Discrete Fourier transform

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Discrete Complex exponentials

- Discrete complex exponential of discrete frequency k and duration N

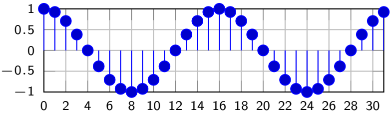
$$e_{kN}(n) = \frac{1}{\sqrt{N}}\; e^{j2\pi kn/N} = \frac{1}{\sqrt{N}}\; \exp(j2\pi kn/N)$$

- The complex exponential is explicitly given by

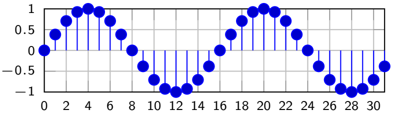
$$e^{j2\pi kn/N} = \cos(2\pi kn/N) + j\sin(2\pi kn/N)$$

- Real part is a discrete cosine and imaginary part a discrete sine

$\text{Re}\left(e^{j2\pi kn/N}\right)$, with $k = 2$ and $N = 32$



$\text{Im}\left(e^{j2\pi kn/N}\right)$, with $k = 2$ and $N = 32$



Equivalent frequencies

Theorem

If $k - l = N$ the signals $e_{kN}(n)$ and $e_{lN}(n)$ coincide for all n, i.e.,

$$e_{kN}(n) = \frac{e^{j2\pi kn/N}}{\sqrt{N}} = \frac{e^{j2\pi ln/N}}{\sqrt{N}} = e_{lN}(n)$$

- Exponentials with frequencies k and l are equivalent if $k - l = N$

Orthogonality

Theorem

Complex exponentials with nonequivalent frequencies are orthogonal. I.e.

$$\langle e_{kN}, e_{lN} \rangle = 0$$

when $k - l < N$. *E.g., when* $k = 0, \ldots N-1$, *or* $k = -N/2+1, \ldots, N/2$.

- Signals of canonical sets are "unrelated." Different rates of change
- Also note that the energy is $\|e_{kN}\|^2 = \langle e_{kN}, e_{kN} \rangle = 1$
- Exponentials with frequencies $k = 0, 1, \ldots, N-1$ are orthonormal

$$\langle e_{kN}, e_{lN} \rangle = \delta(l-k)$$

- They are an orthonormal basis of signal space with N samples

Conjugate frequencies

Theorem

Opposite frequencies k *and* $-k$ *yield conjugate signals:* $e_{-kN} = e^*_{kN}(n)$

Proof.

- Just use the definitions to write the chain of equalities

$$e_{-kN}(n) = \frac{e^{j2\pi(-k)n/N}}{\sqrt{N}} = \frac{e^{-j2\pi kn/N}}{\sqrt{N}} = \left[\frac{e^{j2\pi kn/N}}{\sqrt{N}}\right]^* = e^*_{kN}(n) \qquad \square$$

- Opposite frequencies $\Rightarrow$ Same real part. Opposite imaginary part
  $\Rightarrow$ The cosine is the same, the sine changes sign

Physical meaning

- Of the N canonical frequencies, only $N/2+1$ are distinct.

$$\begin{array}{ccccc} 0, & 1, & \ldots, & N/2-1 & N/2 \\ & -1, & \ldots, & -N/2+1 & \\ & N-1, & \ldots, & N/2+1 & \end{array}$$

- Frequencies 0 and $N/2$ have no counterpart. Others have conjugates
- Canonical set $-N/2+1, \ldots, -1, 0, 1, \ldots, N/2$ easier to interpret
- Reasonable $\Rightarrow$ Can't have more than $N/2$ oscillations in N samples
- With sampling frequency f_s and signal duration $T = NT_s = N/f_s$

  $\Rightarrow$ Discrete frequency k $\Rightarrow$ frequency $f_0 = \dfrac{k}{T} = \dfrac{k}{NT_s} = \dfrac{k}{N} f_s$
- Frequencies from 0 to $N/2 \leftrightarrow f_s/2$ have physical meaning

  $\Rightarrow$ Negative frequencies are conjugates of the positive frequencies

Discrete Fourier transform

Discrete complex exponentials

Discrete Fourier transform (DFT), definitions and examples

Units of the DFT

DFT inverse

Properties of the DFT

Definition of discrete Fourier transform (DFT)

- Signal x of duration N with elements $x(n)$ for $n = 0, \ldots, N-1$
- X is the discrete Fourier transform (DFT) of x if for all $k \in \mathbb{Z}$

$$X(k) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) \exp(-j2\pi kn/N)$$

- We write $X = \mathcal{F}(x)$. All values of X depend on all values of x
- The argument k of the DFT is referred to as frequency
- DFT is complex even if signal is real $\Rightarrow X(k) = X_R(k) + jX_I(k)$
  $\Rightarrow$ It is customary to focus on magnitude

$$\left|X(k)\right| = \left[X_R^2(k) + X_I^2(k)\right]^{1/2} = \left[X(k)X^*(k)\right]^{1/2}$$

DFT elements as inner products

- Discrete complex exponential (freq. k) $\Rightarrow e_{-kN}(n) = \frac{1}{\sqrt{N}} e^{-j2\pi kn/N}$

- Can rewrite DFT as $\Rightarrow X(k) = \sum_{n=0}^{N-1} x(n) e_{-kN}(n) = \sum_{n=0}^{N-1} x(n) e^{*}_{kN}(n)$

- And from the definition of inner product $\Rightarrow X(k) = \langle x, e_{kN} \rangle$

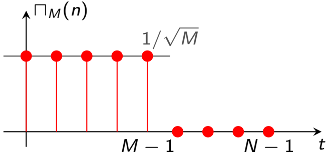
- DFT element $X(k)$ $\Rightarrow$ inner product of $x(n)$ with $e_{kN}(n)$
  - $\Rightarrow$ Projection of $x(n)$ onto complex exponential of frequency k
  - $\Rightarrow$ How much of the signal x is an oscillation of frequency k

DFT of a square pulse (derivation)

- The unit energy square pulse is the signal $\sqcap_M(n)$ that takes values

$$\sqcap_M(n) = \frac{1}{\sqrt{M}} \qquad \text{if } 0 \leq n < M$$

$$\sqcap_M(n) = 0 \qquad \text{if } M \leq n$$



- Since only the first $M-1$ elements of $\sqcap_M(n)$ are not null, the DFT is

$$X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \sqcap_M(n) e^{-j2\pi kn/N} = \frac{1}{\sqrt{N}} \sum_{n=0}^{M-1} \frac{1}{\sqrt{M}} e^{-j2\pi kn/N}$$

- $X(k) =$ sum of first M components of exponential of frequency $-k$

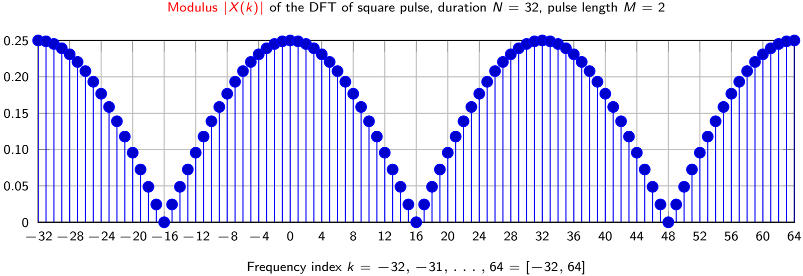
- Can reduce to simpler expression but who cares? $\Rightarrow$ It's just a sum

DFT of a square pulse (illustration)

- Square pulse of length $M = 2$ and overall signal duration $N = 32$

$$X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{1} \frac{1}{\sqrt{2}} e^{-j2\pi kn/N} = \frac{1}{\sqrt{2N}} \left(1 + e^{-j2\pi k/N}\right)$$

- E.g., $X(k) = \dfrac{2}{\sqrt{2N}}$ at $k = 0, \pm N, \ldots$ and $X(k) = 0$ at $k = 0, \pm 3N/2, \ldots$



- This DFT is periodic with period N $\Rightarrow$ true in general

Periodicity of the DFT

- Consider frequencies k and $k+N$. The DFT at $k+N$ is

$$X(k+N) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi(k+N)n/N}$$

- Complex exponentials of freqs. k and $k+N$ are equivalent. Then

$$X(k+N) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = X(k)$$

- DFT values N apart are equivalent $\Rightarrow$ DFT has period N

- Suffices to look at N consecutive frequencies $\Rightarrow$ canonical sets
  - $\Rightarrow$ Computation $\Rightarrow k \in [0, N-1]$
  - $\Rightarrow$ Interpretation $\Rightarrow k \in [-N/2, N/2]$ (actually, $N+1$ freqs.)
  - $\Rightarrow$ Related by chop and shift $\Rightarrow [-N/2, -1] \sim [N/2, N-1]$

- DFT of the square pulse highlighting frequencies $k \in [0, N-1]$

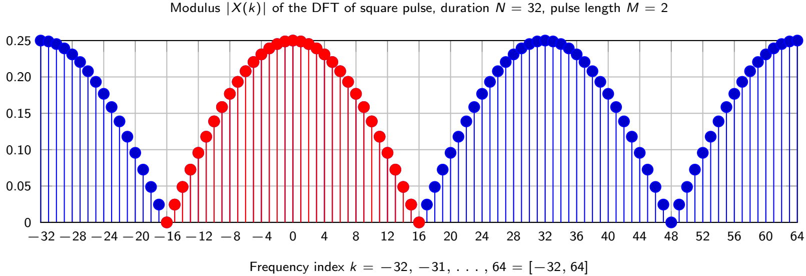
Modulus $|X(k)|$ of the DFT of square pulse, duration $N = 32$, pulse length $M = 2$

Frequency index $k = -32, -31, \ldots, 64 = [-32, 64]$

- Frequencies larger than $N/2$ have no clear physical meaning

Canonical set $k \in [-N/2, N/2]$

- DFT of the square pulse highlighting frequencies $k \in [-N/2, N/2]$
- Negative freq. $-k$ has the same interpretation as positive freq. k
- One redundant element $\Rightarrow X(-N/2) = X(N/2)$. Just convenient

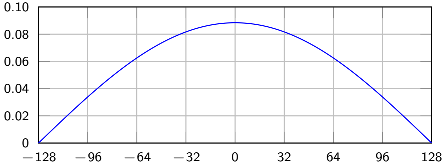


- Obtain frequencies $k \in [-N/2, -1]$ from frequencies $[N/2, N-1]$

Pulses of different length

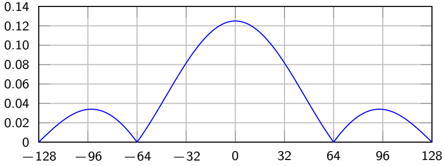
- The DFT X gives information on how fast the signal x changes

DFT modulus of square pulse, duration $N = 256$, pulse length $M = 2$



Frequency index $k = -128, -127, \ldots, 128 = [-128, 128]$

DFT modulus of square pulse, duration $N = 256$, pulse length $M = 4$



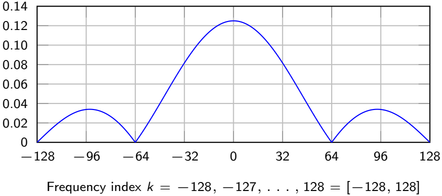
Frequency index $k = -128, -127, \ldots, 128 = [-128, 128]$

- For length $M = 2$ have weight at high frequencies
- Length $M = 4$ concentrates weight at lower frequencies
- Pulse of length $M = 2$ changes more than a pulse of length $M = 4$

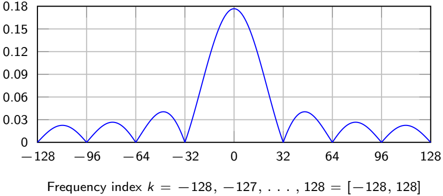
More DFTs of pulses of different length

- The lengthier the pulse the less it changes $\Rightarrow$ DFT concentrates at zero freq.

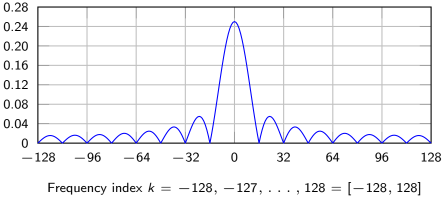
DFT modulus of square pulse, duration $N = 256$, pulse length $M = 4$



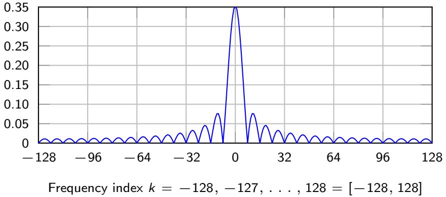
DFT modules of square pulse, duration $N = 256$, pulse length $M = 8$



DFT modulus of square pulse, duration $N = 256$, pulse length $M = 16$

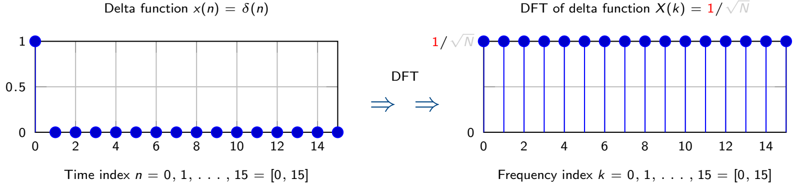


DFT modulus of square pulse, duration $N = 256$, pulse length $M = 32$



- The delta function is $\delta(0) = 1$ and $\delta(n) = 0$, else. Then, the DFT is

$$X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \delta(n) e^{-j2\pi kn/N} = \frac{1}{\sqrt{N}} \delta(0) e^{-j2\pi k0/N} = \frac{1}{\sqrt{N}}$$



- Only the N values $k \in [0, 15]$ shown. DFT defined for all k but periodic
- Observe that the energy is conserved $\|X\|^2 = \|\delta\|^2 = 1$

DFT of a shifted delta function

- For shifted delta $\delta(n_0 - n_0) = 1$ and $\delta(n - n_0) = 0$ otherwise. Thus

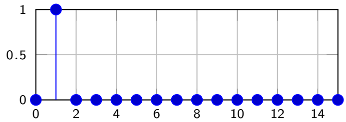
$$X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \delta(n - n_0) e^{-j2\pi kn/N} = \frac{1}{\sqrt{N}} \delta(n_0 - n_0) e^{-j2\pi k n_0/N}$$

- Of course $\delta(n_0 - n_0) = \delta(0) = 1$, implying that

$$X(k) = \frac{1}{\sqrt{N}} e^{-j2\pi k n_0/N} = e_{-n_0 N}(k)$$

- Complex exponential of frequency $-n_0$ (below, $N = 16$ and $n_0 = 1$)

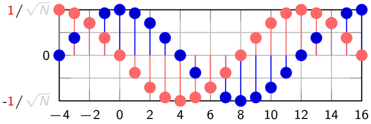
Shifted delta function $x(n) = \delta(n - n_0)$



Time index $n = 0, 1, \ldots, 15 = [0, 15]$

DFT $\Rightarrow$ $\Rightarrow$

DFT $X(k) = \frac{1}{\sqrt{N}} e^{-j2\pi k n_0/N} = e_{-n_0 N}(k)$

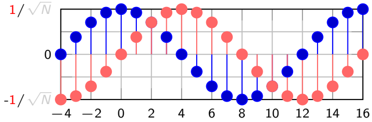


Frequency index $k = 0, 1, \ldots, 15 = [0, 15]$

DFT of a complex exponential

- Complex exponential of freq. k_0 $\Rightarrow x(n) = \frac{1}{\sqrt{N}} e^{j2\pi k_0 n/N} = e_{k_0 N}(n)$
- Use inner product form of DFT definition $\Rightarrow X(k) = \langle e_{k_0 N}, e_{kN} \rangle$
- Orthonormality of complex exponentials $\Rightarrow \langle e_{k_0 N}, e_{kN} \rangle = \delta(k - k_0)$

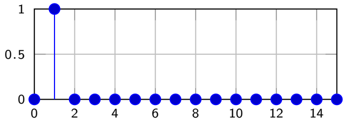
Complex exponential $x(n) = \frac{1}{\sqrt{N}} e^{j2\pi k_0 n/N} = e_{k_0 N}(n)$



Time index $n = 0, 1, \ldots, 15 = [0, 15]$

DFT $\Rightarrow \Rightarrow$

DFT is shifted delta function $X(k) = \delta(k - k_0)$

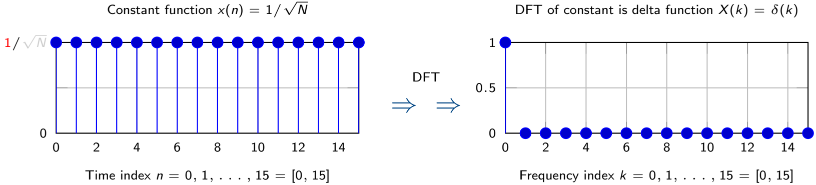


Frequency index $n = 0, 1, \ldots, 15 = [0, 15]$

- DFT of exponential $e_{k_0 N}(n)$ is shifted delta $X(k) = \delta(k - k_0)$

DFT of a constant

- Constant function $x(n) = 1/\sqrt{N}$ (it has unit energy) and $k = 0$
  $\Rightarrow$ Complex exponential with frequency $k_0 = 0$ $\Rightarrow x(n) = e_{0N}$
- Use inner product form of DFT definition $\Rightarrow X(k) = \langle e_{0N}, e_{kN} \rangle$
- Complex exponential orthonormality $\Rightarrow \langle e_{0N}, e_{kN} \rangle = \delta(k-0) = \delta(k)$



- DFT of constant $x(n) = 1/\sqrt{N}$ is delta function $X(k) = \delta(k)$

Observations

- DFT of a signal captures its rate of change
- Signals that change faster have more DFT weight at high frequencies

- DFT conserves energy (all have unit energy in our examples)
- Energy of DFT $X = \mathcal{F}(x)$ is the same as energy of the signal x
- Indeed, an important property we will show

- Duality of signal - transform pairs (signals and DFTs come in pairs)
- DFT of delta is a constant. DFT of constant is a delta
- DFT of exponential is shifted delta. DFT of shifted delta is exponential
- Indeed, a fact that follows from the form of the inverse DFT

Units of the DFT

Discrete complex exponentials

Discrete Fourier transform (DFT), definitions and examples

Units of the DFT

DFT inverse

Properties of the DFT

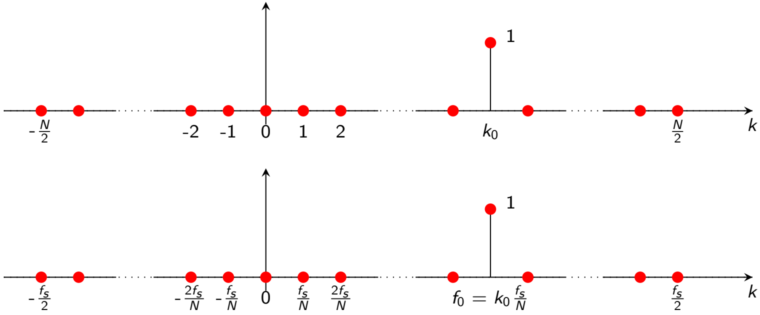
Units

- Sampling time T_s, sampling frequency f_s, signal duration $T = NT_s$
- Discrete frequency $k \Rightarrow k$ oscillations in time NT_s = Period NT_s/k
- Discrete frequency k equivalent to real frequency $f_k = \dfrac{k}{NT_s} = k\dfrac{f_s}{N}$
- In particular, $k = N/2$ equivalent to $\Rightarrow f_{N/2} = \dfrac{N/2 f_s}{N} = \dfrac{f_s}{2}$
- Set of frequencies $k \in [-N/2, N/2]$ equivalent to real frequencies ...
  - $\Rightarrow$ That lie between $-f_s/2$ and $f_s/2$
  - $\Rightarrow$ Are spaced by f_s/N (difference between frequencies f_k and f_{k+1})
- Interval width given by sampling frequency. Resolution given by N

Units in DFT of a discrete complex exponential

- Complex exponential of frequency $f_0 = k_0 f_s/N$
  $\Rightarrow$ Discrete frequency k_0 and DFT $\Rightarrow X(k) = \delta(k - k_0)$

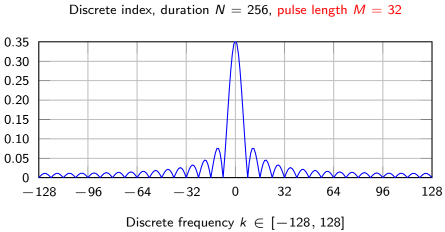
- But frequency k_0 corresponds to frequency f_0 $\Rightarrow X(f) = \delta(f - f_0)$

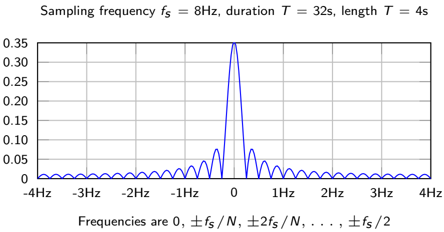


- True only when frequency $f_0 = (k_0/N)f_s$ is a multiple of f_s/N

Units in DFT of a square pulse

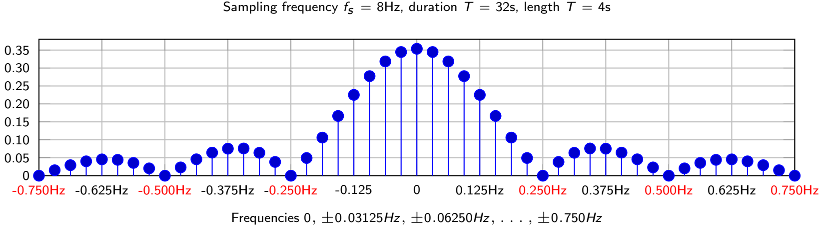
- Square pulse of length $T_0 = 4$s observed during a total of $T = 32$s.
- Sampled every $T_s = 125$ms $\Rightarrow$ Sample frequency $f_s = 8$Hz
- Total number of samples $\Rightarrow N = T/T_s = 256$
- Maximum frequency $k = N/2 = 128 \leftrightarrow f_k = f_{N/2} = f_s/2 = 4Hz$
- Fequency resolution $f_s/N = 8\text{Hz}/256 = 0.03125\text{Hz}$





Units in DFT of a square pulse

- Interval between freqs. $\Rightarrow f_s/N = 8\text{Hz}/256 = 1/32 = 0.03125\text{Hz}$
  $\Rightarrow$ 32 equally spaced frees for each 1Hz interval = 8 every 0.125 Hz.



- Zeros of DFT are at frequencies 0.250Hz, 0.500 Hz, 0.750 Hz, . . .
  $\Rightarrow$ Thus, zeros are at frequencies are $1/T_0, 2/T_0, 3/T_0, \ldots$

- Most (a lot) of the DFT energy is between freqs. $-1/T_0$ and $1/T_0$

DFT inverse

Definition of DFT inverse

- Given a Fourier transform X, the inverse (i)DFT $x = \mathcal{F}^{-1}(X)$ is

$$x(n) := \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(k) \exp(j2\pi kn/N)$$

- Same as DFT but for sign in the exponent (also, sum over k, not n)

- Any summation over N consecutive frequencies works as well. E.g.,

$$x(n) = \frac{1}{\sqrt{N}} \sum_{k=-N/2+1}^{N/2} X(k) e^{j2\pi kn/N}$$

- Because for a DFT X we know that it must be $X(k+N) = X(k)$

iDFT is, indeed, the inverse of the DFT

Theorem

The inverse DFT of the DFT of x is the signal x $\Rightarrow \mathcal{F}^{-1}[\mathcal{F}(x)] = x$

- Every signal x can be written as a sum of complex exponentials

$$x(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} = \frac{1}{\sqrt{N}} \sum_{k=-N/2+1}^{N/2} X(k) e^{j2\pi kn/N}$$

- Coefficient multiplying $e^{j2\pi kn/N}$ is $X(k) = k$th element of DFT of x

$$X(k) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

Proof of DFT inverse formula

Proof.

- Let $X = \mathcal{F}(x)$ be the DFT of x. Let $\tilde{x} = \mathcal{F}^{-1}(X)$ be the iDFT of X.

  $\Rightarrow$ We want to show that $\tilde{x} \equiv x$

- From the definition of the iDFT of $X \Rightarrow \tilde{x}(\tilde{n}) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(k) e^{j2\pi k\tilde{n}/N}$

- From the definition of the DFT of $x \Rightarrow X(k) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$

- Substituting expression for $X(k)$ into expression for $\tilde{x}(\tilde{n})$ yields

$$\tilde{x}(\tilde{n}) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left[\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \right] e^{j2\pi k\tilde{n}/N}$$

Proof of DFT inverse formula

Proof.

- Exchange summation order to sum first over k and then over n

$$\tilde{x}(\tilde{n}) = \sum_{n=0}^{N-1} x(n) \left[\sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} e^{j2\pi k\tilde{n}/N} \frac{1}{\sqrt{N}} e^{-j2\pi kn/N} \right]$$

- Pulled $x(n)$ out because it doesn't depend on k

- Innermost sum is the inner product between $e_{\tilde{n}N}$ and e_{nN}. Orthonormality:

$$\sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} e^{j2\pi k\tilde{n}/N} \frac{1}{\sqrt{N}} e^{-j2\pi kn/N} = \delta(\tilde{n} - n)$$

- Reducing to $\Rightarrow \tilde{x}(\tilde{n}) = \sum_{n=0}^{N-1} x(n)\delta(\tilde{n} - n) = x(\tilde{n})$

- Last equation is true because only term $n = \tilde{n}$ is not null in the sum □

Inverse DFT as inner product

- Discrete complex exponential (freq. n) $\Rightarrow e_{nN}(k) = \frac{1}{\sqrt{N}} e^{j2\pi kn/N}$

- Rewrite iDFT as $\Rightarrow x(n) = \sum_{k=0}^{N-1} X(k) e_{nN}(k) = \sum_{k=0}^{N-1} X(k) e^{*}_{-nN}(k)$

- And from the definition of inner product $\Rightarrow x(n) = \langle X, e_{nN} \rangle$

- iDFT element $X(k)$ $\Rightarrow$ inner product of $X(k)$ with $e_{-nN}(k)$

- Different from DFT, this is not the most useful interpretation

- Signal as sum of exponentials $\Rightarrow x(n) = \frac{1}{\sqrt{N}} \sum_{k=-N/2+1}^{N/2} X(k) e^{j2\pi kn/N}$

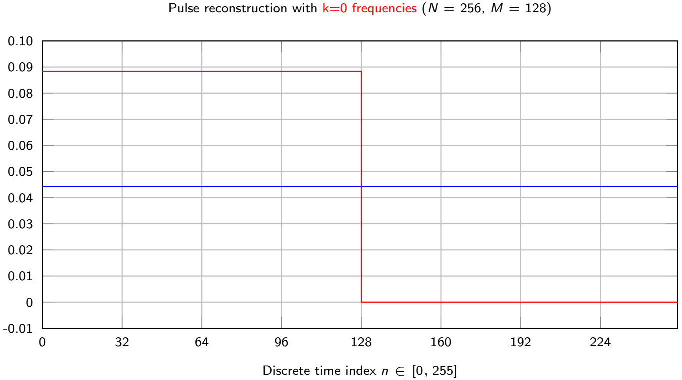
- Expand the sum inside out from $k=0$ to $k=\pm1$, to $k=\pm2$, ...

$$
\begin{array}{lllllll}
x(n) = & X(0) & e^{j2\pi 0n/N} & & & \text{constant} \\
& +X(1) & e^{j2\pi 1n/N} & +X(-1) & e^{-j2\pi 1n/N} & \text{single oscillation} \\
& +X(2) & e^{j2\pi 2n/N} & +X(-2) & e^{-j2\pi 2n/N} & \text{double oscillation} \\
& \vdots & \vdots & \vdots & \vdots & \vdots \\
& +X\left(\frac{N}{2}-1\right) & e^{j2\pi\left(\frac{N}{2}-1\right)n/N} & +X\left(-\frac{N}{2}+1\right) & e^{-j2\pi\left(\frac{N}{2}-1\right)n/N} & \left(\frac{N}{2}-1\right)-\text{oscillation} \\
& +X\left(\frac{N}{2}\right) & e^{j2\pi\left(\frac{N}{2}\right)n/N} & & & \frac{N}{2}-\text{oscillation}
\end{array}
$$

- Start with slow variations and progress on to add faster variations

Reconstruction of square pulse

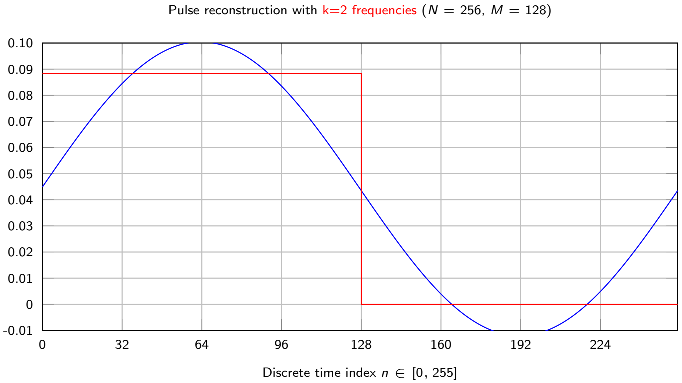
- Consider square pulse of duration $N = 256$ and length $M = 128$
- Reconstruct with frequency $k = 0$ only (DC component)



- Bound to be not very good $\Rightarrow$ Just the average signal value

Reconstruction of square pulse

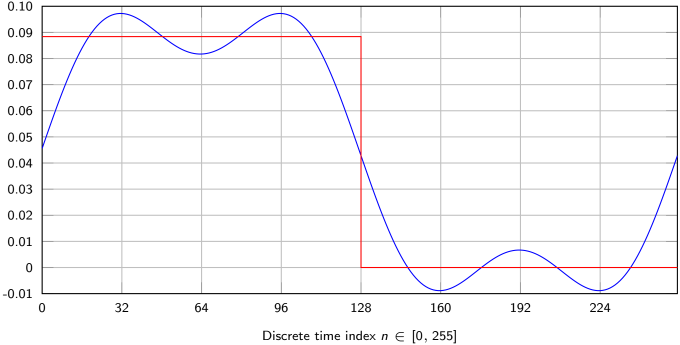
- Consider square pulse of duration $N = 256$ and length $M = 128$
- Reconstruct with frequencies $k = 0$, $k = \pm 1$, and $k = \pm 2$



- Not too bad, sort of looks like a pulse $\Rightarrow$ only 3 frequencies

Reconstruction of square pulse

- Consider square pulse of duration $N = 256$ and length $M = 128$
- Reconstruct with frequencies up to $k = 4$

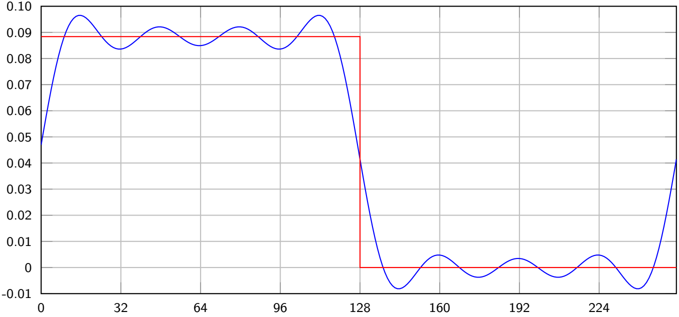


- Starts to look like a good approximation

Reconstruction of square pulse

- Consider square pulse of duration $N = 256$ and length $M = 128$
- Reconstruct with frequencies up to $k = 8$

Pulse reconstruction with k=8 frequencies ($N = 256$, $M = 128$)



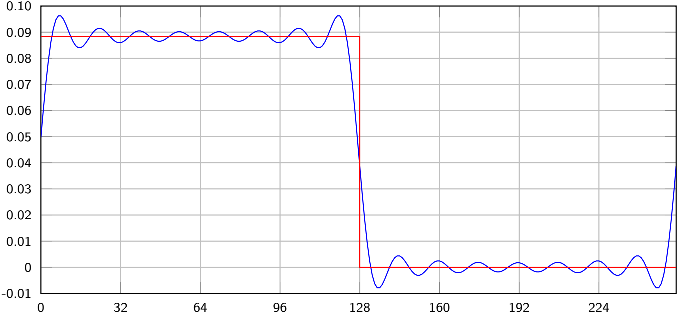
Discrete time index $n \in [0, 255]$

- Good approximation of the $N = 256$ values with 9 DFT coefficients

Reconstruction of square pulse

- Consider square pulse of duration $N = 256$ and length $M = 128$
- Reconstruct with frequencies up to $k = 16$

Pulse reconstruction with k=16 frequencies ($N = 256$, $M = 128$)

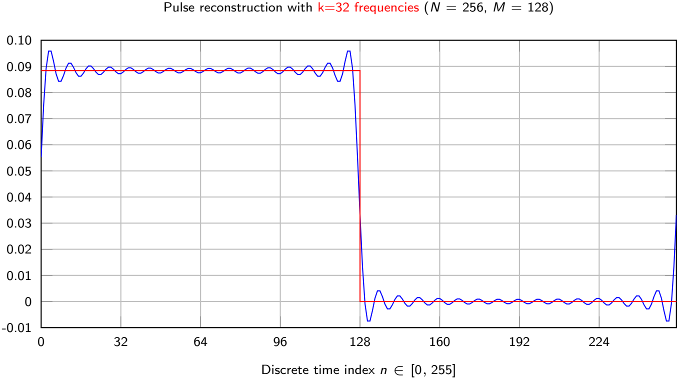


Discrete time index $n \in [0, 255]$

- Compression $\Rightarrow$ Store $k + 1 = 17$ DFT values instead of $N = 128$ samples

Reconstruction of square pulse

- Consider square pulse of duration $N = 256$ and length $M = 128$
- Reconstruct with frequencies up to $k = 32$



- Can tradeoff less compression for better signal accuracy

Spectrum (re)shaping

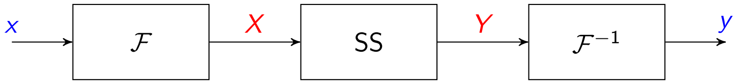
(1) Start with a signal x with elements $x(n)$. Compute DFT X as

$$X(k) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

(2) (Re)shape spectrum $\Rightarrow$ Transform DFT X into DFT Y

(3) With DFT Y available, recover signal y with inverse DFT

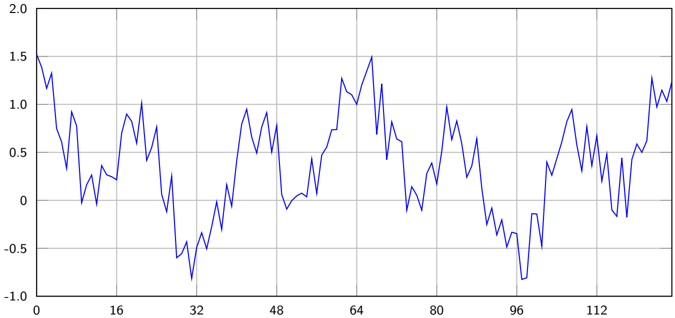
$$y(n) := \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} Y(k) e^{j2\pi kn/N}$$



Spectrum reshaping to remove noise

- An application of spectrum reshaping is to clean a noisy signal
- Signal with some underlying trend (good) and some noise (bad)

Original signal $x(n)$. It moves randomly, but not that much

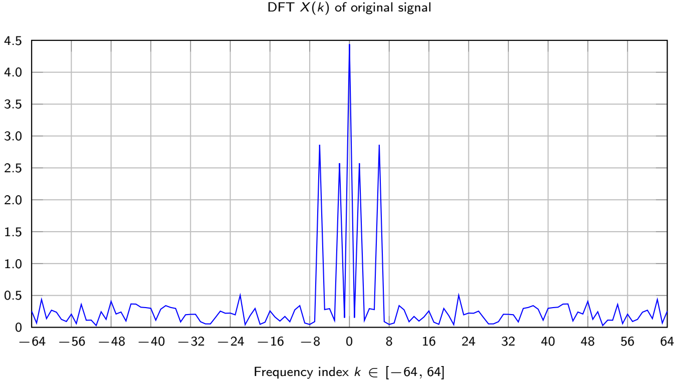


Discrete time index $n \in [0, 128]$

- Which is which? $\Rightarrow$ Not clear $\Rightarrow$ Let's look at the spectrum (DFT)

Spectrum reshaping to remove noise

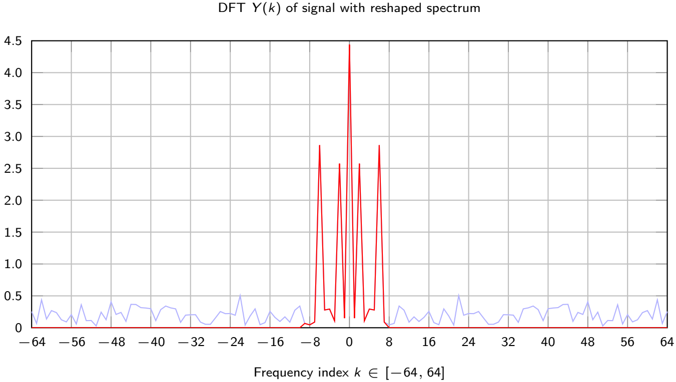
- An application of spectrum reshaping is to clean a noisy signal
- Now the trend (spikes) is clearly separated from the noise (the floor)



- How do we remove the noise? $\Rightarrow$ Reshape the spectrum

Spectrum reshaping to remove noise

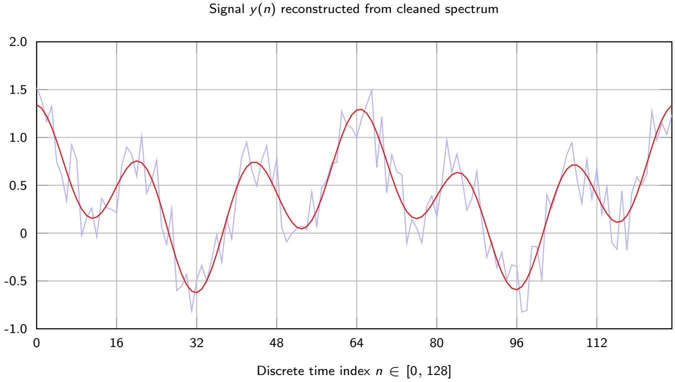
- An application of spectrum reshaping is to clean a noisy signal
- Remove freqs. larger than 8 $\Rightarrow Y(k) = 0$ for $k > 8$, $Y(k) = X(k)$ else



- How do we recover the trend? $\Rightarrow$ Inverse DFT

Spectrum reshaping to remove noise

- An application of spectrum reshaping is to clean a noisy signal
- Inverse DFT of reshaped specturm $Y(k)$ yields cleaned signal $y(n)$



- The trend now is clearly visible. Noise has been removed

Properties of the DFT

Three important properties of DFTs

- DFTs of real signals (no imaginary part) are conjugate symmetric

$$X(-k) = X^*(k)$$

- Signals of unit energy have transforms of unit energy
- More generically, the DFT preserves energy (Parseval's theorem)

$$\sum_{n=0}^{N-1} |x(n)|^2 = \|x\|^2 = \|X\|^2 = \sum_{k=0}^{N-1} |X(k)|^2$$

- The DFT operator is a linear operator

$$\mathcal{F}(ax + by) = a\mathcal{F}(x) + b\mathcal{F}(y)$$

Symmetry

Theorem

The DFT $X = \mathcal{F}(x)$ of a real signal x is conjugate symmetric

$$X(-k) = X^*(k)$$

- Can recover all DFT components from those with freqs. $k \in [0, N/2]$
- What about components with freqs. $k \in [-N/2, -1]$?
  $\Rightarrow$ Conjugates of those with freqs $k \in [0, N/2]$
- Other elements are equivalent to one in $[-N/2, N/2]$ (periodicity)

Proof of symmetry property

Proof.

- Write the DFT $X(-k)$ using its definition

$$X(-k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi(-k)n/N}$$

- When the signal is real, its conjugate is itself $\Rightarrow x(n) = x^*(n)$
- Conjugating a complex exponential $\Rightarrow$ changing the exponent's sign

- Can then rewrite $\Rightarrow X(-k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x^*(n) \left(e^{-j2\pi kn/N}\right)^*$

- Sum and multiplication can change order with conjugation

$$X(-k) = \left[\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}\right]^* = X^*(k) \qquad \square$$

Energy conservation

Theorem (Parseval)

Let $X = \mathcal{F}(x)$ be the DFT of signal x. The energies of x and X are the same, i.e.,

$$\sum_{n=0}^{N-1} |x(n)|^2 = \|x\|^2 = \|X\|^2 = \sum_{k=0}^{N-1} |X(k)|^2$$

- In energy of DFT, any set of consecutive freqs. would do. E.g.,

$$\|X\|^2 = \sum_{k=0}^{N-1} |X(k)|^2 = \sum_{k=-N/2+1}^{N/2} |X(k)|^2$$

Proof of Parseval's Theorem

Proof.

- From the definition of the energy of $X \Rightarrow \|X\|^2 = \sum_{k=0}^{N-1} X(k)X^*(k)$

- From the definition of the DFT of x $\Rightarrow X(k) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$

- Substitute expression for $X(k)$ into one for $\|X\|^2$ (observe conjugation)

$$\|X\|^2 = \sum_{k=0}^{N-1} \left[\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \right] \left[\frac{1}{\sqrt{N}} \sum_{\tilde{n}=0}^{N-1} x^*(\tilde{n}) e^{+j2\pi k\tilde{n}/N} \right]$$

Proof of Parseval's Theorem

Proof.

- Distribute product and exchange order of summations $\Rightarrow$ sum over k first

$$\|X\|^2 = \sum_{n=0}^{N-1} \sum_{\tilde{n}=0}^{N-1} x(n) x^*(\tilde{n}) \left[\sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} e^{-j2\pi kn/N} \frac{1}{\sqrt{N}} e^{+j2\pi k\tilde{n}/N} \right]$$

- Pulled $x(n)$ and $x^*(\tilde{n})$ out because they don't depend on k

- Innermost sum is the inner product between $e_{\tilde{n}N}$ and e_{nN}. Orthonormality:

$$\sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} e^{-j2\pi kn/N} \frac{1}{\sqrt{N}} e^{+j2\pi k\tilde{n}/N} = \langle e_{\tilde{n}N}, e_{nN} \rangle = \delta(\tilde{n} - n)$$

- Thus $\Rightarrow \|X\|^2 = \sum_{n=0}^{N-1} \sum_{\tilde{n}=0}^{N-1} x(n) x^*(\tilde{n}) \delta(\tilde{n} - n) = \sum_{n=0}^{N-1} x(n) x^*(n) = \|x\|^2$

- True because only terms $n = \tilde{n}$ are not null in the sum

□

Linearity

Theorem

The DFT of a linear combination of signals is the linear combination of the respective DFTs of the individual signals,

$$\mathcal{F}(ax + by) = a\mathcal{F}(x) + b\mathcal{F}(y).$$

- In particular...
  - $\Rightarrow$ Adding signals $(z = x + y)$ $\Rightarrow$ Adding DFTs $(Z = X + Y)$
  - $\Rightarrow$ Scaling signals$(y = ax)$ $\Rightarrow$ Scaling DFTs $(Y = aX)$

Proof.

- Let $Z := \mathcal{F}(ax + by)$. From the definition of the DFT we have

$$Z(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \Big[ax(n) + by(n)\Big] e^{-j2\pi kn/N}$$

- Expand the product, reorder terms, identify the DFTs of x and y

$$Z(k) = \frac{a}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} + \frac{b}{\sqrt{N}} \sum_{n=0}^{N-1} y(n) e^{-j2\pi kn/N}$$

- First sum is DFT $X = \mathcal{F}(x)$. Second sum is DFT $Y = \mathcal{F}(y)$

$$Z(k) = aX(k) + bY(k)$$

□

DFT of a discrete cosine

- DFT of discrete cosine of freq. k_0 $\Rightarrow x(n) = \frac{1}{\sqrt{N}} \cos(2\pi k_0 n/N)$
- Can write cosine as a sum of discrete complex exponentials

$$x(n) = \frac{1}{2\sqrt{N}} \left[e^{j2\pi k_0 n/N} + e^{-j2\pi k_0 n/N} \right] = \frac{1}{2} \left[e_{k_0 N}(n) + e_{-k_0 N}(n) \right]$$

- From linearity of DFTs $\Rightarrow X = \mathcal{F}(x) = \frac{1}{2} \left[\mathcal{F}(e_{k_0 N}) + \mathcal{F}(e_{-k_0 N}) \right]$
- DFT of complex exponential e_{kN} is delta function $\delta(k - k_0)$. Then

$$X(k) = \frac{1}{2} \left[\delta(k - k_0) + \delta(k + k_0) \right]$$

- A pair of deltas at positive and negative frequency k_0

DFT of a discrete sine

- DFT of discrete sine of freq. k_0 $\Rightarrow x(n) = \frac{1}{\sqrt{N}} \sin(2\pi k_0 n/N)$

- Can write sine as a difference of discrete complex exponentials

$$x(n) = \frac{1}{2j\sqrt{N}} \left[e^{j2\pi k_0 n/N} - e^{-j2\pi k_0 n/N} \right] = \frac{-j}{2} \left[e_{k_0 N}(n) - e_{-k_0 N}(n) \right]$$

- From linearity of DFTs $\Rightarrow X = \mathcal{F}(x) = \frac{j}{2} \left[\mathcal{F}(e_{-k_0 N}) - \mathcal{F}(e_{k_0 N}) \right]$

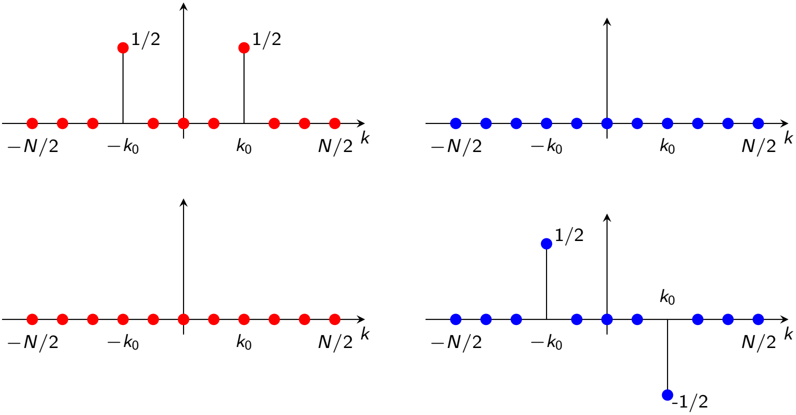
- DFT of complex exponential e_{kN} is delta function $\delta(k - k_0)$. Then

$$X(k) = \frac{j}{2} \left[\delta(k + k_0) - \delta(k - k_0) \right]$$

- Pair of opposite complex deltas at positive and negative frequency k_0

DFT of discrete cosine and discrete sine

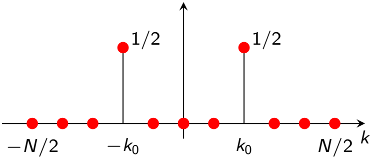
- Cosine has real part only (top). Sine has imaginary part only (bottom)

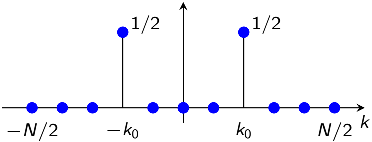


- Cosine is symmetric around $k = 0$. Sine is antisymmetric around $k = 0$.

DFT of discrete cosine and discrete sine (more)

- Real and imaginary parts are different but the moduli are the same





- Cosine and sine are essentially the same signal (shifted versions)
  - $\Rightarrow$ The moduli of their DFTs are identical
  - $\Rightarrow$ Phase difference captured by phase of complex number $X(\pm k_0)$