Week 11: Gaussian processes
White Gaussian noise

White Gaussian noise (WGN) is likely the most common stochastic model used in engineering applications. A stochastic process $X(t)$ is said to be WGN if $X(\tau)$ is normally distributed for each τ and values $X(t_1)$ and $X(t_2)$ are independent for $t_1 \neq t_2$. The first assumption refers to the "Gaussian" and the second one to the "white[1]." Clearly, this is a very simple model: it merely represents the drawing of independent normal random variables at different time instants. Still, it can be used to model complex stochastic systems. To see this, we will first develop a model of WGN. Then, we will see some non-trivial applications of this process.

Before proceeding, though, we need some preliminary definitions. First, let a Gaussian process be one for which the probability distribution of any linear combination of its values is normally distributed. Formally,

$$\sum_{i=1}^{N} a_i X(t_i) \sim \mathcal{N}(\mu, \sigma^2), \tag{1}$$

for arbitrary coefficients a_i, times t_i, and number of terms N. Then, define the mean value function as

$$\mu(t) = \mathbb{E}\left[X(t)\right] \tag{2}$$

and the autocorrelation function as

$$R_X(t_1, t_2) = \mathbb{E}\left[X(t_1)X(t_2)\right]. \tag{3}$$

For $t_1 = t_2 = t$, (3) is sometimes called the the process power (or variance) function:

$$P(t) = R(t,t) = \mathbb{E}\left[X^2(t)\right].$$

Gaussian processes have the appealing property of being completely determined by their mean value and autocorrelation functions. Notice that all of these definitions apply both to continuous and discrete time processes.

For this problem we also need to define the Dirac delta $\delta(t)$. If you took ESE 224, you have had headaches about this already. The Dirac delta is often thought as a function that is 0 everywhere and infinite at 0, i.e., $\delta(0) = \infty$ and $\delta(t) = 0$ for $t \neq 0$. This is a bad way to think about it. First, this is not a valid definition of a function. Second, we will be integrating deltas and if we use this definition we will have to deal with the fact that it has infinite magnitude at a point of measure zero. Regardless of whether you use Riemann sums or Lebesgue integrals, you will have an issue evaluating this integral because $0 \times \infty$ is undefined. The Dirac delta is actually a *distribution*, a generalization of functions, and it is defined through the integral of its product with an arbitrary

[1] *White noise* is a traditional term in signal processing to refer to stochastic processes made of independent random variables. The reason for this name is that the spectrum of these stochastic processes (signals) is flat, i.e., all frequencies have the same magnitude. It just so happens that this is what the spectrum of white light looks like. Take a look at the material from ESE 224 if you need to refresh your memory on spectra and Fourier transforms.

function $f(t)$. Formally, $\delta(t)$ satisfies

$$\int_a^b f(t)\delta(t) = \begin{cases} f(0) & \text{for } a < 0 < b \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Using deltas, we can define a zero-mean WGN as a Gaussian process $W(t)$ with

$$\mu_W(t) = 0, \quad \text{for all } t, \qquad \text{and} \qquad R_W(t_1, t_2) = \sigma^2 \delta(t_1 - t_2). \tag{5}$$

Notice that since the autocorrelation function of $W(t)$ is not really a function (it involves the Dirac delta), WGN *cannot* model any real physical phenomena. Nonetheless, it is a convenient abstraction to generate processes that *can* model real physical phenomena.

A Independence. Using the definitions in (5), show that values $W(t_1)$ and $W(t_2)$ are independent for $t_1 \neq t_2$.

B The integral of WGN. Define the process $X(t)$ as the integral of $W(\tau)$ from 0 to t, i.e.,

$$X(t) = \int_0^t W(\tau) d\tau. \tag{6}$$

Show that $X(t)$ is normally distributed for all t and compute its mean and autocorrelation functions. What is the probability of $X(t) > a$ for arbitrary a and $t > 0$?

C Discrete time representation of WGN. Define the discrete time process $W_h(n)$ as the integral of $W(t)$ over the interval $[nh, (n+1)h)$, i.e.,

$$W_h(n) = \int_{nh}^{(n+1)h} W(\tau) d\tau. \tag{7}$$

Compute the mean value function $\mu_{W_h}(n)$ and the autocorrelation function $R_{W_h}(n_1, n_2)$ of this process.

D Simulating $X(t)$. Use the result of Part C to produce a discrete time simulation of $X_h(n)$ of the process $X(t)$ from Part B. Run your simulation for a duration of $T = 10$ with step size $h = 0.01$ and $\sigma^2 = 1$.