# Stochastic Systems Analysis and Simulations

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Presentations

Class description and contents

Gambling

# Who are us, where to find me, lecture times

- Alejandro Ribeiro, Luiz Chamon, Fernando Gama
- Walnut 3401 floor 4B. https://alelab.seas.upenn.edu/
- Teaching assistants: Vinicius Lima
- Class email: ese303@seas.upenn.edu.
- Don't write to our personal addresses. Please.
- We also have a separate grader
- We meet on the Berger auditorium in Skirkanich Hall
- Mondays, Wednesdays, Fridays 10 am to 11 am
- My office hours, Wednesdays and Fridays from 11 am to 12 pm
- Anytime, as long as you have something interesting to tell me
- https://ese303.seas.upenn.edu/

# Prerequisites

- Probability theory
- Stochastic processes are time-varying random entities
- If unknown, need to learn as we go
- Will cover in first seven lectures

- Linear algebra
- Vector matrix notation, systems of linear equations, eigenvalues

- Programming in Matlab
- Needed for homework.
- If you know programming you can learn Matlab in one afternoon
- But it has to be this afternoon

- Differential equations, Fourier transforms
- Appear here and there. Should not be a problem

# Homework and grading

- 14 homework sets in 14 weeks
- Collaboration accepted, welcomed, and encouraged
- Sets graded as 0 (bad), 1 (good), 2 (very good) and 3 (outstanding)
- We'll use the 3 sparingly. Goal is to earn 28 homework points

- First Midterm examination starts on Monday October 7, 36 points
- Take home due on Wednesday October 9
- Work independently. No collaboration, no discussion

- Second midterm is on Monday December 9 worth 36 points

- At least 60 points are required for passing.
- C requires at least 70 points. B at least 80. A at least 90
- Goal is for everyone to earn an A

# Textbooks

- Textbook for the class is (older or newer editions acceptable)
- Sheldon M. Ross "Introduction to Probability Models", Academic Press, whatever ed.

- Same topics at advanced level (more rigor, includes proofs)
- Sheldon Ross "Stochastic Processes", John Wiley & sons, 2nd ed.

- Stohastic processes in systems biology
- Darren J. Wilkinson "Stochastic Modelling for Systems Biology", Chapman & Hall/CRC, 1st ed.
- Part on simulation of chemical reactions taken from here

- Use of stochastic processes in finance
- Masaaki Kijima "Stochastic Processes with Applications to Finance", Chapman & Hall/CRC, 1st ed.

# This is not a programming class

- Just that. Not a programming class

# Difficulty

- This class has a reputation for been hard and demanding
  - $\Rightarrow$ I do not entirely agree but I take the point
- On the other hand, the quality ratings are very good
  - $\Rightarrow$ S. Reid Warren Jr. Award (2012)
  - $\Rightarrow$ Lindback Award for Distinguished Teaching (2017)
  - $\Rightarrow$ Collected the two possible teaching awards in 8 years
- This is, really, a great class. You will do things that look like magic
- Also, the class is front loaded. It will become easier after the break.
  - $\Rightarrow$ Don't drop it! You will enjoy it.

# Some remarks on teaching methods

- I ask questions to individual students (cold calling).
  - $\Rightarrow$ Absolutely zero premium (penalty) on right (wrong) answer
- Do these questions serve any purpose?
  - $\Rightarrow$ I need to gauge what you understand of what I say
  - $\Rightarrow$ There are different ways of explaining ideas
  - $\Rightarrow$ Spatial memory associates parts of the room with concepts
  - $\Rightarrow$ People remember conversations better than lectures
- Can anyone explain to me why this is called cold calling?

# Class contents

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# Stochastic systems

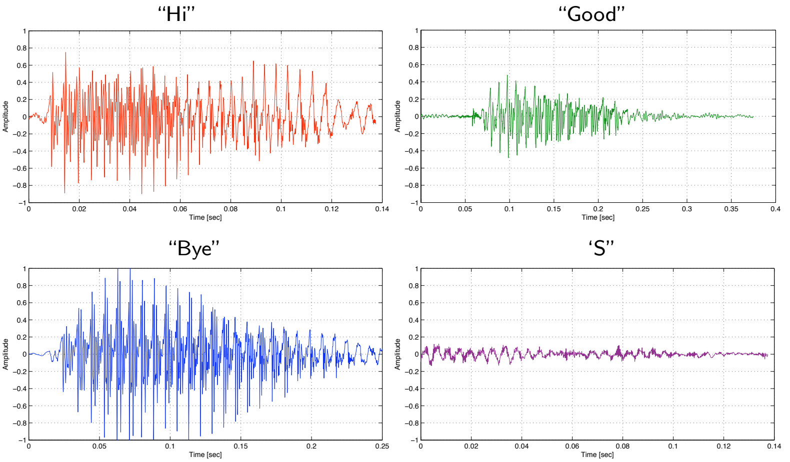
- Anything random that evolves in time
- Time can be discrete $(0, 1, \ldots)$ or continuous

- More formally, assign a function to a random event
- Compare with "random variable assigns a value to a random event"
- Generalizes concept of random vector to functions
- Or generalizes the concept of function to random settings

- Can interpret a stochastic process as a set of random variables
- Not always the most appropriate way of thinking

# A voice recognition system

- Random event $\sim$ word spoken. Stochastic process $\sim$ the waveform
  - Try the file speech_signals.m

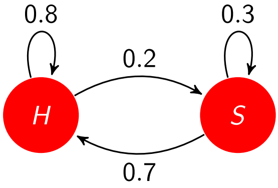


# Five blocks

- Probability theory review (6 lectures)
  - Probability spaces
  - Conditional probability: time $n+1$ given time $n$, future given past ...
  - Limits in probability, almost sure limits: behavior as $t \to \infty$ ...
  - Common probability distributions (binomial, exponential, Poisson, Gaussian)
- Stochastic processes are complicated entities
- Restrict attention to particular classes that are somewhat tractable
- Markov chains (9 lectures)
- Continuous time Markov chains (12 lectures)
- Stationary random processes (9 lectures)
- Midterm covers up to Markov chains

# Markov chains

- A set of states $1, 2, \ldots$. At time $n$, state is $X_n$
- Memoryless property
  - $\Rightarrow$ Probability of next state $X_{n+1}$ depends on current state $X_n$
  - $\Rightarrow$ But not on past states $X_{n-1}$, $X_{n-2}$, ...
- Can be happy ($X_n = 0$) or sad ($X_n = 1$)
- Happiness tomorrow affected by happiness today only
- Whether happy or sad today, likely to be happy tomorrow
- But when sad, a little less likely so

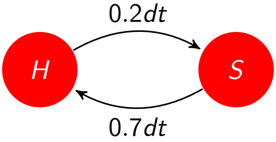



- Classification of states, ergodicity, limiting distributions
- Google's page rank, machine learning, virus propagation, queues ...

# Continuous time Markov chains

- A set of states $1, 2, \ldots$. Continuous time index $t$
- Transition between states can happen at any time
- Future depends on present but is independent of the past

- Probability of changing state in an infinitesimal time $dt$




- Poisson processes, exponential distributions, transition probabilities, Kolmogorov equations, limit distributions
- Chemical reactions, queues, communication networks, weather forecasting ...

# Stationary random processes

- Continuous time $t$, continuous state $x(t)$, not necessarily memoryless
- System has a steady state in a random sense
- Prob. distribution of $x(t)$ constant or becomes constant as $t$ grows
- Brownian motion, white noise, Gaussian processes, autocorrelation, power spectral density.
- Black Scholes model for option pricing, speech, noise in electric circuits, filtering and equalization ...

# An interesting betting game

- There is a certain game in a certain casino in which ...

  $\Rightarrow$ your chances of winning are $p > 1/2$
- You place \$1 bets
  (a) With probability $p$ you gain \$1 and
  (b) With probability $(1 - p)$ you lose your \$1 bet
- The catch is that you either
  (a) Play until you go broke (lose all your money)
  (b) Keep playing forever
- You start with an initial wealth of $\$w_0$
- Shall you play this game?

# Modeling

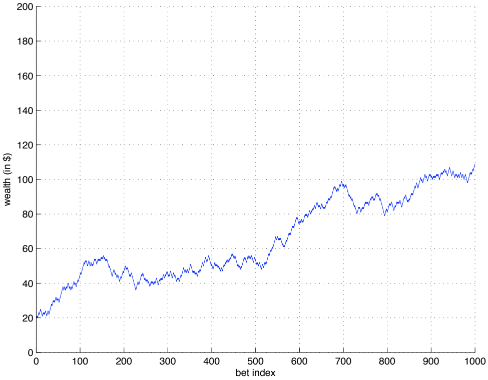
- Let $t$ be a time index (number of bets placed)
- Denote as $x(t)$ the outcome of the bet at time $t$
  - $x(t) = 1$ if bet is won (with probability $p$)
  - $x(t) = 0$ if bet is lost (probability $(1-p)$)
- $x(t)$ is called a Bernoulli random varible with parameter $p$
- Denote as $w(t)$ the player's wealth at time $t$
- At time $t = 0$, $w(0) = w_0$
- At times $t > 0$ wealth $w(t)$ depends on past wins and losses
- More specifically we have
  - When bet is won $w(t) = w(t-1) + 1$
  - When bet is lost $w(t) = w(t-1) - 1$

$t = 0$; $w(t) = w_0$; $max_t = 10^3$; `// Initialize variables`
% repeat while not broke up to time $max_t$
**while** $(w(t) > 0)$ & $(t < max_t)$ **do**
    $x(t) =$ random('bino',1,$p$); % Draw Bernoulli random variable
    **if** $x(t) == 1$ **then**
        $w(t+1) = w(t) + b$; % If $x = 1$ wealth increases by $b$
    **else**
        $x(t+1) = w(t) - b$; % If $x = 0$ wealth decreases by $b$
    **end**
    $t = t + 1$;
**end**

- Initial wealth $w_0 = 20$, bet $b = 1$, win probability $p = 0.55$
- Shall we play?

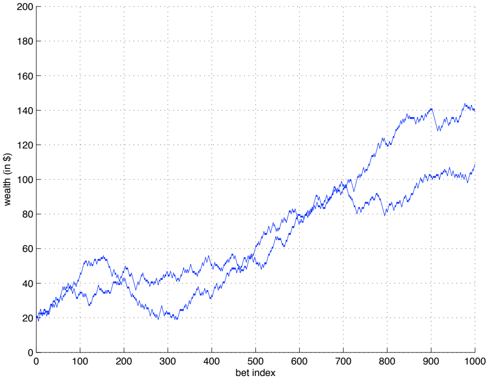
# One lucky player

- She didn't go broke. After $t = 1000$ bets, her wealth is $w(t) = 109$
- Less likely to go broke now because wealth increased



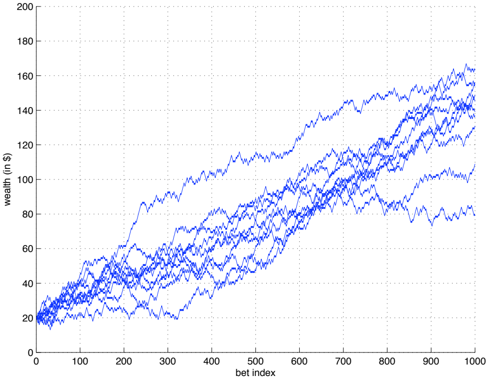
# Two lucky players

- Wealths are $w_1(t) = 109$ and $w_2(t) = 139$
- Increasing wealth seems to be a pattern



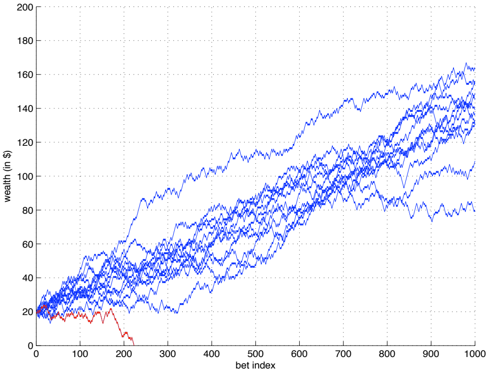
# Ten lucky players

- Wealths $w_j(t)$ between 78 and 139
- Increasing wealth is definitely a pattern



# One unlucky player

- But this does not mean that all players will turn out as winners
- The twelfth player $j = 12$ goes broke



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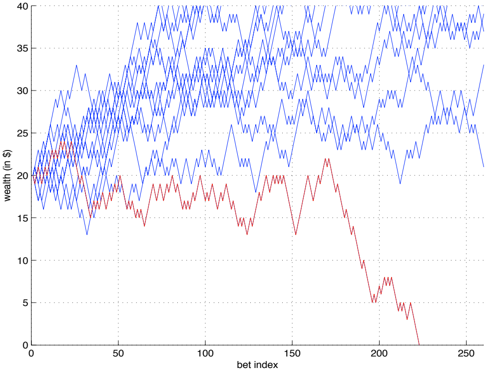
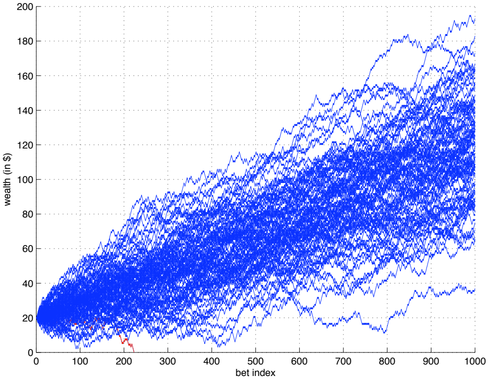


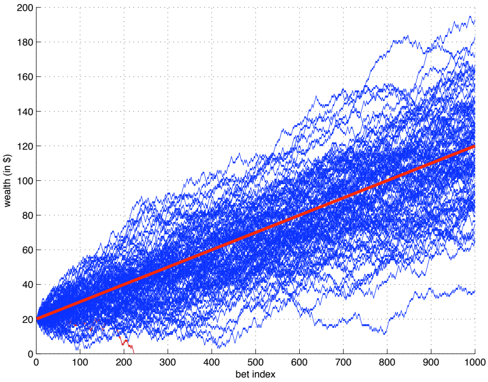
# One hundred players

- Only one player ($j = 12$) goes broke
- All other players end up with substantially more money



# Average tendency

- It is not difficult to find a line estimating the average of $w(t)$
- $\bar{w}(t) \approx w_0 + (2p - 1)t \approx w_0 + 0.1t$



# Where does the average tendency comes from?

- To discover average tendency notice that for all times $t$ we can write

$$W(t+1) = W(t) + \Big(2X(t) - 1\Big)$$

- Taking expectation on both sides and using linearity of expectations

$$\mathbb{E}\left[W(t+1)\right] = \mathbb{E}\left[W(t)\right] + \Big(2\mathbb{E}\left[X(t)\right] - 1\Big)$$

- The expected value of $X(t)$ is

$$\mathbb{E}\left[X(t)\right] \;=\; 1 \times \mathrm{P}\left(X(t) = 1\right) + 0 \times \mathrm{P}\left(X(t) = 1\right) \;=\; p$$

- Which yields $\Rightarrow \mathbb{E}\left[W(t+1)\right] = \mathbb{E}\left[W(t)\right] + (2p-1)$

- Applying recursively $\Rightarrow \mathbb{E}\left[W(t+1)\right] = w_0 + (2p-1)t$

# Analysis of outcomes: mean

- For a more accurate analysis analyze simulation's outcome

- Consider $J$ experiments
- For each experiment, there is a wealth history $w_j(t)$
- We can estimate the average outcome as

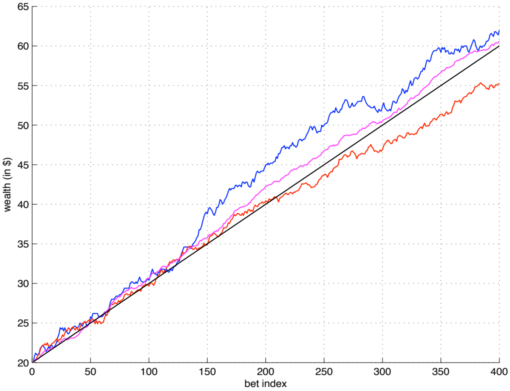
$$\bar{w}_J(t) = \frac{1}{J}\sum_{j=1}^{J} w_j(t)$$

- $\bar{w}_J(t)$ is called the sample average

- Do not confuse $\bar{w}_J(t)$ with $\mathbb{E}[w(t)]$
  - $\bar{w}_J(t)$ is computed from experiments, it is a random quantity in itself
  - $\mathbb{E}[w(t)]$ is a property of the random variable $w(t)$
  - We will see later that for large $J$, $\bar{w}_J(t) \to \mathbb{E}[w(t)]$

# Analysis of outcomes: mean

- Expected value $\mathbb{E}\left[w(t)\right]$ in black (approximation)
- Sample average for $J = 10$ (blue), $J = 20$ (red), and $J = 100$ (magenta)



# Histogram

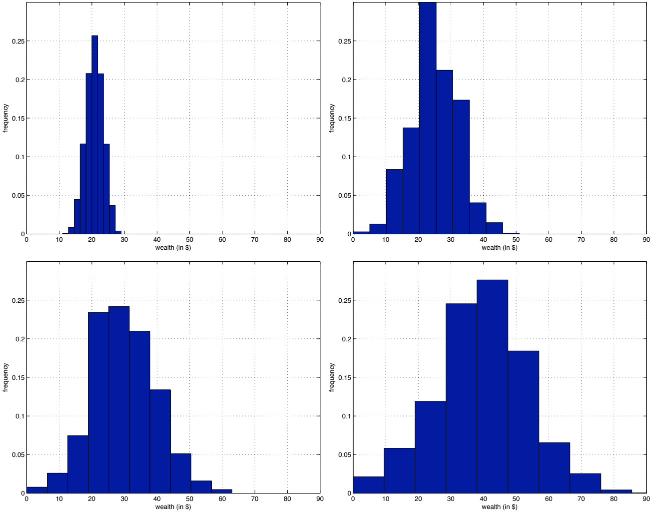
- There is more information in the simulation's output
- Estimate the probability distribution function (pdf) $\Rightarrow$ Histogram
- Consider a set of points $w^{(0)}, \ldots, w^{(N)}$
- Indicator function of the event $w^{(n)} \leq w_j < w^{(n+1)}$
  - $\mathbb{I}\left[w^{(n)} \leq w_j < w^{(n+1)}\right] = 1$ when $w^{(n)} \leq w_j < w^{(n+1)}$
  - $\mathbb{I}\left[w^{(n)} \leq w_j < w^{(n+1)}\right] = 0$ else
- Histogram is then defined as

$$H\left[t; w^{(n)}, w^{(n+1)}\right] = \frac{1}{J}\sum_{j=1}^{J} \mathbb{I}\left[w^{(n)} \leq w_j(t) < w^{(n+1)}\right]$$

- Fraction of experiments with wealth $w_j(t)$ between $w^{(n)}$ and $w^{(n+1)}$

# Histogram

- The pdf broadens and shifts to the right ($t = 10, 50, 100, 200$)



# What is this class about

- Analysis and simulation of stochastic systems

  $\Rightarrow$ A system that evolves in time with some randomness

- They are usually quite complex $\Rightarrow$ Simulations

- We will learn how to model stochastic systems, e.g.,
  - $x(t)$ Bernoulli with parameter $p$
  - $w(t) = w(t-1) + 1$ when $x(t) = 1$
  - $w(t) = w(t-1) - 1$ when $x(t) = 0$

- ... how to analyze, e.g., $\mathbb{E}\left[W(t)\right] = w_0 + t(2p-1)$

- ... and how to interpret simulations and experiments, e.g,
  - Average tendency through sample average