the U. S. national economy (tax rates, interest rates, reinvestment policies, etc.); let $\mathbf{y}$ represent a set of economic indicators (cost of living, unemployment rate, growth rate, etc.). The system model $\mathbf{T}$ must describe the economic laws which relate $\mathbf{y}$ to $\mathbf{x}$.

Example 2. A Baking Process. Suppose $\mathbf{x}$ is the weight of a sample of clay before a baking process and $\mathbf{y}$ is the weight after baking. Then the system model $\mathbf{T}$ must describe the chemical and thermodynamic laws insofar as they relate $\mathbf{x}$ and $\mathbf{y}$.

Example 3. A Positioning System. Suppose the system of interest is an armature-controlled motor which is used to position a piece of equipment. Let $\mathbf{x}$ represent the armature voltage, a function of time; let $\mathbf{y}$ be the shaft position, another function of time. The system model $\mathbf{T}$ should describe the manner in which the dynamic system relates the function $\mathbf{y}$ to the function $\mathbf{x}$.

The variables in the economic system of Example 1 clearly separate into input (or independent) variables and output (or system condition) variables. In Example 2, both the independent and dependent variables describe the condition of the system. Yet we can view the condition before baking as the input to the system and view the condition after baking as the output. The dynamic system of Example 3 is reciprocal; $\mathbf{x}$ and $\mathbf{y}$ are mutually related by $\mathbf{T}$. Since the system is used as a motor, we view the armature voltage $\mathbf{x}$ as the input to the system and the shaft position $\mathbf{y}$ as the output. We could, as well, use the machine as a dc generator; then we would view the shaft position as the input and the armature voltage as the output.

The notation $\mathbf{Tx} = \mathbf{y}$ that we introduced in (1.1) implies that the model $\mathbf{T}$ does something to the vector $\mathbf{x}$ to yield the vector $\mathbf{y}$. As a result, we may feel inclined to call $\mathbf{x}$ the input and $\mathbf{y}$ the output. Yet in Section 1.3 we note that equations are sometimes expressed in an inverse form. The positions of the variables in an equation do not determine whether they are independent or dependent variables. Furthermore, we can see from Example 3 that the input and output of a system in some instances may be determined arbitrarily. In general, we treat one of the vectors in the equation $\mathbf{Tx} = \mathbf{y}$ as the input and the other as the output. However, unless we are exploring a problem for which the input is clearly defined, we use the terms input and output loosely in reference to the known and unknown variables, respectively.

Transformations on Vector Spaces

Our present purpose is to make more precise the vaguely defined model $\mathbf{T}$ introduced in (1.1) and illustrated above.

Definition. A **transformation** or **function** $\mathbf{T}$: $\mathcal{S}_1 \rightarrow \mathcal{S}_2$ is a rule that

associates with each element of the set $\mathcal{S}_1$ a unique element from the set $\mathcal{S}_2$*. The set $\mathcal{S}_1$ is called the **domain** of **T**; $\mathcal{S}_2$ is the **range of definition** of **T**.

Our attention is directed primarily toward transformations where $\mathcal{S}_1$ and $\mathcal{S}_2$ are linear spaces. We speak of **T**: $\mathcal{V} \rightarrow \mathcal{W}$ as a transformation from the vector space $\mathcal{V}$ into the vector space $\mathcal{W}$. An **operator** is another term for a transformation between vector spaces. We use this term primarily when the domain and range of definition are identical; we speak of **T**: $\mathcal{V} \rightarrow \mathcal{V}$ as an operator **on** $\mathcal{V}$. If $\mathcal{S}_{\mathcal{V}}$ is a subset of $\mathcal{V}$, we denote by $\mathbf{T}(\mathcal{S}_{\mathcal{V}})$ the set of all vectors **Tx** in $\mathcal{W}$ for which **x** is in $\mathcal{S}_{\mathcal{V}}$; we refer to $\mathbf{T}(\mathcal{S}_{\mathcal{V}})$ as the **image** of $\mathcal{S}_{\mathcal{V}}$ under T. The range of **T** is $\mathbf{T}(\mathcal{V})$, the image of $\mathcal{V}$ under **T**. The **nullspace of T** is the set of all vectors **x** in $\mathcal{V}$ such that $\mathbf{Tx} = \boldsymbol{\theta}_{\mathcal{W}}$ ($\boldsymbol{\theta}_{\mathcal{W}}$ is the zero vector in the space $\mathcal{W}$). If $\mathcal{S}_{\mathcal{W}}$ is a subset of $\mathcal{W}$, we call the set of vectors **x** in $\mathcal{V}$ for which **Tx** is in $\mathcal{S}_{\mathcal{W}}$ the **inverse image** of $\mathcal{S}_{\mathcal{W}}$. Thus the nullspace of **T** is the inverse image of the set $\{\boldsymbol{\theta}_{\mathcal{W}}\}$. See Figure 2.6.

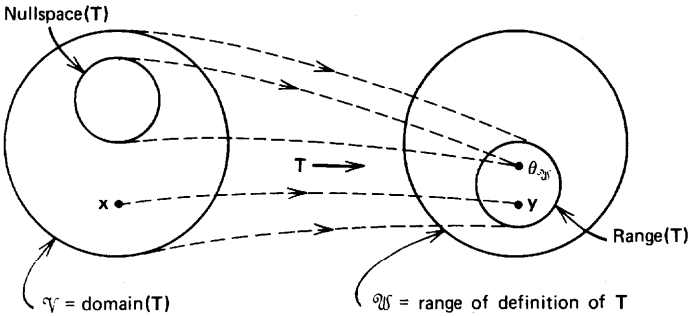

Figure 2.6. Abstract illustration of a transformation **T**.

Example 4. A Transformation Define **T**: $\mathcal{R}^2 \rightarrow \mathcal{R}^1$ by

$$\begin{aligned} \mathbf{T}(\xi_1, \xi_2) &\triangleq \sqrt{\xi_1^2 + \xi_2^2} - 1 \quad \text{for } \xi_1^2 + \xi_2^2 \geq 1 \\ &\triangleq 0 \qquad\qquad\quad \text{for } \xi_1^2 + \xi_2^2 < 1 \end{aligned} \tag{2.20}$$

Physically, the vector **Tx** can be interpreted as the distance between **x** and the unit circle in the two-dimensional arrow space. The variables ξ_1 and ξ_2 are "dummy" variables; they merely assist us in cataloguing the "values" of **T** in the defining

*In the modeling process we use the function concept twice: once as a vector—a model for the condition of a system—and once as a relation between input and output vectors—a model for the system itself. In order to avoid confusion, we use the term function in referring to vectors in a vector space, but the term transformation in referring to the relation between vectors.

equation; we can use any other symbols in their place without changing the definition of $\mathbf{T}$. The range of $\mathbf{T}$ is the set of positive numbers in $\mathcal{R}^1$. The nullspace of $\mathbf{T}$ is the set consisting of all vectors in the domain $\mathcal{R}^2$ which satisfy $\xi_1^2 + \xi_2^2 \leq 1$.

Suppose we wish to solve the equation $\mathbf{Tx} = 1$ for the transformation of Example 4. In effect, we ask which points in the arrow space are a unit distance from the unit circle—all points on the circle of radius 2. The solution is not unique because $\mathbf{T}$ assigns to the single number 1 in $\mathcal{R}^1$ more than one vector in $\mathcal{R}^2$. The equation $\mathbf{Tx} = -1$, on the other hand, has no solution because $\mathbf{T}$ does not assign the number -1 in $\mathcal{R}^1$ to any vector in $\mathcal{R}^2$. We now proceed to specify the properties of a transformation which are necessary in order that the transformation be uniquely reversible.

Definition. Let $\mathbf{T}: \mathcal{V} \to \mathcal{W}$. Then $\mathbf{T}$ is **one-to-one** if

$$\mathbf{x}_1 \neq \mathbf{x}_2 \quad \Rightarrow \quad \mathbf{T}\mathbf{x}_1 \neq \mathbf{T}\mathbf{x}_2 \tag{2.21}$$

for all $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathcal{V}$; that is, if $\mathbf{T}$ does not assign more than one $\mathbf{x}$ in $\mathcal{V}$ to a single $\mathbf{y}$ in $\mathcal{W}$.

If $\mathbf{T}$ is one-to-one, any solution to $\mathbf{Tx} = \mathbf{y}$ is unique. It might appear that the effect of $\mathbf{T}$ is reversible if $\mathbf{T}$ is one-to-one. The nonreversibility of $\mathbf{T}$ in Example 4, however, arises only in part because $\mathbf{T}$ is not one-to-one. In general, there may be vectors in the range of definition $\mathcal{W}$ which are not associated in any way with vectors in $\mathcal{V}$. In point of fact, **range(T)** consists precisely of those vectors $\mathbf{y}$ in $\mathcal{W}$ for which the equation $\mathbf{Tx} = \mathbf{y}$ is solvable. Unless we know which vectors are in **range(T)**, we cannot reverse the transformation.

Definition. Let $\mathbf{T}: \mathcal{V} \to \mathcal{W}$. Then $\mathbf{T}$ is **onto** if

$$\mathbf{range(T)} = \mathcal{W} \tag{2.22}$$

That is, $\mathbf{T}$ is onto if every vector $\mathbf{y}$ in $\mathcal{W}$ is associated with at least one vector $\mathbf{x}$ in $\mathcal{V}$.

Definition. If a transformation is one-to-one and onto, then it is **invertible**—it can be reversed uniquely. If T: $\mathcal{V} \to \mathcal{W}$ is invertible, we define the **inverse of T** to be the transformation $\mathbf{T}^{-1}: \mathcal{W} \to \mathcal{V}$ which associates with each $\mathbf{y}$ in $\mathcal{W}$ the unique vector $\mathbf{x}$ in $\mathcal{V}$ for which $\mathbf{Tx} = \mathbf{y}$. See (2.29) for another characterization of $\mathbf{T}^{-1}$.

Example 5. The Identity Operator, I. Let $\mathcal{V}$ be a vector space. Define the operator $\mathbf{I}$ on $\mathcal{V}$ by

$$\mathbf{Ix} \triangleq \mathbf{x} \tag{2.23}$$

for all $\mathbf{x}$ in $\mathcal{V}$. The nullspace of $\mathbf{I}$ is $\boldsymbol{\theta}_{\mathcal{V}}$. Range $(\mathbf{I}) = \mathcal{V}$; thus $\mathbf{I}$ is onto. Furthermore, $\mathbf{I}$ is one-to-one. Therefore, the identity operator is invertible.

Example 6. The Zero Transformation, $\boldsymbol{\Theta}$. Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces. Define $\boldsymbol{\Theta}: \mathcal{V} \rightarrow \mathcal{W}$ by

$$\boldsymbol{\Theta}\mathbf{x} \triangleq \boldsymbol{\theta}_{\mathcal{W}} \tag{2.24}$$

for all $\mathbf{x}$ in $\mathcal{V}$. The nullspace of $\boldsymbol{\Theta}$ is $\mathcal{V}$. The range of $\boldsymbol{\Theta}$ is $\boldsymbol{\theta}_{\mathcal{W}}$. The zero transformation is neither one-to-one nor onto. It is clearly not invertible.

Example 7. A Transformation on a Function Space. Define $\mathbf{T}: \mathcal{C}(a,b) \rightarrow \mathcal{R}^1$ by

$$\mathbf{Tf} \triangleq \int_a^b \mathbf{f}^2(t)\,dt \tag{2.25}$$

for all $\mathbf{f}$ in $\mathcal{C}(a, b)$. This transformation specifies an integral-square measure of the size of the function $\mathbf{f}$; this measure is used often in judging the performance of a control system. The function $\mathbf{f}$ is a dummy variable used to define $\mathbf{T}$; the scalar t is a dummy variable used to define $\mathbf{f}$. In order to avoid confusion, we must carefully distinguish between the concept of the function $\mathbf{f}$ in the vector space $\mathcal{C}(a, b)$ and the concept of the transformation $\mathbf{T}$ which relates each function $\mathbf{f}$ in $\mathcal{C}(a, b)$ to a vector in $\mathcal{R}^1$. The transformation acts on the whole function $\mathbf{f}$—we must use all values of $\mathbf{f}$ to find $\mathbf{Tf}$. The range of $\mathbf{T}$ is the set of positive numbers in $\mathcal{R}^1$; thus $\mathbf{T}$ is not onto the range of definition $\mathcal{R}^1$. The nullspace of $\mathbf{T}$ is the single vector $\boldsymbol{\theta}_{\mathcal{V}}$. If we define $\mathbf{f}_1$ and $\mathbf{f}_2$ by $\mathbf{f}_1(t) = 1$ and $\mathbf{f}_2(t) = -1$, then $\mathbf{Tf}_1 = \mathbf{Tf}_2$; therefore $\mathbf{T}$ is not one-to-one.

The transformations of Examples 4 and 7 are scalar valued; that is, the range of definition in each case is the space of scalars. We call a scalar-valued transformation a **functional.** Most functionals are not one-to-one.

Example 8. A Transformation for a Dynamic System. Let $\mathcal{C}^2(a, b)$ be the space of functions which have continuous second derivatives on $[a, b]$. Define $\mathbf{L}: \mathcal{C}^2(a, b) \rightarrow \mathcal{C}(a,b)$ by

$$(\mathbf{Lf})(t) \triangleq \mathbf{f}''(t) + \alpha(\mathbf{f}(t) + 0.01\mathbf{f}^3(t)) \tag{2.26}$$

for all $\mathbf{f}$ in $\mathcal{C}^2(a, b)$ and all t in $[a, b]$. This transformation is a model for a particular mass-spring system in which the spring is nonlinear. The comments under Example 7 concerning the dummy variables $\mathbf{f}$ and t apply here as well. As usual, the definition is given in terms of scalars, functions evaluated at t. Again, $\mathbf{L}$ acts on the whole function $\mathbf{f}$. Even in this example we cannot determine any value of the function $\mathbf{Lf}$ without using an "interval" of values of $\mathbf{f}$, because the derivative

function $\mathbf{f}'$ is defined in terms of a limit of values of $\mathbf{f}$ in the neighborhood of t:

$$\mathbf{f}'(t) \triangleq \lim_{\Delta t \to 0} \frac{\mathbf{f}(t+\Delta t)-\mathbf{f}(t)}{\Delta t}$$

The nullspace of $\mathbf{L}$ consists in all solutions of the nonlinear differential equation, $\mathbf{Lf} = \boldsymbol{\theta}_{\mathcal{W}}$; restated in terms of the values of $\mathbf{Lf}$, this equation is

$$\mathbf{f}''(t) + \alpha(\mathbf{f}(t) + 0.01\mathbf{f}^3(t)) = 0 \qquad a \leqslant t \leqslant b$$

To determine these solutions is not a simple task. By selecting $\mathcal{C}(a,b)$ as the range of definition, we ask that the function $\mathbf{Lf}$ be continuous; since $\mathbf{Lf}$ represents a force in the mass-spring system described by (2.26) continuity seems a practical assumption. By choosing $\mathcal{C}^2(a,b)$ as the domain, we guarantee that $\mathbf{Lf}$ is continuous. Yet the range of $\mathbf{L}$ is not clear. It is in the range of definition, but is it equal to the range of definition? In other words, can we solve the nonlinear differential equation $\mathbf{Lf} = \mathbf{u}$ for *any* continuous $\mathbf{u}$? The function $\mathbf{f}$ represents the displacement versus time in the physical mass-spring system. The function $\mathbf{u}$ represents the force applied to the system as a function of time. Physical intuition leads us to believe that for given initial conditions there is a unique displacement pattern $\mathbf{f}$ associated with each continuous forcing pattern $\mathbf{u}$. Therefore, $\mathbf{L}$ should be onto. On the other hand, since no initial conditions are specified, we expect two degrees of freedom in the solution to $\mathbf{Lf} = \mathbf{u}$ for each continuous $\mathbf{u}$. Thus the dimension of nullspace $(\mathbf{L})$ is two, and $\mathbf{L}$ is not one-to-one.

Combining Transformations

The transformation introduced in Example 8 is actually a composite of several simpler transformations. In developing a model for a system, we usually start with simple models for portions of the system, and then combine the parts into the total system model. Suppose $\mathbf{T}$ and $\mathbf{U}$ are both transformations from $\mathcal{V}$ into $\mathcal{W}$. We define the transformation $a\mathbf{T} + b\mathbf{U}$: $\mathcal{V} \to \mathcal{W}$ by

$$(a\mathbf{T} + b\mathbf{U})\mathbf{x} \triangleq a\mathbf{T}\mathbf{x} + b\mathbf{U}\mathbf{x} \tag{2.27}$$

for all $\mathbf{x}$ in $\mathcal{V}$. If $\mathbf{G}$: $\mathcal{W} \to \mathcal{U}$, we define the transformation $\mathbf{GT}$: $\mathcal{V} \to \mathcal{U}$ by

$$(\mathbf{GT})\mathbf{x} \triangleq \mathbf{G}(\mathbf{T}\mathbf{x}) \tag{2.28}$$

for all $\mathbf{x}$ in $\mathcal{V}$. Equations (2.27) and (2.28) define **linear combination** and **composition** of transformations, respectively.

Example 9. Composition of Matrix Multiplications. Define $\mathbf{G}: \mathcal{R}^3 \to \mathcal{R}^2$ by

$$\mathbf{G}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} \triangleq \begin{pmatrix}1 & 0 & 1\\ 2 & 1 & 3\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}$$

and $\mathbf{T}: \mathcal{R}^2 \to \mathcal{R}^3$ by

$$\mathbf{T}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} \triangleq \begin{pmatrix}2 & 2\\ 1 & 2\\ 3 & 1\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}$$

Then $\mathbf{GT}: \mathcal{R}^2 \to \mathcal{R}^2$ is described by

$$\begin{aligned}\mathbf{GT}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} &= \mathbf{G}\begin{pmatrix}2 & 2\\ 1 & 2\\ 3 & 1\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\\ &= \begin{pmatrix}1 & 0 & 1\\ 2 & 1 & 3\end{pmatrix}\begin{pmatrix}2 & 2\\ 1 & 2\\ 3 & 1\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\\ &= \begin{pmatrix}5 & 3\\ 14 & 9\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\end{aligned}$$

Exercise 1. Let $\mathbf{T}: \mathcal{V} \to \mathcal{W}$. Show that $\mathbf{T}$ is invertible if and only if $\mathcal{V} = \mathcal{W}$ and there is a transformation $\mathbf{T}^{-1}: \mathcal{W} \to \mathcal{V}$ such that

$$\mathbf{T}^{-1}\mathbf{T} = \mathbf{T}\mathbf{T}^{-1} = \mathbf{I} \tag{2.29}$$

Exercise 2. Suppose $\mathbf{G}$ and $\mathbf{T}$ of (2.26) are invertible. Show that

$$(\mathbf{GT})^{-1} = \mathbf{T}^{-1}\mathbf{G}^{-1} \tag{2.30}$$

The composition (or product) of two transformations has two nasty characteristics. First, unlike scalars, transformations usually **do not commute;** that is, $\mathbf{GT} \neq \mathbf{TG}$. As illustrated in Example 9, $\mathbf{G}$ and $\mathbf{T}$ generally do not even act on the same vector space, and $\mathbf{TG}$ has no meaning. Even if $\mathbf{G}$ and $\mathbf{T}$ both act on the same space, we must not expect commutability, as demonstrated by the following matrix multiplications:

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Commutable operators do exist. In fact, since any operator commutes with itself, we can write $\mathbf{G}^2$, as we do in Example 10 below, without being ambiguous. Operators which commute act much like scalars in their behavior toward each other (see P&C 4.29).

If two scalars satisfy $ab = 0$, then either $a = 0$, $b = 0$, or both. The second matrix multiplication above demonstrates that this property does not extend even to simple transformations. This second difficulty with the composition of transformations is sometimes called the existence of **divisors of zero.** If $\mathbf{GT} = \mathbf{\Theta}$ and $\mathbf{G} \neq \mathbf{\Theta}$, we cannot conclude that $\mathbf{T} = \mathbf{\Theta}$; the cancellation laws of algebra do not apply to transformations. The difficulty lies in the fact that for transformations there is a "gray" region between being invertible and being zero. The range of $\mathbf{T}$ can lie in the nullspace of $\mathbf{G}$.

Example 10. Linear Combination and Composition of Transformations. The space $\mathcal{C}^n(a, b)$ consists in all functions with continuous nth derivatives on $[a, b]$. Define $\mathbf{G}: \mathcal{C}^n(a, b) \rightarrow \mathcal{C}^{n-1}(a, b)$ by $\mathbf{Gf} \triangleq \mathbf{f}'$ for all $\mathbf{f}$ in $\mathcal{C}^n(a, b)$. Then $\mathbf{G}^2: \mathcal{C}^2(a, b) \rightarrow \mathcal{C}(a, b)$ is well defined. Let $\mathbf{U}: \mathcal{C}^2(a, b) \rightarrow \mathcal{C}(a, b)$ be defined by $(\mathbf{Uf})(t) \triangleq \mathbf{f}(t) + 0.01\mathbf{f}^3(t)$ for all $\mathbf{f}$ in $\mathcal{C}^2(a, b)$ and all t in $[a, b]$. The transformation $\mathbf{L}$ of Example 8 can be described by $\mathbf{L} \triangleq \mathbf{G}^2 + \alpha\mathbf{U}$.

As demonstrated by the above examples, the domain and range of definition are essential parts of the definition of a transformation. This importance is emphasized by the notation $\mathbf{T}: \mathcal{V} \rightarrow \mathcal{W}$. The spaces $\mathcal{V}$ and $\mathcal{W}$ are selected to fit the structure of the situation we wish to model. If we pick a domain that is too large, the operator will not be one-to-one. If we pick a range of definition that is too large, the operator will not be onto. Thus both $\mathcal{V}$ and $\mathcal{W}$ affect the invertibility of $\mathbf{T}$. We apply loosely the term *finite (infinite) dimensional transformation* to those transformations that act on a finite (infinite) dimensional domain.

2.4 Linear Transformations

One of the most common and useful transformations is the matrix multiplication introduced in Chapter 1. It is well suited for automatic computation using a digital computer. Let $\mathbf{A}$ be an $m \times n$ matrix. We define $\mathbf{T}: \mathcal{R}^{n \times 1} \rightarrow \mathcal{R}^{m \times 1}$ by

$$\mathbf{Tx} \triangleq \mathbf{Ax} \tag{2.3 1}$$

for all $\mathbf{x}$ in $\mathcal{R}^{n \times 1}$. We distinguish carefully between $\mathbf{T}$ and $\mathbf{A}$. $\mathbf{T}$ is not $\mathbf{A}$, but rather *multiplication* by $\mathbf{A}$. The nullspace of $\mathbf{T}$ is the set of solutions to

the matrix equation $\mathbf{Ax}=\boldsymbol{\theta}$. Even though $\mathbf{T}$ and $\mathbf{A}$ are conceptually different, we sometimes refer to the nullspace of $\mathbf{T}$ as the nullspace of $\mathbf{A}$. Similarly, we define $\text{range}(\mathbf{A}) \triangleq \text{range}(\mathbf{T})$.

Suppose $\mathbf{A}$ is square $(m = n)$ and invertible; then the equation $\mathbf{Tx} = \mathbf{Ax} = \mathbf{y}$ has a unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$ for each $\mathbf{y}$ in $\mathfrak{M}^{n\times 1}$. But $\mathbf{T}^{-1}$ is defined as precisely that transformation which associates with each $\mathbf{y}$ in $\mathfrak{M}^{n\times 1}$ the unique solution to the equation $\mathbf{Tx}=\mathbf{y}$. Therefore, $\mathbf{T}$ is invertible, and $\mathbf{T}^{-1}$: $\mathfrak{M}^{m\times 1}\to\mathfrak{M}^{n\times 1}$ is given by $\mathbf{T}^{-1}\mathbf{y} \triangleq \mathbf{A}^{-1}\mathbf{y}$.

The properties of matrix multiplication (Appendix 1) are such that $\mathbf{A}(a\mathbf{x}_1 + b\mathbf{x}_2) = a\mathbf{Ax}_1 + b\mathbf{Ax}_2$. That is, matrix multiplication preserves linear combinations. This property of matrix multiplication allows **superposition** of solutions to a matrix equation: if $\mathbf{x}_1$ solves $\mathbf{Ax}=\mathbf{y}_1$ and $\mathbf{x}_2$ solves $\mathbf{Ax}=\mathbf{y}_2$, then the solution to $\mathbf{Ax}=\mathbf{y}_1+\mathbf{y}_2$ is $\mathbf{x}_1+\mathbf{x}_2$. From one or two input-output relationships we can infer others. Many other familiar transformations preserve linear combinations and allow superposition of solutions.

Definition. The transformation $\mathbf{T}: \mathcal{V}\to\mathcal{W}$ is **linear** if

$$\mathbf{T}(a\mathbf{x}_1 + b\mathbf{x}_2) = a\mathbf{Tx}_1 + b\mathbf{Tx}_2 \tag{2.32}$$

for all vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathcal{V}$ and all scalars a and b.

Example 1. Integration. Define $\mathbf{T}: \mathcal{C}(0, 1)\to\mathcal{C}(0, 1)$ by

$$(\mathbf{Tf})(t) \triangleq \int_0^t \mathbf{f}(s)\,ds \tag{2.33}$$

for all $\mathbf{f}$ in $\mathcal{C}(0, 1)$ and all t in $[0, 1]$. The linearity of this indefinite integration operation is a fundamental fact of integral calculus; that is,

$$\int_0^t [a\mathbf{f}_1(s) + b\mathbf{f}_2(s)]\,ds = a\int_0^t \mathbf{f}_1(s)\,ds + b\int_0^t \mathbf{f}_2(s)\,ds$$

The operator (2.33) is a special case of the linear integral operator $\mathbf{T}: \mathcal{C}(a, b)\to \mathcal{C}(c, d)$ defined by

$$(\mathbf{Tf})(t) \triangleq \int_a^b k(t,s)\mathbf{f}(s)\,ds \tag{2.34}$$

for all $\mathbf{f}$ in $\mathcal{C}(a,b)$ and all t in $[c,d]$. We can substitute for the domain $\mathcal{C}(a,b)$ any other space of functions for which the integral exists. We can use any range of definition which includes the integrals (2.34) of all functions in the domain. The function k is called the **kernel** of the integral transformation. Another special case of (2.34) is $\mathbf{T}: \mathcal{L}_2(-\infty, \infty)\to\mathcal{L}_2(-\infty, \infty)$ defined by

$$(\mathbf{Tf})(t) \triangleq \int_{-\infty}^{\infty} \mathbf{g}(t-s)\mathbf{f}(s)\,ds$$

for some $\mathbf{g}$ in $\mathfrak{L}_2(-\infty, \infty)$, all $\mathbf{f}$ in $\mathfrak{L}_2(-\infty, \infty)$, and all t in $(-\infty, \infty)$. This $\mathbf{T}$ is known as the convolution of $\mathbf{f}$ with the function $\mathbf{g}$. It arises in connection with the solution of linear constant-coefficient differential equations (Appendix 2).

The integral transformation (2.34) is the analogue for function spaces of the matrix multiplication (2.31). That matrix transformation can be expressed

$$(\mathbf{Tx})_i \triangleq \sum_{j=1}^{n} \mathbf{A}_{ij}\xi_j \qquad i = 1, \ldots, m \tag{2.35}$$

for all vectors $\mathbf{x}$ in $\mathfrak{M}^{n\times 1}$. The symbol ξ_j represents the jth element of $\mathbf{x}$; the symbol $(\mathbf{Tx})_i$ means the ith element of $\mathbf{Tx}$. In (2.35) the matrix is treated as a function of two discrete variables, the row variable i and the column variable j. In analogy with the integral transformation, we call the matrix multiplication [as viewed in the form of (2.35)] a **summation transformation;** we refer to the function $\mathbf{A}$ (with values $\mathbf{A}_{ij}$) as the **kernel** of the summation transformation.

Example 2. Differentiation Define $\mathbf{D}: \mathcal{C}^1(a, b) \to \mathcal{C}(a, b)$ by

$$(\mathbf{Df})(t) \triangleq \mathbf{f}'(t) \triangleq \lim_{\Delta t \to 0} \frac{\mathbf{f}(t+\Delta t) - \mathbf{f}(t)}{\Delta t} \tag{2.36}$$

for all $\mathbf{f}$ in $\mathcal{C}^1(a,b)$ and all t in $[a,b]$; $\mathbf{f}'(t)$ is the slope of the graph of $\mathbf{f}$ at t; $\mathbf{f}'$ (or $\mathbf{Df}$) is the whole "slope" function. We also use the symbols $\dot{\mathbf{f}}$ and $\mathbf{f}^{(1)}$ in place of $\mathbf{Df}$. We can substitute for the above domain and range of definition any pair of function spaces for which the derivatives of all functions in the domain lie in the range of definition. Thus we could define $\mathbf{D}$ on $\mathcal{C}(a, b)$ if we picked a range of definition which contains the appropriate discontinuous functions. The nullspace of $\mathbf{D}$ is $\mathbf{span}\{\mathbf{1}\}$, where $\mathbf{1}$ is the function defined by $\mathbf{1}(t) = 1$ for all t in $[a,b]$. It is well known that differentiation is linear; $\mathbf{D}(c_1\mathbf{f}_1 + c_2\mathbf{f}_2) = c_1\mathbf{Df}_1 + c_2\mathbf{Df}_2$.

We can define more general differential operators in terms of (2.36). The general linear constant-coefficient differential operator $\mathbf{L}: \mathcal{C}^n(a, b) \to \mathcal{C}(a, b)$ is defined, for real scalars $\{a_i\}$, by

$$\mathbf{L} \triangleq \mathbf{D}^n + a_1\mathbf{D}^{n-1} + \cdots + a_n\mathbf{I} \tag{2.37}$$

where we have used (2.27) and (2.28) to combine transformations. A **variable-coefficient** (or "time-varying") extension of (2.37) is the operator $\mathbf{L}: \mathcal{C}^n(a, b) \to \mathcal{C}(a,b)$ defined by*

$$(\mathbf{Lf})(t) \triangleq g_0(t)\mathbf{f}^{(n)}(t) + g_1(t)\mathbf{f}^{(n-1)}(t) + \cdots + g_n(t)\mathbf{f}(t) \tag{2.37}$$

*Note that we use boldface print for some of the functions in (2.38) but not for others. As indicated in the Preface, we use boldface print only to emphasize the vector or transformation interpretation of an object. We sometimes describe the same function both ways, $\mathbf{f}$ and f.

for all $\mathbf{f}$ in $\mathcal{C}^n(a, b)$ and all t in $[a, b]$. (We have denoted the kth derivative $\mathbf{D}^k\mathbf{f}$ by $\mathbf{f}^{(k)}$.) If the interval $[a, b]$ is finite, if the functions g_i are continuous, and if $g_0(t) \neq 0$ on $[a, b]$, we refer to (2.38) as a regular nth-order differential operator. [With $g_0(t) \neq 0$, we would lose no generality by letting $g_0(t) = 1$ in (2.38).] We can apply the differential operators (2.37) and (2.38) to other function spaces than $\mathcal{C}^n(a, b)$.

Example 3. Evaluation of a Function. Define $\mathbf{T}: \mathcal{C}(a, b) \to \mathcal{R}^1$ by

$$\mathbf{Tf} \triangleq \mathbf{f}(t_1) \tag{2.39}$$

for all $\mathbf{f}$ in the function space $\mathcal{C}(a, b)$. In this example, $\mathbf{f}$ is a dummy variable, but t_1 is not. The transformation is a *linear functional* called "evaluation at t_1." The range of $\mathbf{T}$ is $\mathcal{R}^1$; $\mathbf{T}$ is onto. The nullspace of $\mathbf{T}$ is the set of continuous functions which pass through zero at t_1. Because many functions have the same value at t_1, $\mathbf{T}$ is not one-to-one. This functional can also be defined using some other function space for its domain.

Example 4. A One-Sided Laplace Transform, $\mathfrak{L}$. Suppose $\mathcal{W}$ is the space of complex-valued functions defined on the positive-real half of the complex plane. (See Example 10, Section 2.1.) Let $\mathcal{V}$ be the space of functions which are defined and continuous on $[0, \infty]$ and for which $e^{-ct}|f(t)|$ is bounded for some constant c and all values of t greater than some finite number. We define the one-sided Laplace transform $\mathfrak{L}: \mathcal{V} \to \mathcal{W}$ by

$$(\mathfrak{L}\mathbf{f})(s) \triangleq \int_0^\infty e^{-st}\mathbf{f}(t)\,dt \tag{2.40}$$

for all complex s with $\mathbf{real}(s) > 0$. The functions in $\mathcal{V}$ are such that (2.40) converges for $\mathbf{real}(s) > 0$. We sometimes denote the transformed function $\mathfrak{L}\mathbf{f}$ by $\mathbf{F}$. This integral transform, like that of (2.34), is linear. The Laplace transform is used to convert linear constant-coefficient differential equations into linear algebraic equations. •

Exercise 1. Suppose the transformations $\mathbf{T}$, $\mathbf{U}$, and G of (2.27) and (2.28) are linear and $\mathbf{T}$ is invertible. Show that the transformations $a\mathbf{T} + b\mathbf{U}$, $\mathbf{GT}$, and $\mathbf{T}^{-1}$ are also linear.

Exercise 2. Let $\mathcal{V}$ be an n-dimensional linear space with basis $\mathcal{X}$. Define $\mathbf{T}: \mathcal{V} \to \mathcal{M}^{n \times 1}$ by

$$\mathbf{Tx} \triangleq [\mathbf{x}]_{\mathcal{X}} \tag{2.41}$$

Show that $\mathbf{T}$, the process of taking coordinates, is a linear, invertible transformation.

*It can be shown that $[\mathfrak{L}(\mathbf{Df})](s) = s(\mathfrak{L}\mathbf{f})(s) - \mathbf{f}(0^+)$, where $\mathbf{f}(0^+)$ is the limit of $\mathbf{f}(t)$ as $t \to 0$ from the positive side of 0.

The vector space $\mathcal{V}$ of Exercise 2 is equivalent to $\mathcal{M}^{n\times 1}$ in every sense we might wish. The linear, invertible transformation is the key. We say two vector spaces $\mathcal{V}$ and $\mathcal{W}$ are **isomorphic** (or equivalent) if there exists an invertible linear transformation from $\mathcal{V}$ into $\mathcal{W}$. Each real n-dimensional vector space is isomorphic to each other real n-dimensional space and, in particular, to the real space $\mathcal{M}^{n\times 1}$. A similar statement can be made using complex scalars for each space. Infinite-dimensional spaces also exhibit isomorphism. In Section 5.3 we show that all well behaved infinite-dimensional spaces are isomorphic to l_2.

Nullpace and Range—Keys to Invertibility

Even *linear* transformations may have troublesome properties. In point of fact, the example in which we demonstrate *noncommutability* and *noncancellation* of products of transformations uses linear transformations (matrix multiplications). Most difficulties with a linear transformation can be understood through investigation of the range and nullspace of the transformation.*

Let $\mathbf{T}: \mathcal{V}\rightarrow\mathcal{W}$ be linear. Suppose $\mathbf{x}_h$ is a vector in the nullspace of $\mathbf{T}$ (any solution to $\mathbf{Tx} = \boldsymbol{\theta}$); we call $\mathbf{x}_h$ a homogeneous solution for the transformation $\mathbf{T}$. Denote by $\mathbf{x}_p$ a particular solution to the equation $\mathbf{Tx} = \mathbf{y}$. (An $\mathbf{x}_p$ exists if and only if $\mathbf{y}$ is in range(T).) Then $\mathbf{x}_p + \alpha\mathbf{x}_h$ is also a solution to $\mathbf{Tx} = \mathbf{y}$ for any scalar α. One of the most familiar uses of the principle of superposition is in obtaining the general solution to a linear differential equation by combining particular and homogeneous solutions. The general solution to any linear operator equation can be obtained in this manner.

Example 5. The General Solution to a Matrix Equation. Define the linear operator $\mathbf{T}: \mathcal{M}^{2\times 1}\rightarrow\mathcal{M}^{2\times 1}$ by

$$\mathbf{T}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \triangleq \begin{pmatrix}2 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$$

Then the equation

$$\mathbf{Tx} = \begin{pmatrix}2 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}2\\ 2\end{pmatrix} \triangleq \mathbf{y} \tag{2.42}$$

has as its general solution $\mathbf{x} = (\xi_1\ 2\ -2\xi_1)$. A particular solution is $\mathbf{x}_p = (1\ 0)^T$. The nullspace of $\mathbf{T}$ consists in the vector $\mathbf{x}_h = (-1\ 2)^T$ and all its multiples. The general solution can be expressed as $\mathbf{x} = \mathbf{x}_p + \alpha\mathbf{x}_h$ where α is arbitrary. Figure 2.7 shows an

*See Sections 4.4 and 4.6 for further insight into noncancellation and noncommutability of linear operators.

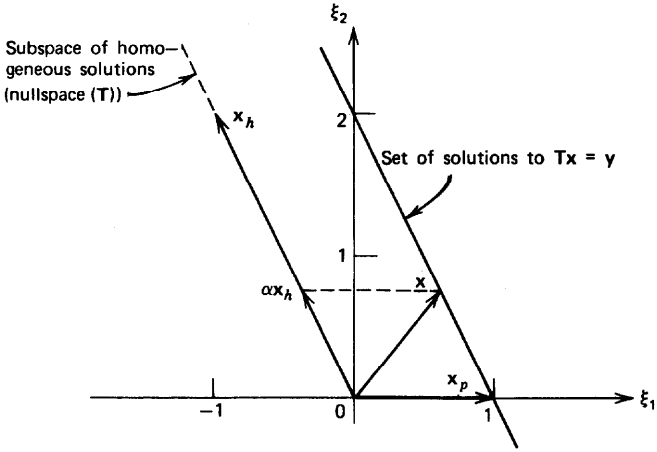

Figure 2.7. Solutions to the linear equation of Example 5.

arrow-space equivalent of these vectors. The nullspace of $\mathbf{T}$ is a subspace of $\mathcal{R}^{2\times 1}$. The general solution (the set of all solutions to $\mathbf{Tx} = \mathbf{y}$) consists of a line in $\mathcal{R}^{2\times 1}$; specifically, it is the nullspace of $\mathbf{T}$ shifted by the addition of any particular solution.

The nullspace of a linear transformation is always a subspace of the domain $\mathcal{V}$. The freedom in the general solution to $\mathbf{Tx} = \mathbf{y}$ lies only in **nullspace(T)**, the subspace of homogeneous solutions. For if $\hat{\mathbf{x}}_p$ is another particular solution to $\mathbf{Tx} = \mathbf{y}$, then

$$\mathbf{T}(\mathbf{x}_p - \hat{\mathbf{x}}_p) = \mathbf{T}\mathbf{x}_p - \mathbf{T}\hat{\mathbf{x}}_p = \mathbf{y} - \mathbf{y} = \boldsymbol{\theta}$$

The difference between $\mathbf{x}_p$ and $\hat{\mathbf{x}}_p$ is a vector in **nullspace(T)**. If **nullspace(T)** = $\boldsymbol{\theta}$, there is no freedom in the solution to $\mathbf{Tx} = \mathbf{y}$; it is unique.

Definition. A transformation $\mathbf{G}: \mathcal{V} \to \mathcal{W}$ is nonsingular if **nullspace(G)** = $\boldsymbol{\theta}$.

Exercise 3. Show that a *linear* transformation is one-to-one if and only if it is nonsingular.

Because a linear transformation $\mathbf{T}: \mathcal{V} \to \mathcal{W}$ preserves linear combinations, it necessarily transforms $\boldsymbol{\theta}_{\mathcal{V}}$ into $\boldsymbol{\theta}_{\mathcal{W}}$. Furthermore, $\mathbf{T}$ acts on the vectors in $\mathcal{V}$ by subspaces—whatever $\mathbf{T}$ does to $\mathbf{x}$ it does also to $c\mathbf{x}$, where c is any scalar. The set of vectors in $\mathcal{V}$ which are taken to zero, for example, is the subspace which we call **nullspace(T)**. Other subspaces of $\mathcal{V}$ are "rotated" or "stretched" by $\mathbf{T}$. This fact becomes more clear during our discussion of spectral decomposition in Chapter 4.

Example 6. The Action of a Linear Transformation on Subspaces. Define $\mathbf{T}$: $\mathcal{R}^3 \rightarrow \mathcal{R}^2$ by $\mathbf{T}(\xi_1, \xi_2, \xi_3) \triangleq (\xi_3, 0)$. The set $\{\mathbf{x}_1 = (1,0,0), \mathbf{x}_2 = (0,1,0)\}$ forms a basis for **nullspace(T)**. By adding a third independent vector, say, $\mathbf{x}_3 = (1, 1, 1)$, we obtain a basis for the domain $\mathcal{R}^3$. The subspace spanned by $\{\mathbf{x}_1, \mathbf{x}_2\}$ is annihilated by $\mathbf{T}$. The subspace spanned by $\{\mathbf{x}_3\}$ is transformed by $\mathbf{T}$ into a subspace of $\mathcal{R}^2$—the range of $\mathbf{T}$. The vector $\mathbf{x}_3$ itself is transformed into a basis for **range(T)**. Because $\mathbf{T}$ acts on the vectors in $\mathcal{R}^3$ by subspaces, the dimension of **nullspace(T)** is a measure of the degree to which $\mathbf{T}$ acts like zero; the dimension of **range(T)** indicates the degree to which $\mathbf{T}$ acts invertible. Specifically, of the three dimensions in $\mathcal{R}^3$, $\mathbf{T}$ takes two to zero. The third dimension of $\mathcal{R}^3$ is taken into the one-dimensional **range(T)**.

The characteristics exhibited by Example 6 extend to any linear transformation on a finite-dimensional space, Let $\mathbf{T}$: $\mathcal{V} \rightarrow \mathcal{W}$ be linear with $\dim(\mathcal{V}) = n$. We call the dimension of **nullspace(T)** the **nullity of T**. The **rank of T** is the dimension of **range(T)**. Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be a basis for **nullspace(T)**. Pick vectors $\{\mathbf{x}_{k+1}, \ldots, \mathbf{x}_n\}$ which extend the basis for **nullspace(T)** to a basis for $\mathcal{V}$ (P&C 2.9). We show that $\mathbf{T}$ takes $\{\mathbf{x}_{k+1}, \ldots, \mathbf{x}_n\}$ into a basis for **range(T)**. Suppose $\mathbf{x} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$ is an arbitrary vector in $\mathcal{V}$. The linear transformation $\mathbf{T}$ annihilates the first k components of $\mathbf{x}$. Only the remaining $n-k$ components are taken into **range(T)**. Thus the vectors $\{\mathbf{T}\mathbf{x}_{k+1}, \ldots, \mathbf{T}\mathbf{x}_n\}$ must span **range(T)**. To show that these vectors are independent, we use the test (2.11):

$$\xi_{k+1}(\mathbf{T}\mathbf{x}_{k+1}) + \cdots + \xi_n(\mathbf{T}\mathbf{x}_n) = \boldsymbol{\theta}_{\mathcal{W}}$$

Since $\mathbf{T}$ is linear,

$$\mathbf{T}(\xi_{k+1}\mathbf{x}_{k+1} + \cdots + \xi_n\mathbf{x}_n) = \boldsymbol{\theta}_{\mathcal{W}}$$

Then $\xi_{k+1}\mathbf{x}_{k+1} + \cdots + \xi_n\mathbf{x}_n$ is in **nullspace(T)**, and

$$\xi_{k+1}\mathbf{x}_{k+1} + \cdots + \xi_n\mathbf{x}_n = d_1\mathbf{x}_1 + \cdots + d_k\mathbf{x}_k$$

for some $\{d_i\}$. The independence of $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ implies $d_1 = \cdots = d_k = \xi_{k+1} = \ldots = \xi_n = 0$; thus $\{\mathbf{T}\mathbf{x}_{k+1}, \ldots, \mathbf{T}\mathbf{x}_n\}$ is an independent set and is a basis for **range(T)**.

We have shown that a linear transformation T acting on a finite-dimensional space $\mathcal{V}$ obeys a "conservation of dimension" law:

$$\dim(\mathcal{V}) = \mathbf{rank(T)} + \mathbf{nullity(T)} \tag{2.43}$$

Nullity(T) is the "dimension" annihilated by $\mathbf{T}$. **Rank(T)** is the "dimension" $\mathbf{T}$ retains. If **nullspace(T)** = $\{\boldsymbol{\theta}\}$, then **nullity(T)** = 0 and **rank(T)** = $\dim(\mathcal{V})$. If, in addition, $\dim(\mathcal{W}) = \dim(\mathcal{V})$, then **rank(T)** = $\dim(\mathcal{W})$ ($\mathbf{T}$ is

onto), and $\mathbf{T}$ is invertible. A linear $\mathbf{T}: \mathcal{V} \rightarrow \mathcal{W}$ cannot be invertible unless $\dim(\mathcal{W}) = \dim(\mathcal{V})$.

We sometimes refer to the vectors $\mathbf{x}_{k+1}, \ldots, \mathbf{x}_n$ as **progenitors of the range of T**. Although the nullspace and range of $\mathbf{T}$ are unique, the space spanned by the progenitors is not; we can add any vector in nullspace to any progenitor without changing the basis for the range (see Example 6).

The Near Nullpace

In contrast to mathematical analysis, mathematical *computation* is not clear-cut. For example, a set of equations which is mathematically invertible can be so "nearly singular" that the inverse cannot be computed to an acceptable degree of precision. On the other hand, because of the finite number of significant digits used in the computer, a mathematically singular system will be indistinguishable from a "nearly singular" system. The phenomenon merits serious consideration.

The matrix operator of Example 5 is singular. Suppose we modify the matrix slightly to obtain the nonsingular, but "nearly singular" matrix equation

$$\begin{pmatrix} 2 & 1 \\ 2 & 1+\epsilon \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{2.44}$$

where ϵ is small. Then the arrow space diagram of Figure 2.7 must also be modified to show a pair of almost parallel lines. (Figure 1.7 of Section 1.5 is the arrow space diagram of essentially this pair of equations.) Although the solution (the intersection of the nearly parallel lines) is unique, it is difficult to compute accurately; the nearly singular equations are very ill conditioned. Slight errors in the data and roundoff during computing lead to significant uncertainty in the computed solution, even if the computation is handled carefully (Section 1.5). The uncertain component of the solution lies essentially in the nullspace of the operator; that is, it is almost parallel to the nearly parallel lines in the arrow-space diagram. The above pair of nearly singular algebraic equations might represent a nearly singular system. On the other hand, the underlying system might be precisely singular; the equations in the model of a singular system may be only *nearly singular* because of inaccuracies in the data. Regardless of which of these interpretations is correct, determining the "near nullspace" of the matrix is an important part of the analysis of the system. If the underlying system is singular, a description of the near nullspace is a description of the *freedom* in the solutions for the system. If the underlying system is just nearly singular, a description of the near nullspace is a description of the *uncertainty* in the solution.

Definition. Suppose $\mathbf{T}$ is a *nearly singular* linear operator on a vector space $\mathcal{V}$. We use the term **near nullspace of T** to mean those vectors that are taken *nearly* to zero by $\mathbf{T}$; that is, those vectors which $\mathbf{T}$ drastically reduces in "size."*

In the two-dimensional example described above, the near nullspace consists in vectors which are *nearly* parallel to the vector $\mathbf{x} = (-1\ 2)^T$. The near nullspace of $\mathbf{T}$ is *not a subspace* of $\mathcal{V}$. Rather, it consists in a set of vectors which are *nearly* in a subspace of $\mathcal{V}$. We can think of the near nullspace as a "fuzzy" subspace of $\mathcal{V}$.

We now present a method, referred to as **inverse iteration,** for describing the near nullspace of a nearly singular operator $\mathbf{T}$ acting on a vector space $\mathcal{V}$. Let $\mathbf{x}_0$ be an arbitrary vector in $\mathcal{V}$. Assume $\mathbf{x}_0$ contains a component which is in the near nullspace of $\mathbf{T}$. (If it does not, such a component will be introduced by roundoff during the ensuing computation.) Since $\mathbf{T}$ reduces such components drastically, compared to its effect on the other components of $\mathbf{x}_0$, $\mathbf{T}^{-1}$ must drastically emphasize such components. Therefore, if we solve $\mathbf{T}\mathbf{x}_1 = \mathbf{x}_0$ (in effect determining $\mathbf{x}_1 = \mathbf{T}^{-1}\mathbf{x}_0$), the computed solution $\mathbf{x}_1$ contains a *significant* component in the near nullspace of $\mathbf{T}$. (This component is the error vector which appears during the solution of the nearly singular equation.) The inverse iteration method consists in iteratively solving $\mathbf{T}\mathbf{x}_{k+1} = \mathbf{x}_k$. After a few iterations, $\mathbf{x}_k$ is dominated by its near-nullspace component; we use $\mathbf{x}_k$ as a partial basis for the near nullspace of $\mathbf{T}$. (The number of iterations required is at the discretion of the analyst. We are not looking for a precisely defined subspace, but rather, a subspace that is fuzzy.) By repeating the above process for several different starting vectors $\mathbf{x}_0$, we usually obtain a set of vectors which spans the near nullspace of $\mathbf{T}$.

Example 7. Describing a Near Nullspace. Define a linear operator $\mathbf{T}$ on $\mathcal{M}^{2\times 1}$ by means of the nearly singular matrix multiplication described above:

$$\mathbf{T}\mathbf{x} \triangleq \begin{pmatrix} 2 & 1 \\ 2 & 1+\epsilon \end{pmatrix}\mathbf{x}$$

For this simple example we can invert $\mathbf{T}$ explicitly

$$\mathbf{T}^{-1}\mathbf{x} = \frac{1}{2\epsilon}\begin{pmatrix} 1+\epsilon & -1 \\ -2 & 2 \end{pmatrix}\mathbf{x}$$

We apply the inverse iteration method to the vector $\mathbf{x}_0 = (1\ 1)^T$; o f course, we have no roundoff in our computations:

$$\mathbf{x}_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \frac{1}{2\epsilon}\begin{pmatrix} (1+\epsilon)/2 \\ -1 \end{pmatrix}, \quad \mathbf{x}_3 = \frac{1}{(2\epsilon)^2}\begin{pmatrix} (\epsilon^2+2\epsilon+3)/2 \\ -(\epsilon+3) \end{pmatrix}, \ldots$$

*In Section 4.2 we describe the near nullspace more precisely as the eigenspace for the smallest eigenvalue of $\mathbf{T}$.

If ϵ is small, say $\epsilon = 0.01$, then

$$\mathbf{x}_2 = 50\begin{pmatrix} 0.505 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_3 = (50)^2\begin{pmatrix} 1.51 \\ -3.01 \end{pmatrix}$$

After only three iterations, the sequence $\mathbf{x}_k$ has settled; the vector $\mathbf{x}_3$ provides a good description of the near nullspace of $\mathbf{T}$. If $\epsilon = 0$, $\mathbf{T}$ is singular; $\mathbf{x}_3$ lies almost in the nullspace of this singular operator (Figure 2.7). Were we to try other starting vectors $\mathbf{x}_0$, we would obtain other vectors $\mathbf{x}_k$ nearly parallel to $(-1\ 2)^T$. This near nullspace of $\mathbf{T}$ should be considered one-dimensional.

We note from Example 7 that the vector $\mathbf{x}_k$ in the inverse iteration grows drastically in size. Practical computer implementations of inverse iteration include normalization of $\mathbf{x}_k$ at each step in order to avoid numbers too large for the computer. A description for a two-dimensional near nullspace is sought in P&C 2.26. In Section 4.2 we analyze the inverse iteration more precisely in terms of eigenvalues and eigenvectors. Forsythe [2.3] gives some interesting examples of the treatment of nearly singular operators.

The Role of Linear Transformations

The purpose of modeling a system is to develop insight concerning the system, to develop an intuitive feel for the input-output relationship. In order to decide whether or not a particular model, linear or nonlinear, is a good model, we must compare the input-output relationship of the model with the corresponding, but measurable, input-output relationship of the system being modeled. If the model and the system are sufficiently in agreement for our purposes, we need not distinguish between the system and the model.

Almost all physical systems are to some degree nonlinear. Yet most systems act in a nearly linear manner if the range of variation of the variables is restricted. For example, the current through a resistor is essentially proportional to the applied voltage if the current is not large enough to heat the resistor significantly. We are able to develop adequate models for a wide variety of static and dynamic physical systems using only linear transformations. For linear models there is available a vast array of mathematical results; most mathematical analysis is linear analysis. Furthermore, the analysis or optimization of a *nonlinear* system is usually based on linearization (Chapters 7 and 8). Even in solving a nonlinear equation for a given input, we typically must resort to repetitive linearization.

The examples and exercises of this section have demonstrated the variety of familiar transformations which are linear: matrix multiplication, differentiation, integration, etc. We introduce other linear transformations

as we need them. The next few chapters pertain only to linear transformations. In Chapter 3 we focus on the peculiarities of linear differential systems. In Chapter 4 we develop the concepts of spectral decomposition of linear systems. The discussion of infinite-dimensional systems in Chapter 5 is also directed toward linear systems. Because we use the symbols $\mathbf{T}$ and $\mathbf{U}$ so much in reference to linear transformations, hereinafter we employ the symbols $\mathbf{F}$ and $\mathbf{G}$ to emphasize concepts which apply as well to nonlinear transformations. We begin to examine nonlinear concepts in Chapter 6. We do not return fully to the subject of nonlinear systems, however, until we introduce the concepts of linearization and repetitive linearization in Chapters 7 and 8.

2.5 Matrices of Linear Transformations

By the process of picking an ordered basis for an n-dimensional vector space $\mathcal{V}$, we associate with each vector in $\mathcal{V}$ a unique $n \times 1$ column matrix. In effect, we convert the vectors in $\mathcal{V}$ into an equivalent set of vectors which are suitable for matrix manipulation and, therefore, automatic computation by computer. By taking coordinates, we can also convert a linear equation, $\mathbf{Tx} = \mathbf{y}$, into a matrix equation. Suppose $\mathbf{T}: \mathcal{V} \rightarrow \mathcal{W}$ is a linear transformation, $\dim(\mathcal{V}) = n$, and $\dim(\mathcal{W}) = m$. Pick as bases for $\mathcal{V}$ and $\mathcal{W}$ the sets $\mathcal{X} \triangleq \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ and $\mathcal{Y} \triangleq \{\mathbf{y}_1, \ldots, \mathbf{y}_m\}$, respectively. The vectors $\mathbf{x}$ in $\mathcal{V}$ and $\mathbf{Tx}$ in $\mathcal{W}$ can be represented by their coordinate matrices $[\mathbf{x}]_{\mathcal{X}}$ and $[\mathbf{Tx}]_{\mathcal{Y}}$. The vectors $\mathbf{x}$ and $\mathbf{Tx}$ are linearly related (by the linear transformation $\mathbf{T}$). By (2.41), we know that a vector and its coordinates are also linearly related. Therefore, we expect $[\mathbf{x}]_{\mathcal{X}}$ and $[\mathbf{Tx}]_{\mathcal{Y}}$ to be linearly related as well. Furthermore, we intuitively expect the linear relation between the $n \times 1$ matrix $[\mathbf{x}]_{\mathcal{X}}$ and the $m \times 1$ matrix $[\mathbf{Tx}]_{\mathcal{Y}}$ to be multiplication by an $m \times n$ matrix. We denote this matrix by $[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}$ and refer to it as the **matrix of T relative to the ordered bases $\mathcal{X}$ and $\mathcal{Y}$**; it must satisfy

$$[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}[\mathbf{x}]_{\mathcal{X}} \triangleq [\mathbf{Tx}]_{\mathcal{Y}} \tag{2.45}$$

for all x in $\mathcal{V}$. Assume we can find such a matrix. Then by taking coordinates (with respect to $\mathcal{Y}$) of each side of the linear equation $\mathbf{Tx} = \mathbf{y}$, we convert the equation to the equivalent matrix equation.

$$[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}[\mathbf{x}]_{\mathcal{X}} = [\mathbf{y}]_{\mathcal{Y}} \tag{2.46}$$

We will show that we can represent any linear transformation of $\mathcal{V}$ into $\mathcal{W}$ by a matrix multiplication by selecting bases for $\mathcal{V}$ and $\mathcal{W}$—we can

convert any linear equation involving finite-dimensional vector spaces into a matrix equation. We first show how to determine the matrix of $\mathbf{T}$, then we show that it satisfies the defining equation (2.45) for all vectors $\mathbf{x}$ in $\mathcal{V}$.

Example 1. Determining the Matrix of a Linear Transformation Let $\mathbf{x} = (\xi_1, \xi_2, \xi_3)$, an arbitrary vector in $\mathcal{R}^3$. Define $\mathbf{T}: \mathcal{R}^3 \to \mathcal{R}^2$ by

$$\mathbf{T}(\xi_1, \xi_2, \xi_3) \triangleq (2\xi_2 - \xi_1, \xi_1 + \xi_2 + \xi_3)$$

We now find $[\mathbf{T}]_{\mathcal{E}_3 \mathcal{E}_2}$, where $\mathcal{E}_3$ and $\mathcal{E}_2$ are the standard bases for $\mathcal{R}^3$ and $\mathcal{R}^2$, respectively. By (2.45), we have

$$[\mathbf{T}]_{\mathcal{E}_3 \mathcal{E}_2}[(\xi_1, \xi_2, \xi_3)]_{\mathcal{E}_3} = [(2\xi_2 - \xi_1, \xi_1 + \xi_2 + \xi_3)]_{\mathcal{E}_2}$$

for all vectors (ξ_1, ξ_2, ξ_3), or

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 2\xi_2 - \xi_1 \\ \xi_1 + \xi_2 + \xi_3 \end{pmatrix} \tag{2.47}$$

where we have used $\{a_{ij}\}$ to represent the elements of $[\mathbf{T}]_{\mathcal{E}_3 \mathcal{E}_2}$. By making three independent choices of the scalars ξ_1, ξ_2, and ξ_3, we could convert this matrix equation into six equations in the six unknowns $\{a_{ij}\}$. However, by using a little ingenuity, we reduce this effort. Think of the matrix multiplication in terms of the columns of $[\mathbf{T}]_{\mathcal{E}_3 \mathcal{E}_2}$. The ith element of $[\mathbf{x}]_{\mathcal{E}_3}$ multiplies the ith column of $[\mathbf{T}]_{\mathcal{E}_3 \mathcal{E}_2}$. If we choose $\mathbf{x} = (1, 0, 0)$, then $[(1, 0, 0)]_{\mathcal{E}_3} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and (2.47) becomes

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

We have found the first column of $[\mathbf{T}]_{\mathcal{E}_3 \mathcal{E}_2}$ directly. We obtain the other two columns of $[\mathbf{T}]_{\mathcal{E}_3 \mathcal{E}_2}$ from (2.47) by successive substitution of $\mathbf{x} = (0, 1, 0)$ and $\mathbf{x} = (0, 0, 1)$. The result is

$$[\mathbf{T}]_{\mathcal{E}_3 \mathcal{E}_2} = \begin{pmatrix} -1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

In Example 1 we avoided the need for simultaneous equations by substituting the basis vectors $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, and $\boldsymbol{\varepsilon}_3$ into (2.47) to pick out the columns of $[\mathbf{T}]_{\mathcal{E}_3 \mathcal{E}_2}$. This same technique can be used to find the matrix of any linear transformation acting on a finite-dimensional space. We refer again to $\mathbf{T}: \mathcal{V} \to \mathcal{W}$, with $\dim(\mathcal{V}) = n$, $\dim(\mathcal{W}) = m$, $\mathcal{X}$ a basis for $\mathcal{V}$, and $\mathcal{Y}$ a basis for $\mathcal{W}$. If we substitute into (2.45) the vector $\mathbf{x}_i$, the ith vector of

the basis $\mathcal{X}$, we pick out the ith column of $[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}$:

$$[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}[\mathbf{x}_i]_{\mathcal{X}} = [\mathbf{T}]_{\mathcal{X}\mathcal{Y}} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} = i\text{th column of } [\mathbf{T}]_{\mathcal{X}\mathcal{Y}} = [\mathbf{T}\mathbf{x}_i]_{\mathcal{Y}}$$

We can find each column of $[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}$ independently. The only computational effort is that in determining the coordinate matrices $[\mathbf{T}\mathbf{x}_i]_{\mathcal{Y}}$. Therefore,

$$[\mathbf{T}]_{\mathcal{X}\mathcal{Y}} = ([\mathbf{T}\mathbf{x}_1]_{\mathcal{Y}} \vdots [\mathbf{T}\mathbf{x}_2]_{\mathcal{Y}} \vdots \cdots \vdots [\mathbf{T}\mathbf{x}_n]_{\mathcal{Y}}) \tag{2.48}$$

Example 2. The Matrix of a Linear Operator. Define the differential operator $\mathbf{D}: \mathcal{P}^3 \to \mathcal{P}^3$ as in (2.36). The set $\mathcal{R} \triangleq \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$, where $\mathbf{f}_1(t) = 1$, $\mathbf{f}_2(t) = t$, $\mathbf{f}_3(t) = t^2$, is a natural basis for $\mathcal{P}^3$. We use (2.48) to find

$$\begin{aligned} [\mathbf{D}]_{\mathcal{R}\mathcal{R}} &= ([\mathbf{D}\mathbf{f}_1]_{\mathcal{R}} \vdots [\mathbf{D}\mathbf{f}_2]_{\mathcal{R}} \vdots [\mathbf{D}\mathbf{f}_3]_{\mathcal{R}}) \\ &= ([\boldsymbol{\theta}]_{\mathcal{R}} \vdots [\mathbf{f}_1]_{\mathcal{R}} \vdots [2\mathbf{f}_2]_{\mathcal{R}}) \\ &= \begin{pmatrix} 0 \vdots 1 \vdots 0 \\ 0 \vdots 0 \vdots 2 \\ 0 \vdots 0 \vdots 0 \end{pmatrix} \end{aligned}$$

From the method used to determine $[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}$ in (2.48), we know that this matrix correctly represents the action of $\mathbf{T}$ on the basis vectors $\{\mathbf{x}_i\}$. We now show that the matrix (2.48) also represents correctly the action of $\mathbf{T}$ on all other vectors in $\mathcal{V}$. An arbitrary vector $\mathbf{x}$ in $\mathcal{V}$ may be written in terms of the basis vectors for $\mathcal{V}$:

$$\mathbf{x} = \sum_{i=1}^{n} \xi_i \mathbf{x}_i$$

Since the transformation $\mathbf{T}$ is linear,

$$\mathbf{T}\mathbf{x} = \sum_{i=1}^{n} \xi_i \mathbf{T}\mathbf{x}_i$$

Because the process of taking coordinates is linear [see (2.41)],

$$[\mathbf{Tx}]_{\mathcal{Y}} = \sum_{i=1}^{n} \xi_i [\mathbf{Tx}_i]_{\mathcal{Y}}$$

$$= ([\mathbf{Tx}_1]_{\mathcal{Y}} \vdots \cdots \vdots [\mathbf{Tx}_n]_{\mathcal{Y}}) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$$

$$= [\mathbf{T}]_{\mathcal{X}\mathcal{Y}} [\mathbf{x}]_{\mathcal{X}}$$

Thus, continuing Example 2 above, if $\mathbf{f}$ is the arbitrary vector defined by $\mathbf{f}(t) \triangleq \xi_1 + \xi_2 t + \xi_3 t^2$, then

$$(\mathbf{Df})(t) = \xi_2 + 2\xi_3 t, [\mathbf{f}]_{\mathcal{X}} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}, [\mathbf{Df}]_{\mathcal{X}} = \begin{pmatrix} \xi_2 \\ 2\xi_3 \\ 0 \end{pmatrix}, \text{ and } [\mathbf{D}]_{\mathcal{X}\mathcal{X}} [\mathbf{f}]_{\mathcal{X}} = [\mathbf{Df}]_{\mathcal{X}}$$

When the domain and range space of $\mathbf{T}$ are identical, and the same basis is used for both spaces (as it is in Example 2), we sometimes refer to the matrix $[\mathbf{T}]_{\mathcal{X}\mathcal{X}}$ as the **matrix of the operator $\mathbf{T}$ relative to the basis $\mathcal{X}$.**

We expect the matrix of a linear transformation to possess the basic characteristics of that transformation. The only basic characteristics of a linear transformation that we have discussed thus far are its rank and nullity. The picking of coordinate systems $\mathcal{X}$ and $\mathcal{Y}$ converts the transformation equation $\mathbf{Tx} = \mathbf{y}$ to a precisely equivalent matrix equation, $[\mathbf{Tx}]_{\mathcal{Y}} = [\mathbf{T}]_{\mathcal{X}\mathcal{Y}} [\mathbf{x}]_{\mathcal{X}} = [\mathbf{y}]_{\mathcal{Y}}$; for every $\mathbf{x}$ and $\mathbf{y}$ in the one equation, there is a unique $[\mathbf{x}]_{\mathcal{X}}$ and $[\mathbf{y}]_{\mathcal{Y}}$ in the other. The dimensions of the nullspace and range of the transformation "multiplication by $[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}$" must be the same, therefore, as the dimensions of the nullspace and range of $\mathbf{T}$. We speak loosely of the rank and nullity of $[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}$ when we actually mean the rank and nullity of the transformation "multiplication by $[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}$." We refer to the nullity and rank of a matrix as if it were the matrix of a linear transformation. The nullspace and range of matrix multiplications are explored in P&C 2.19; the problem demonstrates that for an $m \times n$ matrix $\mathbf{A}$,

$$\text{rank}(\mathbf{A}) = \text{the number of independent columns of } \mathbf{A}$$

$$= \text{the number of independent rows of } \mathbf{A}$$

$$\text{nullity}(\mathbf{A}) = n - \text{rank}(\mathbf{A})$$

$$\text{nullity}(\mathbf{A}^{\mathrm{T}}) = m - \text{rank}(\mathbf{A})$$

Once again referring to Example 2, we see that the nullity of $\mathbf{D}$ is 1 [the vector $\mathbf{f}_1$ is a basis for nullspace(D)]. The nullity of $[\mathbf{D}]_{\mathcal{R}\mathcal{R}}$ is also 1 ($[\mathbf{D}]_{\mathcal{R}\mathcal{R}}$ contains one dependent column). The matrix $[\mathbf{D}]_{\mathcal{R}\mathcal{R}}$ does possess the same nullity and rank as the operator $\mathbf{D}$.

It is apparent that determination of the matrix of a transformation reduces to the determination of coordinate matrices for the set of vectors $\{\mathbf{T}\mathbf{x}_i\}$ of (2.48). We found in Section 2.2 that determination of the coordinate matrix of a vector $\mathbf{x}$ with respect to a basis $\mathcal{X} = \{\mathbf{x}_i\}$ can be reduced to performing elimination on the matrix equation (2.17):

$$[\mathbf{x}]_{\mathcal{R}} = ([\mathbf{x}_1]_{\mathcal{R}} \vdots \cdots \vdots [\mathbf{x}_n]_{\mathcal{R}})[\mathbf{x}]_{\mathcal{X}}$$

where $\mathcal{R}$ is a natural basis for the space $\mathcal{V}$ of which $\mathbf{x}$ is a member (i.e., a basis with respect to which coordinates can be determined by inspection).

Exercise 1. Show that $[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}$ of (2.48) can be obtained by the row reduction

$$\left([\mathbf{y}_1]_{\mathcal{R}} \vdots \cdots \vdots [\mathbf{y}_n]_{\mathcal{R}} \vdots [\mathbf{T}\mathbf{x}_1]_{\mathcal{R}} \vdots \cdots \vdots [\mathbf{T}\mathbf{x}_n]_{\mathcal{R}}\right) \rightarrow \left(\mathbf{I} \vdots [\mathbf{T}]_{\mathcal{X}\mathcal{Y}}\right) \tag{2.49}$$

where $\mathcal{R}$ is a natural basis for the range of definition $\mathcal{W}$. (Hint: if the elements of $[\mathbf{T}\mathbf{x}_i]_{\mathcal{Y}}$ are denoted by $[\mathbf{T}\mathbf{x}_i]_{\mathcal{Y}} = (c_{1i} \cdots c_{ni})^T$, then $\mathbf{T}\mathbf{x}_i = \sum_j c_{ji}\mathbf{y}_i$, and $[\mathbf{T}\mathbf{x}_i]_{\mathcal{R}} = \sum_j c_{ji}[\mathbf{y}_i]_{\mathcal{R}}$.) Use this approach to find $[\mathbf{T}]_{\mathcal{E}_3\mathcal{E}_2}$ of Example 1.

Example 3. The Matrix of a Matrix Transformation. Let $\mathbf{T}: \mathcal{M}^{n\times 1} \rightarrow \mathcal{M}^{m\times 1}$ be defined by $\mathbf{T}\mathbf{x} \triangleq \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is an $m \times n$ matrix. Denoting the standard bases for $\mathcal{M}^{n\times 1}$ and $\mathcal{M}^{m\times 1}$ by $\mathcal{E}_n$ and $\mathcal{E}_m$, respectively, we find $[\mathbf{T}]_{\mathcal{E}_n\mathcal{E}_m} = \mathbf{A}$. Although $[\mathbf{x}]_{\mathcal{X}}$ and $\mathbf{x}$ are identical in this example, we should distinguish between them, for it is certainly incorrect to equate the matrix $[\mathbf{T}]_{\mathcal{E}_n\mathcal{E}_m}$ to the transformation $\mathbf{T}$.

Suppose $\mathbf{T}: \mathcal{V} \rightarrow \mathcal{W}$ is invertible and linear; $\mathcal{V}$ and $\mathcal{W}$ are finite-dimensional with bases $\mathcal{X}$ and $\mathcal{Y}$, respectively. It follows from (2.45) that

$$[\mathbf{T}^{-1}]_{\mathcal{Y}\mathcal{X}}[\mathbf{y}]_{\mathcal{Y}} = [\mathbf{T}^{-1}\mathbf{y}]_{\mathcal{X}} \tag{2.50}$$

for all y in $\mathcal{W}$. Then, for each x in $\mathcal{V}$,

$$[\mathbf{x}]_{\mathcal{X}} = [\mathbf{T}^{-1}\mathbf{T}\mathbf{x}]_{\mathcal{X}} = [\mathbf{T}^{-1}]_{\mathcal{Y}\mathcal{X}}[\mathbf{T}\mathbf{x}]_{\mathcal{Y}} = [\mathbf{T}^{-1}]_{\mathcal{Y}\mathcal{X}}[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}[\mathbf{x}]_{\mathcal{X}}$$

A similar relationship can be established with T and $\mathbf{T}^{-1}$ reversed. Then as a consequence of (2.29),

$$[\mathbf{T}^{-1}]_{\mathcal{Y}\mathcal{X}} = [\mathbf{T}]_{\mathcal{X}\mathcal{Y}}^{-1} \tag{2.51}$$

Exercise 2. Suppose $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{U}$ are finite-dimensional vector spaces with bases $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$, respectively. Show that

a. If $\mathbf{T}: \mathcal{V} \to \mathcal{W}$ and U: $\mathcal{V} \to \mathcal{W}$ are linear, then

$$[a\mathbf{T} + b\mathbf{U}]_{\mathcal{X}\mathcal{Y}} = a[\mathbf{T}]_{\mathcal{X}\mathcal{Y}} + b[\mathbf{U}]_{\mathcal{X}\mathcal{Y}} \tag{2.52}$$

b. If $\mathbf{T}: \mathcal{V} \to \mathcal{W}$ and $\mathbf{U}: \mathcal{W} \to \mathcal{U}$ are linear, then

$$[\mathbf{UT}]_{\mathcal{X}\mathcal{Z}} = [\mathbf{U}]_{\mathcal{Y}\mathcal{Z}}[\mathbf{T}]_{\mathcal{X}\mathcal{Y}} \tag{2.53}$$

Changes in Coordinate System

In Chapter 4 we discuss coordinate systems which are particularly suitable for analysis of a given linear transformation—coordinate systems for which the matrix of the transformation is diagonal. In preparation for that discussion we now explore the effect of a change of coordinate system on a coordinate matrix $[\mathbf{x}]$ and on the matrix of a transformation $[\mathbf{T}]$.

Suppose $\mathcal{X}$ and $\mathcal{Z}$ are two different bases for an n-dimensional vector space $\mathcal{V}$. We know by (2.41) that the transformations

$$\mathbf{x} \to [\mathbf{x}]_{\mathcal{X}} \quad \text{and} \quad \mathbf{x} \to [\mathbf{x}]_{\mathcal{Z}}$$

are linear and invertible. Thus we expect $[\mathbf{x}]_{\mathcal{X}}$ and $[\mathbf{x}]_{\mathcal{Z}}$ to be related by

$$\mathbf{S}[\mathbf{x}]_{\mathcal{X}} = [\mathbf{x}]_{\mathcal{Z}} \tag{2.54}$$

where $\mathbf{S}$ is an $n \times n$ invertible matrix. In fact, multiplication of $[\mathbf{x}]_{\mathcal{X}}$ by any invertible matrix represents a change from the coordinate system $\mathcal{X}$ to some new coordinate system. We sometimes denote the matrix $\mathbf{S}$ of (2.54) by the symbol $\mathbf{S}_{\mathcal{X}\mathcal{Z}}$, thereby making explicit the fact that $\mathbf{S}$ converts coordinates relative to $\mathcal{X}$ into coordinates relative to $\mathcal{Z}$. Then $(\mathbf{S}_{\mathcal{X}\mathcal{Z}})^{-1} = S_{\mathcal{Z}\mathcal{X}}$.

Determination of the specific change-of-coordinates matrix $\mathbf{S}$ defined in (2.54) follows the same line of thought as that used to determine $[\mathbf{T}]$ in (2.48). By successively substituting into (2.54) the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ from the basis $\mathcal{X}$, we isolate the columns of $\mathbf{S}$: the ith column of $\mathbf{S}$ is $[\mathbf{x}_i]_{\mathcal{Z}}$. Thus the unique invertible matrix $\mathbf{S}$ which transforms coordinate matrices relative to $\mathcal{X}$ into coordinate matrices relative to $\mathcal{Z}$ is

$$\mathbf{S} = \mathbf{S}_{\mathcal{X}\mathcal{Z}} = ([\mathbf{x}_1]_{\mathcal{Z}} \vdots \cdots \vdots [\mathbf{x}_n]_{\mathcal{Z}}) \tag{2.55}$$

where the $\mathbf{x}_i$ are the vectors in the basis $\mathcal{X}$.

Since a change-of-coordinates matrix is always invertible, we determine

from (2.54) that

$$\mathbf{S}^{-1}[\mathbf{x}]_{\mathfrak{Z}} = [\mathbf{x}]_{\mathfrak{X}}$$

and

$$\mathbf{S}^{-1} = \mathbf{S}_{\mathfrak{X}\mathfrak{Z}}^{-1} = \mathbf{S}_{\mathfrak{Z}\mathfrak{X}} = ([\mathbf{z}_1]_{\mathfrak{X}} \vdots \cdots \vdots [\mathbf{z}_n]_{\mathfrak{X}}) \tag{2.56}$$

where the $\mathbf{z}_i$ are the vectors in the basis $\mathfrak{Z}$. If $\mathfrak{Z}$ is a natural basis for the space, then $\mathbf{S}$ can be found by inspection. On the other hand, if $\mathfrak{X}$ is a natural basis, we find $\mathbf{S}^{-1}$ by inspection. It is appropriate to use either (2.55) or (2.56) in determining $\mathbf{S}$. We need both $\mathbf{S}$ and $\mathbf{S}^{-1}$ to allow conversion back and forth between the two coordinate systems. Besides, the placing of $\mathbf{S}$ on the left side of (2.54) was arbitrary.

Example 4. A Change-of-Coordinates Matrix, Let $\mathfrak{S}$ be the standard basis for $\mathcal{R}^3$. Another basis for $\mathcal{R}^3$ is $\mathfrak{Z} = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$, where $\mathbf{z}_1 = (1, 1, 1)$, $\mathbf{z}_2 = (1, 1, 0)$, and $\mathbf{z}_3 = (1, 0, 0)$. Since $\mathfrak{S}$ is a natural basis for $\mathcal{R}^3$, we use (2.56) to find

$$\mathbf{S}^{-1} = ([\mathbf{z}_1]_{\mathfrak{S}} \vdots [\mathbf{z}_2]_{\mathfrak{S}} \vdots [\mathbf{z}_3]_{\mathfrak{S}}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.57}$$

A straightforward elimination (Section 1.5) yields

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \tag{2.58}$$

We note that for an arbitrary vector $\mathbf{x} = (\xi_1, \xi_2, \xi_3)$ in $\mathcal{R}^3$, $[\mathbf{x}]_{\mathfrak{S}} = (\xi_1 \ \xi_2 \ \xi_3)^{\mathrm{T}}$. By (2.54),

$$[\mathbf{x}]_{\mathfrak{Z}} = \mathbf{S}[\mathbf{x}]_{\mathfrak{S}} = (\xi_3 \quad \xi_2 - \xi_3 \quad \xi_1 - \xi_2)^{\mathrm{T}} \tag{2.59}$$

But then,

$$\begin{aligned} \mathbf{x} &= (\xi_3)\mathbf{z}_1 + (\xi_2 - \xi_3)\mathbf{z}_2 + (\xi_1 - \xi_2)\mathbf{z}_3 \\ &= (\xi_3)(1, 1, 1) + (\xi_2 - \xi_3)(1, 1, 0) + (\xi_1 - \xi_2)(1, 0, 0) \\ &= (\xi_1, \xi_2, \xi_3) \end{aligned} \tag{2.60}$$

and the validity of the change of coordinates matrix $\mathbf{S}$ is verified.

If neither $\mathfrak{X}$ nor $\mathfrak{Z}$ is a natural basis, the determination of $\mathbf{S}$ can still be systematized by the introduction of an intermediate step which does involve a natural basis.

Exercise 3. Suppose we need the change-of-coordinates matrix $\mathbf{S}$ such that $\mathbf{S}[\mathbf{x}]_{\mathcal{X}} = [\mathbf{x}]_{\mathcal{Z}}$, where neither $\mathcal{X}$ nor $\mathcal{Z}$ is a natural basis for $\mathcal{V}$. Suppose $\mathcal{N}$ is a natural basis. Show, by introducing an intermediate change to the coordinates $[\mathbf{x}]_{\mathcal{N}}$, that

$$\mathbf{S} = \left([\mathbf{z}_1]_{\mathcal{N}} \vdots \cdots \vdots [\mathbf{z}_n]_{\mathcal{N}}\right)^{-1}\left([\mathbf{x}_1]_{\mathcal{N}} \vdots \cdots \vdots [\mathbf{x}_n]_{\mathcal{N}}\right) \tag{2.61}$$

Example 5. Change of Coordinates via an Intermediate Natural Basis. Two bases for $\mathcal{P}^3$ are $\mathcal{F} \triangleq \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ and $\mathcal{G} \triangleq \{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$, where

$$\mathbf{f}_1(t) = 1, \quad \mathbf{f}_2(t) = 1 + t, \quad \mathbf{f}_3(t) = 1 + t^2$$

$$\mathbf{g}_1(t) = 1 + t, \quad \mathbf{g}_2(t) = t, \quad \mathbf{g}_3(t) = t + t^2$$

To find $\mathbf{S}$ such that $\mathbf{S}[\mathbf{f}]_{\mathcal{F}} = [\mathbf{f}]_{\mathcal{G}}$, we introduce the natural basis $\mathcal{N} \triangleq \{\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3\}$, where $\mathbf{h}_i(t) = t^{i-1}$. Then, by (2.61),

$$\begin{aligned}\mathbf{S} &= ([\mathbf{g}_1]_{\mathcal{N}} \vdots [\mathbf{g}_2]_{\mathcal{N}} \vdots [\mathbf{g}_3]_{\mathcal{N}})^{-1}([\mathbf{f}_1]_{\mathcal{N}} \vdots [\mathbf{f}_2]_{\mathcal{N}} \vdots [\mathbf{f}_3]_{\mathcal{N}}) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

Similarity and Equivalence Transformations

Now that we have a process for changing coordinate systems, we explore the effect of such a change on the matrix of a transformation. Suppose $\mathbf{T}$ is a linear operator on $\mathcal{V}$, and that $\mathcal{X}$ and $\mathcal{Z}$ are two different bases for $\mathcal{V}$. Then $[\mathbf{T}]_{\mathcal{Z}\mathcal{X}}$ is defined by

$$[\mathbf{T}]_{\mathcal{Z}\mathcal{X}}[\mathbf{x}]_{\mathcal{X}} = [\mathbf{T}\mathbf{x}]_{\mathcal{Z}}$$

The change from the $\mathcal{X}$ to the $\mathcal{Z}$ coordinate system is described by

$$\mathbf{S}[\mathbf{x}]_{\mathcal{X}} = [\mathbf{x}]_{\mathcal{Z}}$$

The change-of-coordinates matrix $\mathbf{S}$ also applies to the vector $\mathbf{T}\mathbf{x}$ in $\mathcal{V}$:

$$\mathbf{S}[\mathbf{T}\mathbf{x}]_{\mathcal{X}} = [\mathbf{T}\mathbf{x}]_{\mathcal{Z}}$$

By substituting $[\mathbf{x}]_{\mathfrak{X}}$ and $[\mathbf{Tx}]_{\mathfrak{X}}$ from these last two equations into the defining equation for [Tx],, , we find

$$[\mathbf{T}]_{\mathfrak{X}\mathfrak{X}}\mathbf{S}^{-1}[\mathbf{x}]_{\mathfrak{Z}} = \mathbf{S}^{-1}[\mathbf{Tx}]_{\mathfrak{Z}}$$

or

$$\left(\mathbf{S}[\mathbf{T}]_{\mathfrak{X}\mathfrak{X}}\mathbf{S}^{-1}\right)[\mathbf{x}]_{\mathfrak{Z}} = [\mathbf{Tx}]_{\mathfrak{Z}}$$

But this is the defining equation for $[\mathbf{T}]_{\mathfrak{Z}\mathfrak{Z}}$. It is apparent that

$$[\mathbf{T}]_{\mathfrak{Z}\mathfrak{Z}} = \mathbf{S}[\mathbf{T}]_{\mathfrak{X}\mathfrak{X}}\mathbf{S}^{-1} \tag{2.62}$$

where $\mathbf{S}$ converts from the $\mathfrak{X}$ coordinate system to the $\mathfrak{Z}$ coordinate system. Equation (2.62) describes an invertible linear transformation on $[\mathbf{T}]_{\mathfrak{X}\mathfrak{X}}$ known as a **similarity transformation.** In Section 4.2, we find that a similarity transformation preserves the basic spectral properties of the matrix. It is comforting to know that any two matrix representations of a linear system have the same properties-these properties are inherent in the model, $\mathbf{T}$, and should not be affected by the coordinate system we select.

Example 6. A Similarity Transformation. In Example 2 we found the matrix of the differential operator on $\mathcal{P}^3$ relative to the natural basis for $\mathcal{P}^3$:

$$[\mathbf{D}]_{\mathfrak{N}\mathfrak{N}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Another basis for $\mathcal{P}^3$ is $\mathcal{G} = \{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$, where $\mathbf{g}_1(t) = 1 + t$, $\mathbf{g}_2(t) = t$, and $\mathbf{g}_3(t) = t + t^2$. The change-of-coordinates matrix which relates the two bases $\mathfrak{N}$ and $\mathcal{G}$ is defined by $\mathbf{S}[\mathbf{f}]_{\mathfrak{N}} = [\mathbf{f}]_{\mathcal{G}}$; we find it using (2.56):

$$\mathbf{S}^{-1} = ([\mathbf{g}_1]_{\mathfrak{N}} \vdots [\mathbf{g}_2]_{\mathfrak{N}} \vdots [\mathbf{g}_3]_{\mathfrak{N}})$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The inverse matrix is

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Then, by (2.62),

$$[\mathbf{D}]_{\mathcal{G}\mathcal{G}} = \mathbf{S}[\mathbf{D}]_{\mathcal{X}\mathcal{X}}\mathbf{S}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise 4. Let $\mathbf{T}: \mathcal{V} \to \mathcal{W}$ be a linear transformation. Assume $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional. Let the invertible matrix $\mathbf{S}_{\mathcal{X}\mathcal{F}}$ convert from the basis $\mathcal{X}$ to the basis $\mathcal{F}$ in $\mathcal{V}$. Let the invertible matrix $\mathbf{S}_{\mathcal{Y}\mathcal{G}}$ convert from the basis $\mathcal{Y}$ to the basis $\mathcal{G}$ in $\mathcal{W}$. Show that

$$[\mathbf{T}]_{\mathcal{F}\mathcal{G}} = \mathbf{S}_{\mathcal{Y}\mathcal{G}}[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}\mathbf{S}_{\mathcal{X}\mathcal{F}}^{-1} \tag{2.63}$$

This transformation of the matrix $[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}$ is called an **equivalence transformation.** The similarity transformation (2.42) is a special case. The term "equivalence" is motivated by the fact that $[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}$ and $[\mathbf{T}]_{\mathcal{F}\mathcal{G}}$ are equivalent models of the system. The system equation $\mathbf{Tx} = \mathbf{y}$ is equally well represented by the matrix equations which result from the introduction of *any* coordinate systems for $\mathcal{V}$ and $\mathcal{W}$.

The discussion of matrices of transformations has been limited to transformations on finite-dimensional vector spaces. The primary reason for avoiding the infinite-dimensional counterparts is our inability to speak meaningfully about bases for infinite-dimensional spaces before discussing convergence of an infinite sequence of vectors (Section 5.3). However, matrices of infinite dimension are more difficult to work with (to invert, etc.) than are finite-dimensional matrices.

2.6 Problems and Comments

*2.1 Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be subsets of a vector space $\mathcal{V}$. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be subspaces of $\mathcal{V}$.

(*a*) The **intersection** $\mathcal{S}_1 \cap \mathcal{S}_2$ of the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ is the set of vectors which belong to both $\mathcal{S}_1$ and $\mathcal{S}_2$; if $\mathcal{S}_1 \cap \mathcal{S}_2$ is empty or if $\mathcal{S}_1 \cap \mathcal{S}_2 = \boldsymbol{\theta}$, we say $\mathcal{S}_1$ and $\mathcal{S}_2$ are **disjoint.**

(*b*) The **union** $\mathcal{S}_1 \cup \mathcal{S}_2$ of the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ is the set of vectors which belong either to $\mathcal{S}_1$ or to $\mathcal{S}_2$ or to both.

(*c*) The **sum** $\mathcal{S}_1 + \mathcal{S}_2$ of the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ is the set of vectors of the form $\mathbf{x}_1 + \mathbf{x}_2$, where $\mathbf{x}_1$ is in $\mathcal{S}_1$ and $\mathbf{x}_2$ is in $\mathcal{S}_2$.

(*d*) $\mathcal{W}_1 \cap \mathcal{W}_2$ is a subspace.

(*e*) $\mathcal{W}_1 \cup \mathcal{W}_2$ is usually not a subspace.

(*f*) $\mathcal{W}_1 + \mathcal{W}_2$ is the subspace spanned by $\mathcal{W}_1 \cup \mathcal{W}_2$.

(*g*) $\dim(\mathcal{W}_1) + \dim(\mathcal{W}_2) = \dim(\mathcal{W}_1 + \mathcal{W}_2) + \dim(\mathcal{W}_1 \cap \mathcal{W}_2)$.

2.2 Prove that the real 3-tuple space $\mathcal{R}^3$ introduced in Equation (2.2) is a vector space.

2.3 Determine whether or not the following sets of vectors are linearly independent:

(*a*) The column vectors $(2\ 1\ 0\ 1)^T$, $(1\ 2\ -1\ 1)^T$, and $(3\ 0\ 1\ 1)^T$ in $\mathcal{M}^{4\times 1}$

(*b*) The functions $\mathbf{f}_1(t) = 1 + 2t - t^2$, $\mathbf{f}_2(t) = 2 + 2t + t^2$, and $\mathbf{f}_3(t) = -1 + 3t + t^2$ in $\mathcal{P}^3$.

(*c*) The functions $\mathbf{g}_1(t) = 1 + 2t + t^2 - t^3$, $\mathbf{g}_2(t) = 1 + t - t^2 + t^3$, and $\mathbf{g}_3(t) = 1 + 3t + 3t^2 - 3t^3$ in $\mathcal{P}^4$.

*2.4 *Modulo-2 scalars:* data transmitted by radio or telephone usually consist in strings of binary numbers (ones and zeros). A character or number to be transmitted is represented by a binary code word of length *n*. It is a sequence of these code words which makes up the transmitted string. We can think of the set of all possible code words of length *n* as vectors in a vector space. We call the space a binary linear code (see [2.8]). The scalars used in vector space manipulations can be restricted to binary numbers if ordinary addition of scalars is replaced by **modulo-2 addition:**

$$0+0=0 \qquad 0+1=1$$
$$1+0=1 \qquad 1+1=0$$

The rules for multiplication of scalars need not be changed. One way to check for errors in data transmission is to let the *n*th element of each code word equal the sum (mod-2) of the other elements in the word. If a single error appears in the transmitted word, the *n*th element will fail to give the proper sum.

(*a*) Let $\mathcal{V}$ be the set of 5×1 matrices with the mod-2 scalars as elements. Show that $\mathcal{V}$ is a vector space. (Assume that addition and scalar multiplication of the matrices is based on the mod-2 scalars.)

(*b*) Let $\mathcal{W}$ be the subset of $\mathcal{V}$ consisting in vectors for which the fifth element equals the sum of the other four elements. Show that $\mathcal{W}$ is a subspace of $\mathcal{V}$.

(*c*) Find a basis $\mathcal{X}$ for $\mathcal{W}$. Determine $[\mathbf{x}]_{\mathcal{X}}$, where $\mathbf{x} = (1\ 1\ 0\ 1\ 1)^T$.

(*d*) The subspace $\mathcal{W}$ is a binary linear code. A code can also be described by a "parity check" matrix $\mathbf{P}$ for which the code is the nullspace. Find the parity check matrix for the code $\mathcal{W}$.

2.5 The set of all real $m \times n$ matrices, together with the usual definitions of addition and scalar multiplication of matrices, forms a vector space which we denote by $\mathcal{M}^{m \times n}$. Determine the dimension of this linear space by exhibiting a basis for the space.

*2.6 Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces. With the definition of linear combination of transformations given in (2.27),

(*a*) The set of all transformations from $\mathcal{V}$ into $\mathcal{W}$ forms a vector space.

(*b*) The set $\mathcal{L}(\mathcal{V}, \mathcal{W})$ of all linear transformations from $\mathcal{V}$ into $\mathcal{W}$ forms a subspace of the vector space in (*a*).

(*c*) The set of all linear transformations which take a particular subspace of $\mathcal{V}$ into $\boldsymbol{\theta}_{\mathcal{W}}$ constitutes a subspace of $\mathcal{L}(\mathcal{V}, \mathcal{W})$.

(*d*) If $\dim(\mathcal{V}) = n$ and $\dim(\mathcal{W}) = m$, then $\dim(\mathcal{L}(\mathcal{V}, \mathcal{W})) = mn$.

*2.7 *Exploring linear combinations by row reduction.* Let $\mathcal{Y} \triangleq \{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$ be a set of $m \times 1$ column vectors. The linear combination $\mathbf{y} = c_1\mathbf{y}_1 + \cdots + c_n\mathbf{y}_n$ can be expressed as $\mathbf{y} = \mathbf{A}\mathbf{x}$ by defining $\mathbf{A} \triangleq (\mathbf{y}_1 \vdots \mathbf{y}_2 \vdots \cdots \vdots \mathbf{y}_n)$ and $\mathbf{x} \triangleq (c_1 \cdots c_n)^{\mathrm{T}}$. Row reduction of the matrix $(\mathbf{A} \vdots \mathbf{y})$ for an unspecified vector $\mathbf{y} \triangleq (\eta_1 \cdots \eta_m)^{\mathrm{T}}$, or the equivalent row reduction of $(\mathbf{A} \vdots \mathbf{I})$ for an $m \times m$ matrix $\mathbf{I}$, determines the form of the vectors in span($\mathcal{Y}$) and pinpoints any linear dependency in the set $\mathcal{Y}$. If $\mathcal{Y}$ is linearly independent, the row reduction also determines the coordinates with respect to $\mathcal{Y}$ of each vector y in span($\mathcal{Y}$). Let

$$\mathbf{y} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad \mathbf{y}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad \mathbf{y}_2 = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad \mathbf{y}_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \quad \mathbf{y}_4 = \begin{pmatrix} 3 \\ 3 \\ 9 \end{pmatrix}$$

(*a*) Row reduce $(\mathbf{A} \vdots \mathbf{I})$.

(*b*) Determine the space spanned by $\mathcal{Y}$; that is, determine the relationships that must exist among the elements $\{\eta_i\}$ of $\mathbf{y}$ in order that $\mathbf{y}$ be some linear combination of the vectors in $\mathcal{Y}$. Determine a basis for span($\mathcal{Y}$).

(*c*) Determine which linear combinations of the vectors in $\mathcal{Y}$ equal the specific vector $\mathbf{y}$ given above.

(*d*) The form of span($\mathcal{Y}$) can also be determined by row reduction of $\mathbf{A}^{\mathrm{T}}$. The nonzero rows of the row-reduced matrix constitute a basis for span($\mathcal{Y}$). Any zero rows which appear indicate the linear dependence of the set $\mathcal{Y}$.

2.8 For the following sets of vectors, determine if $\mathbf{y}$ is in $\mathbf{span}\{\mathbf{y}_i\}$. If so, express $\mathbf{y}$ as a linear combination of the vectors $\{\mathbf{y}_i\}$.

$$(a)\quad \mathbf{y}=\begin{pmatrix}9\\3\\7\end{pmatrix},\quad \mathbf{y}_1=\begin{pmatrix}2\\1\\3\end{pmatrix},\quad \mathbf{y}_2=\begin{pmatrix}3\\1\\2\end{pmatrix},\quad \mathbf{y}_3=\begin{pmatrix}4\\1\\2\end{pmatrix}$$

$$(b)\quad \mathbf{y}=\begin{pmatrix}9\\12\\10\\10\end{pmatrix},\quad \mathbf{y}_1=\begin{pmatrix}2\\3\\4\\5\end{pmatrix},\quad \mathbf{y}_2=\begin{pmatrix}3\\4\\3\\5\end{pmatrix},\quad \mathbf{y}_3=\begin{pmatrix}4\\5\\3\\6\end{pmatrix}$$

$$(c)\quad \mathbf{y}=\begin{pmatrix}\eta_1\\\eta_2\\\eta_3\\\eta_4\end{pmatrix},\quad \mathbf{y}_1=\begin{pmatrix}1\\2\\2\\1\end{pmatrix},\quad \mathbf{y}_2=\begin{pmatrix}0\\2\\0\\1\end{pmatrix},\quad \mathbf{y}_3=\begin{pmatrix}-2\\1\\-4\\3\end{pmatrix}$$

2.9 Find a basis for the subspace of $\mathcal{P}^4$ spanned by the functions $\mathbf{f}_1(t)=1+t+2t^2$, $\mathbf{f}_2(t)=2t+t^2+t^3$, and $\mathbf{f}_3(t)=2+3t^2-t^3$. Extend the basis for the subspace to a basis for $\mathcal{P}^4$ by adding appropriate vectors to the basis.

2.10 Find the coordinate matrix of the vector $\mathbf{x}=$ (1, 1, 1) in $\mathcal{R}^3$:

(a) Relative to the basis $\mathcal{X}$ = { (1, 0, 0), (1, -1, 0), (0, 1, -1)).

(b) Relative to the basis $\mathcal{Y}$ = {(1, 1, -l), (1, -1, l), (-1, 1, 1)}.

2.11 Find the coordinate matrix of the function f relative to the basis $\mathcal{G}$ = $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$, where $\mathbf{f}(t) \triangleq t$, $\mathbf{g}_1(t) \triangleq 1+t$, $\mathbf{g}_2(t) \triangleq 1+t^2$, and $\mathbf{g}_3(t)=1-t^2$.

2.12 Find the coordinate matrix of the function $\mathbf{g}$ relative to the basis $\mathcal{F}=\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$, where $\mathbf{f}_1(t) \triangleq 1-t$, $\mathbf{f}_2(t) \triangleq 1-t^2$, $\mathbf{f}_3(t) \triangleq 1+t-t^2$, and $\mathbf{g}(t) \triangleq \xi_1+\xi_2 t+\xi_3 t^2$.

2.13 Find the coordinates of the vector x in $\mathcal{M}^{2\times 2}$ relative to the basis $\mathcal{X}$ = $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$, where

$$\mathbf{x}_1 \triangleq \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix},\quad \mathbf{x}_2 \triangleq \begin{pmatrix}1 & -1\\0 & 0\end{pmatrix},$$

$$\mathbf{x}_3 \triangleq \begin{pmatrix}0 & 0\\1 & 1\end{pmatrix},\quad \mathbf{x}_4 \triangleq \begin{pmatrix}0 & 0\\1 & -1\end{pmatrix},\quad \text{and}\quad \mathbf{x} \triangleq \begin{pmatrix}3 & -1\\-3 & 1\end{pmatrix}$$

2. 14 Let $\mathcal{P}^{2\times 2}$ denote the space of polynomial functions of the form $\mathbf{f}(s, t)=a_{11}+a_{12}s+a_{21}t+a_{22}st$. Find a basis for $\mathcal{P}^{2\times 2}$ which includes the function $\mathbf{f}_1(s, t)=2s-t-1$. Find the coordinate matrix of the general vector $\mathbf{f}$ in $\mathcal{P}^{2\times 2}$ relative to that basis.

2.15 Let $\mathcal{V}$ be the space of continuous functions. Define the forward difference operator Δ_δ: $\mathcal{V} \to \mathcal{V}$ by $(\Delta_\delta \mathbf{f})(t) \triangleq [\mathbf{f}(t+\delta) - \mathbf{f}(t)]/\delta$ for all $\mathbf{f}$ in $\mathcal{V}$ and for all t, where $\delta > 0$ is a fixed real number. Show that Δ_δ is linear.

2.16 *Financial planning:* the financial condition of a family unit at time t can be described by $\mathbf{f}(t) = \mathbf{f}(t - \delta) + a\mathbf{f}(t - \delta) + \mathbf{g}(t)$ where $\mathbf{f}(t)$ is the family savings at time t, $\mathbf{f}(t - \delta)$ is the savings at a previous time $t - \delta$, a is the interest rate per time interval δ, and $\mathbf{g}(t)$ is the deposit at time t. (No deposits occur between $t - \delta$ and t.)

(*a*) Let the time interval δ be 1 month. If we consider t only at monthly intervals, the above financial model can be expressed as the difference equation, $\mathbf{f}(k) = (1 + a)\mathbf{f}(k - 1) + \mathbf{g}(k)$. Given $\mathbf{f}(0) = \$100$, $a = 0.005$ (i.e., 6% compounded monthly), and $\mathbf{g}(k) = \$10$ for $k = 1, 2, \ldots$, determine the savings versus time over 1 year by computing $\mathbf{f}(1)$ from $\mathbf{f}(0)$, $\mathbf{f}(2)$ from $\mathbf{f}(1)$, etc. (This computation is known as "marching.")

(*b*) The above financial model can be rewritten as

$$\frac{\mathbf{f}(t) - \mathbf{f}(t-\delta)}{\delta} = \frac{a}{\delta}\mathbf{f}(t-\delta) + \frac{\mathbf{g}(t)}{\delta}$$

The quantity $b \triangleq a/\delta$ is the interest rate per unit time; $\mathbf{u}(t) \triangleq \mathbf{g}(t)/\delta$ is the deposit rate for the interval. If we let $\delta \to 0$, the model becomes a differential equation, $\dot{\mathbf{f}}(t) = b\mathbf{f}(t) + \mathbf{u}(t)$. Let $\mathbf{f}(0) = \$100$, $b = 0.005$ per month, and $\mathbf{u}(t) = \$10$ per month for $t > 0$; find the savings versus time over 1 year by solving the differential equation. Compare the result with (*a*).

(*c*) An arbitrary nonlinear time-varying differential equation with initial conditions can be approximated by a difference equation in order to obtain an approximate solution via the simple marching technique of (*a*). Approximate the differential equation of (*b*) by using the *forward-difference approximation* $\mathbf{f}(t) \approx (1/\epsilon)(\mathbf{f}(t+\epsilon) - \mathbf{f}(t))$, ϵ = 1 month, and considering t only at monthly intervals. Solve the difference equation for a 1 year period using $\mathbf{f}(0)$, b, and $\mathbf{u}(t)$ as given in (*b*). Compare the result with (*b*). How can the difference approximation be improved?

2.17 The electrostatic potential distribution within a two-dimensional charge free region satisfies Laplace's equation:

$$(\nabla^2 \mathbf{f})(s,t) \triangleq \frac{\partial^2 \mathbf{f}(s,t)}{\partial s^2} + \frac{\partial^2 \mathbf{f}(s,t)}{\partial t^2} = 0$$

For the potential distribution between two parallel plates of spacing d, the model reduces to $\mathbf{f}''(s)=0$ with $\mathbf{f}(0)$ and $\mathbf{f}(d)$ given.

(*a*) Assume the differential operator $\mathbf{D}^2$ acts on $\mathcal{C}^2(0,d)$, a space of twice-differentiable functions. Find the nullspace of $\mathbf{D}^2$, a subspace of $\mathcal{C}^2(0,d)$. The nullspace is the solution space for the above differential equation. Express the solution space in terms of the known boundary values $\mathbf{f}(0)$ and $\mathbf{f}(d)$. What is the dimension of the nullspace of $\mathbf{D}^2$?

(*b*) Define the ***central-difference operator*** $\boldsymbol{\Delta}$ on $\mathcal{C}^2(0,d)$ by

$$(\boldsymbol{\Delta}\mathbf{f})(s) \triangleq \mathbf{f}\left(s+\frac{\delta}{2}\right)-\mathbf{f}\left(s-\frac{\delta}{2}\right)$$

The derivative of $\mathbf{f}$ can be expressed as the limit of the central-difference approximation, $\mathbf{f}'(s)\approx(\boldsymbol{\Delta}\mathbf{f})(s)/\delta$. Verify that $\mathbf{D}^2$, as it acts on $\mathcal{C}^2(0,d)$, can be approximated arbitrarily closely by the second-central-difference approximation, $\mathbf{D}^2 \approx \boldsymbol{\Delta}^2/\delta^2$.

(*c*) Suppose the plate spacing is $d = 5$. Let $\delta = 1$, and evaluate the finite-difference approximation $\boldsymbol{\Delta}^2\mathbf{f} = \boldsymbol{\theta}$ at $s = 1, 2, 3$, and 4 to obtain four algebraic equations in the variables $\mathbf{f}(0)$, $\mathbf{f}(1)$, . . . , $\mathbf{f}(5)$. Formulate these algebraic equations as a 4×6 matrix equation $\mathbf{A}\mathbf{x}=\boldsymbol{\theta}$. Compare this matrix equation with the differential equation $\mathbf{D}^2\mathbf{f} = \boldsymbol{\theta}$; that is, compare the spaces on which the operators act; also compare the dimensions of their solution spaces. Solve the matrix equation in terms of the boundary values $\mathbf{f}(0)$ and $\mathbf{f}(5)$. Compare the discrete solution with the continuous solution found in (*a*).

This problem can also be carried out for the two-dimensional case, where $\mathbf{f}(s, t)$ is given on a closed boundary. The finite-difference approach in (*b*) and (*c*) is widely used in the solution of practical problems of this type. The equations, sometimes numbering as many as 100,000, are solved by iterative computer techniques. See Forsythe and Wasow [2.4].

2.18 According to the trapezoidal rule for approximate integration, if we subdivide the interval $[a, b]$ into n segments of length δ, and denote $\mathbf{g}(a+j\delta)$ by g_j, $j=0, 1, \ldots, n$, then for a continuous $\mathbf{g}$,

$$\int_a^b \mathbf{g}(s)\,ds \approx \frac{\delta}{2}(g_0+g_1)+\frac{\delta}{2}(g_1+g_2)+\cdots+\frac{\delta}{2}(g_{n-1}+g_n)$$

$$=\delta\left(\frac{g_0}{2}+g_1+\cdots+g_{n-1}+\frac{g_n}{2}\right)$$

We can view the trapezoidal rule as an approximation of a function space integral operation by a matrix multiplication $\mathbf{Ax}$, where $\mathbf{A}$ is $1 \times n$ and $\mathbf{x} \triangleq (g_0 \cdots g_n)^T$.

(*a*) Find the matrix $\mathbf{A}$ which expresses the trapezoidal rule for $\delta = 1$ and $n = 5$. Apply the trapezoidal rule to accurately represent the integral of the discontinuous function $\mathbf{g}(s) \triangleq 1$ for $0 < s < 2$, $\mathbf{g}(s) \triangleq 0$ for $2 < s < 5$. Hint: at the discontinuity use the midpoint value, $(g_2^- + g_2^+)/2$.

(*b*) We can also approximate a *general* integral operator by a matrix multiplication. Suppose $(\mathbf{Tf})(t) \triangleq \int_a^b k(t,s)\mathbf{f}(s)\, ds$ for t in $[a, b]$. We can treat the function $k(t, s)\mathbf{f}(s)$ as we did $\mathbf{g}(s)$ in (*a*). Subdivide both the s and t intervals into n segments of length δ, and use the same subscript notation for function values as above. Then if $k(t,s)\mathbf{f}(s)$ is continuous,

$$\left(\frac{k_{j,0}f_0}{2} + k_{j,1}f_1 + \cdots + k_{j,n-1}f_{n-1} + \frac{k_{j,n}f_n}{2}\right)$$

for $j = 0, 1, \ldots, n$. We can approximate the integral operation by a matrix multiplication, $\mathbf{y} = \mathbf{Ax}$, where $\mathbf{x} = (f_0 \cdots f_n)^T$ and $\mathbf{y} = ((\mathbf{Tf})_0 \cdots (\mathbf{Tf})_n)^T$. Find $\mathbf{A}$ for $\delta = 1$, $n = 5$, $a = 0$, $b = 5$, and

$$k(t,s) = 1 \quad \text{for } 0 \leqslant s < t$$
$$= 0 \quad \text{for } t < s \leqslant 5$$

Hint: use midpoint values as in (*a*). Note that the operator is ordinary indefinite integration.

(*c*) Apply the matrix multiplication found in (*b*) to obtain the approximate integral of $\mathbf{f}(s) = 3s^2$. Compare the approximation to the actual integral at the points $t = 0, 1, \ldots, 5$.

***2.19** *Exploring the nullspace and range by row reduction:* Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 2 & 1 & 3 & 0 \\ 4 & 5 & 3 & 8 \end{pmatrix}$$

Multiplication by $\mathbf{A}$ is a linear transformation from $\mathfrak{M}^{4\times 1}$ into $\mathfrak{M}^{3\times 1}$. Multiplication by $\mathbf{A}^T$ is a linear transformation from $\mathfrak{M}^{3\times 1}$ into $\mathfrak{M}^{4\times 1}$. In Section 5.4 we find that if $\mathbf{y}$ is in $\mathbf{range(A)}$ and $\mathbf{x}$ is in $\mathbf{nullspace(A^T)}$, then $\mathbf{x} \cdot \mathbf{y} = 0$ where $\mathbf{x} \cdot \mathbf{y}$ is the dot product of analytic geometry. Furthermore, if $\mathbf{z}$ is in $\mathbf{range(A^T)}$ and $\mathbf{w}$ is in

nullspace(A), then $\mathbf{z} \cdot \mathbf{w} = 0$. By means of these dot product equations, we can use bases for **nullspace($\mathbf{A}^T$)** and **range($\mathbf{A}^T$)** to find bases for **range(A)** and **nullspace(A)**, and vice versa. We can also show that **rank(A) = rank($\mathbf{A}^T$)**. In this problem we obtain **nullspace(A)** and **range(A)** directly from $\mathbf{A}^T$.

(*a*) Row reduce $(\mathbf{A} \vdots \mathbf{I})$. Use the results of the row reduction to determine bases for **nullspace(A)** and **range(A)**.

(*b*) Row reduce $(\mathbf{A}^T \vdots \mathbf{I})$. Show that the nonzero rows in the left-hand block of the row-reduced matrix constitute a basis for **range(A)**. Show that the rows of the right-hand block which correspond to zero rows of the left-hand block of the row-reduced matrix constitute a basis for **nullspace(A)**.

2.20 Define $\mathbf{T}: \mathcal{P}^3 \to \mathcal{C}(0, 1)$ by $(\mathbf{Tf})(t) \triangleq \int_0^1 k(t,s)\mathbf{f}(s)\,ds$ for all $\mathbf{f}$ in $\mathcal{P}^3$, where

$$k(t,s) = t(1-s) \text{ for } t \leqslant s$$
$$= (1-t)s \text{ for } t \geqslant s$$

Find a basis for **range(T)**. Describe **nullspace(T)**.

2.21 Let $\mathcal{W}$ be the space of polynomial functions $\mathbf{f}$ of the form $\mathbf{f}(s, t) \triangleq c_1 + c_2 s + c_3 t + c_4 st$ for all s and t. Define $\mathbf{T}: \mathcal{W} \to \mathcal{W}$ by $(\mathbf{Tf})(s, t) \triangleq (\partial/\partial s)\mathbf{f}(s, t)$ for all $\mathbf{f}$ in $\mathcal{W}$.

(*a*) Find a basis for the range of **T**.

(*b*) Determine the rank and nullity of **T**.

2.22 Define $\mathbf{T}: \mathcal{M}^{2\times 2} \to \mathcal{M}^{2\times 2}$ by

$$\mathbf{T}\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \triangleq \begin{pmatrix} c_1 - c_2 & c_1 \\ c_2 & c_4 - c_3 \end{pmatrix}$$

for all choices of the scalars c_1, c_2, c_3, and c_4. Find **nullspace(T)** and **range(T)** by exhibiting a basis for each.

2.23 *Expected value:* the throws of a single die constitute an experiment. Let $\mathcal{V}$ be the space of random variables defined on this experiment. We can think of the probability mass function $\omega(\sigma)$ as the relative frequency with which the outcome σ occurs: $\omega(\sigma) = \frac{1}{6}$ for $\sigma = 1, 2, \ldots, 6$.

(*a*) A random variable $\mathbf{x}$ in $\mathcal{V}$ associates a value $\mathbf{x}(\sigma)$ with each possible outcome of the experiment. The value which $\mathbf{x}$ associates with an actual trial of the experiment is called a sample value of $\mathbf{x}$. The probability mass function $\omega_{\mathbf{x}}(x)$ specifies the relative frequency with which the sample value x occurs during repeated trials. Find $\omega_{\mathbf{y}}(y)$ for the random

variable $\mathbf{y}$ defined by $\mathbf{y}(\sigma) \triangleq 2$ for $\sigma = 1$ or 2 and by $\mathbf{y}(\sigma) \triangleq 0$ for $\sigma = 3, 4, 5$, or 6.

(*b*) The *expected value* of $\mathbf{x}$ is the average, over many trials, of the sample values of $\mathbf{x}$. Thus

$$\mathbf{E}(\mathbf{x}) = \sum_x x\,\omega_{\mathbf{x}}(x) = \sum_\sigma \mathbf{x}(\sigma)\omega(\sigma)$$

Find $\mathbf{E}(\mathbf{y})$ for the random variable $\mathbf{y}$ given in (*a*).

(*c*) Show that the functional $\mathbf{E}\colon \mathcal{V} \to \mathcal{R}$ is linear. Pick a basis $\mathcal{X}$ for $\mathcal{V}$. Let $\mathcal{E} \triangleq \{(1)\}$ be a basis for $\mathcal{R}$. Find $[\mathbf{y}]_{\mathcal{X}}$ and $[\mathbf{E}]_{\mathcal{X}\mathcal{E}}$, where $\mathbf{y}$ is the random variable in (*a*).

(*d*) If $\mathbf{f}\colon \mathcal{V} \to \mathcal{V}$ then $\mathbf{f}(\mathbf{x})$ is a random variable. Express $\mathbf{E}(\mathbf{f}(\mathbf{x}))$ in terms of $\omega(\sigma)$. Find $\mathbf{E}(\mathbf{y}^2)$ for the random variable $\mathbf{y}$ given in (*a*). If $\mathbf{g}\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, can $\mathbf{E}$ be applied to $\mathbf{g}(\mathbf{x}, \mathbf{y})$?

2.24 *Hadamard matrices:* let $\mathbf{f}(s)$ represent the light intensity versus position in one line of a television picture. Let the $n \times 1$ column vector $\mathbf{x}$ be a discrete approximation to $\mathbf{f}$. Then $\mathbf{x}$ can be viewed as a one-dimensional photograph. Suppose the data $\mathbf{x}$ must be transmitted for remote viewing. One way to reduce the effect of transmission errors and to reduce the amount of data transmitted is to transmit, instead, a transformed version of $\mathbf{x}$. A computationally simple transformation is the Hadamard transform—multiplication by a Hadamard matrix. A symmetric Hadamard matrix $\mathbf{H}$ consists in plus and minus ones, and satisfies $\mathbf{H}^{-1} = \mathbf{H}$ (see [2.9]). Denote the transformed vector by $\mathbf{X} = \mathbf{H}\mathbf{x}$. Let $n = 8$ and

$$\mathbf{H} = \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

The Hadamard transform spreads throughout the elements of $\mathbf{X}$ the information which is concentrated in a single element of $\mathbf{x}$; it concentrates information which is spread out.

(*a*) Determine the effect of $\mathbf{H}$ on the photographs $\mathbf{x} = (11111111)^{\mathrm{T}}$ and $\mathbf{x} = \boldsymbol{\varepsilon}_i$, where $\boldsymbol{\varepsilon}_i$ is the ith standard basis vector for $\mathcal{M}^{8\times 1}$.

(*b*) Find the transform of the photograph $\mathbf{x} = (2\ 2\ 2\ 3\ 2\ 2\ 2\ 2)^T$. Assume that an error during transmission of $\mathbf{X}$ reduces the third element of X to zero. Determine the effect of the error on the reconstructed photograph.

(*c*) The inverse transform, $\mathbf{x} = \mathbf{HX}$, can be interpreted as an expansion of $\mathbf{x}$ in terms of the columns of $\mathbf{H}$. The columns of $\mathbf{H}$ are analogous to sinusoidal functions; the number of zero crossings corresponds to frequency. Let $\mathbf{x}$ be the photograph in (*b*). Determine the effect on the reconstructed photograph of not transmitting the highest frequency component of $\mathbf{X}$ (i.e., the effect of making the second element of $\mathbf{X}$ zero). Determine the effect on the reconstructed photograph of eliminating the zero frequency component (i.e., the effect of making the first component of $\mathbf{X}$ zero).

2.25 The space $\mathcal{C}^1(0, \text{cc})$ consists in the continuously differentiable functions on $[0, \infty]$. Define the Cartesian product space $\mathcal{V}$ by $\mathcal{V} \triangleq \underset{1}{\mathcal{C}^1(0,\infty)} \times \cdots \times \underset{n-1}{\mathcal{C}^1(0,\infty)}$. Denote the vector-valued functions in $\mathcal{V}$ by $\mathbf{x}$. We can treat the values of $\mathbf{x}$ as vectors in $\mathcal{R}^{n \times 1}$; that is, $\mathbf{x}(t) = (\mathbf{f}_1(t) \cdots \mathbf{f}_n(t))^T$, where $\mathbf{f}_i$ is in $\mathcal{C}^1(0, \infty)$. Let $\mathbf{A}$ be a real $n \times n$ matrix. Define the linear transformation $\mathbf{T}: \mathcal{V} \to \mathcal{W}$ by

$$(\mathbf{Tx})(t) = \begin{pmatrix} \dot{\mathbf{f}}_1(t) \\ \vdots \\ \dot{\mathbf{f}}_n(t) \end{pmatrix} - \mathbf{A}\begin{pmatrix} \mathbf{f}_1(t) \\ \vdots \\ \mathbf{f}_n(t) \end{pmatrix} = \dot{\mathbf{x}}(t) - \mathbf{Ax}(t)$$

This transformation is central to the state-space analysis of dynamic systems.

(*a*) Determine an appropriate range of definition $\mathcal{W}$ for $\mathbf{T}$.

(*b*) Find a basis for nullspace if $n = 2$ and

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

2.26 Assume $\epsilon < \delta \ll 1$. Then the following matrix is nearly singular:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \epsilon & 0 \\ 1 & 0 & \delta \end{pmatrix}$$

Use inverse iteration to find a basis for the near nullspace of $\mathbf{A}$.

2.27 Define $\mathbf{T}: \mathcal{R}^2 \to \mathcal{R}^2$ by $\mathbf{T}(\xi_1, \xi_2) \triangleq (\xi_1 + 2\xi_2, \xi_1 - 2\xi_2)$ for all (ξ_1, ξ_2) in $\mathcal{R}^2$. Let $\mathcal{X} = \{(1, 1), (1, -1)\}$. Find $[\mathbf{T}]_{\mathcal{X}\mathcal{X}}$.

2.28 Define $\mathbf{T}:\mathcal{R}^3 \to \mathcal{R}^2$ by

$$\mathbf{T}(\xi_1, \xi_2, \xi_3) \triangleq (\xi_1 + \xi_2, 2\xi_3 - \xi_1)$$

(*a*) Determine $[\mathbf{T}]_{\mathcal{E}_3\,\mathcal{E}_2}$, the matrix of $\mathbf{T}$ relative to the standard bases for $\mathcal{R}^3$ and $\mathcal{R}^2$.

(*b*) Determine $[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}$, where $\mathcal{X}$ = {(1, 0, -l), (1, 1, 1), (1, 0, 0)} and $\mathcal{Y}$ = {(1, 0), (1, 1)}.

2.29 Define $\mathbf{T}: \mathcal{R}^2 \to \mathcal{R}^3$ by $\mathbf{T}(\xi_1, \xi_2) \triangleq (\xi_1 + \xi_2, \xi_1 - \xi_2, 2\xi_2)$ for all (ξ_1, ξ_2) in $\mathcal{R}^2$. Let $\mathcal{X}$ = {(1, 1), (1, -1)) and $\mathcal{Y}$ = {(1, 1, -1), (1, -1, 1), (-1, 1,1)}. Find $[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}$.

2.30 Let $\mathcal{P}^{2\times 2}$ denote the space of polynomial functions of the form $\mathbf{f}(s,t) = a_{11} + a_{12}s + a_{21}t + a_{22}st$. Define $\mathbf{T}: \mathcal{P}^{2\times 2} \to \mathcal{W}$ by

$$(\mathbf{Tf})(s,t) = \int_0^s \mathbf{f}(\sigma, t)\, d\sigma$$

where $\mathcal{W}$ = **range(T)**.

(*a*) Find bases, $\mathcal{F}$ for $\mathcal{P}^{2\times 2}$ and $\mathcal{G}$ for $\mathcal{W}$.

(*b*) Find $[\mathbf{T}]_{\mathcal{F}\mathcal{G}}$.

(*c*) Determine $\mathbf{T}^{-1}$ and $[\mathbf{T}^{-1}]_{\mathcal{G}\mathcal{F}}$. How else might $[\mathbf{T}^{-1}]_{\mathcal{G}\mathcal{F}}$ be obtained?

2.31 The sets $\mathcal{X}$ = {(1, -1, 0), (1, 0, 1), (1, 1, 1)} and $\mathcal{Y}$ = {(1, 1, 0), (0, 1, 1), (1, -1, 1)} are bases for $\mathcal{R}^3$. Find the change of coordinates matrix $\mathbf{S}_{\mathcal{X}\mathcal{Y}}$ which converts coordinates relative to $\mathcal{X}$ into coordinates relative to $\mathcal{Y}$.

2.32 Let $\mathbf{g}_1(t) = 1 - t$, $\mathbf{g}_2(t) = 1 - t^2$, and $\mathbf{g}_3(t) = 1 + t - t^2$. The set $\mathcal{G}$ = $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ is a basis for $\mathcal{P}^3$. Another basis is $\mathcal{F}$ = $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ where $\mathbf{f}_k(t) = t^{k-1}$.

(*a*) Find $[\mathbf{f}]_{\mathcal{G}}$ for the arbitrary vector $\mathbf{f}(t) = \xi_1 + \xi_2 t + \xi_3 t^2$.

(*b*) Find the coordinate-transformation matrix $\mathbf{S}$ such that $[\mathbf{f}]_{\mathcal{G}} = \mathbf{S}[\mathbf{f}]_{\mathcal{F}}$.

2.33 Define $\mathbf{T}: \mathcal{R}^2 \to \mathcal{R}^3$ by $\mathbf{T}(\xi_1, \xi_2) \triangleq (\xi_2 - \xi_1, \xi_1, 2\xi_1 - \xi_2)$ for all (ξ_1, ξ_2) in $\mathcal{R}^2$. The sets $\mathcal{X}$ = {(1, l), (1, -l)} and $\mathcal{Z}$ ={(1, 2), (2, l)} are bases for $\mathcal{R}^2$. The sets $\mathcal{Y}$ = {(1, 1, -1), (1, -1, 1), (-1, 1, 1)} and $\mathcal{K}$ = {(1, 1, 1), (0, 1, 1), (0, 0, 1)} are bases for $\mathcal{R}^3$.

(*a*) Find $[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}$.

(*b*) Find the coordinate transformations $\mathbf{S}_{\mathcal{X}\mathcal{Z}}$ and $\mathbf{S}_{\mathcal{Y}\mathcal{K}}$.

(*c*) Use the answers to (*a*) and (*b*) to compute $[\mathbf{T}]_{\mathcal{Z}\mathcal{K}}$ by means of an equivalence transformation.

2.34 Define $\mathbf{T}: \mathcal{R}^2 \to \mathcal{R}^2$ by $\mathbf{T}(\xi_1, \xi_2) \triangleq (\xi_1 + 2\xi_2, \xi_1 - 2\xi_2)$ for all (ξ_1, ξ_2) in $\mathcal{R}^2$. The sets $\mathcal{X}$ = {(1, 2), (2, 1)} and $\mathcal{Y}$ = {(1, 1), (1, -1)} are bases for $\mathcal{R}^2$.

(*a*) Find $[\mathbf{T}]_{\mathcal{X}\mathcal{X}}$.
(*b*) Find the coordinate transformation $\mathbf{S}_{\mathcal{X}\mathcal{Y}}$.
(*c*) Use the answers to (*a*) and (*b*) to compute $[\mathbf{T}]_{\mathcal{Y}\mathcal{Y}}$ by means of a similarity transformation.

2.35 Multiplication by an invertible matrix can be interpreted either as a linear transformation or as a change of coordinates. Let $\mathcal{X} = \{\mathbf{x}_1, \mathbf{x}_2\}$ be a basis for a two-dimensional space $\mathcal{V}$ and $\mathbf{x}$ a vector in $\mathcal{V}$. Then $[\mathbf{x}_1]_{\mathcal{X}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $[\mathbf{x}_2]_{\mathcal{X}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let

$$[\mathbf{x}]_{\mathcal{X}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

(*a*) Alias interpretation: assume $\mathbf{A}[\mathbf{x}]_{\mathcal{X}} = [\mathbf{x}]_{\mathcal{Y}}$, where $\mathcal{Y} = \{\mathbf{y}_1, \mathbf{y}_2\}$ is a second basis for $\mathcal{V}$. Find $[\mathbf{y}_1]_{\mathcal{X}}$ and $[\mathbf{y}_2]_{\mathcal{X}}$. Sketch $[\mathbf{x}_1]_{\mathcal{X}}$, $[\mathbf{x}_2]_{\mathcal{X}}$, $[\mathbf{x}]_{\mathcal{X}}$, $[\mathbf{y}_1]_{\mathcal{X}}$, and $[\mathbf{y}_2]_{\mathcal{X}}$ as arrows in a plane. What is the relationship between $[\mathbf{x}]_{\mathcal{X}}$ and the basis $\{[\mathbf{y}_1]_{\mathcal{X}}, [\mathbf{y}_2]_{\mathcal{X}}\}$; that is, what is meant by the notation $[\mathbf{x}]_{\mathcal{Y}}$?
(*b*) Alibi interpretation: assume $\mathbf{A}[\mathbf{x}]_{\mathcal{X}} = [\mathbf{T}\mathbf{x}]_{\mathcal{X}}$. Sketch $[\mathbf{x}_1]_{\mathcal{X}}$, $[\mathbf{x}_2]_{\mathcal{X}}$, $[\mathbf{x}]_{\mathcal{X}}$, and $[\mathbf{T}\mathbf{x}]_{\mathcal{X}}$ as arrows in a plane. What is the relationship between $[\mathbf{T}\mathbf{x}]_{\mathcal{X}}$ and the basis $\{[\mathbf{x}_1]_{\mathcal{X}}, [\mathbf{x}_2]_{\mathcal{X}}\}$; that is, what is meant by the notation $[\mathbf{T}\mathbf{x}]_{\mathcal{X}}$?

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