# ESE3700: Circuit-Level Modeling, Design, and Optimization for Digital Systems

Lec 13: March 24, 2025

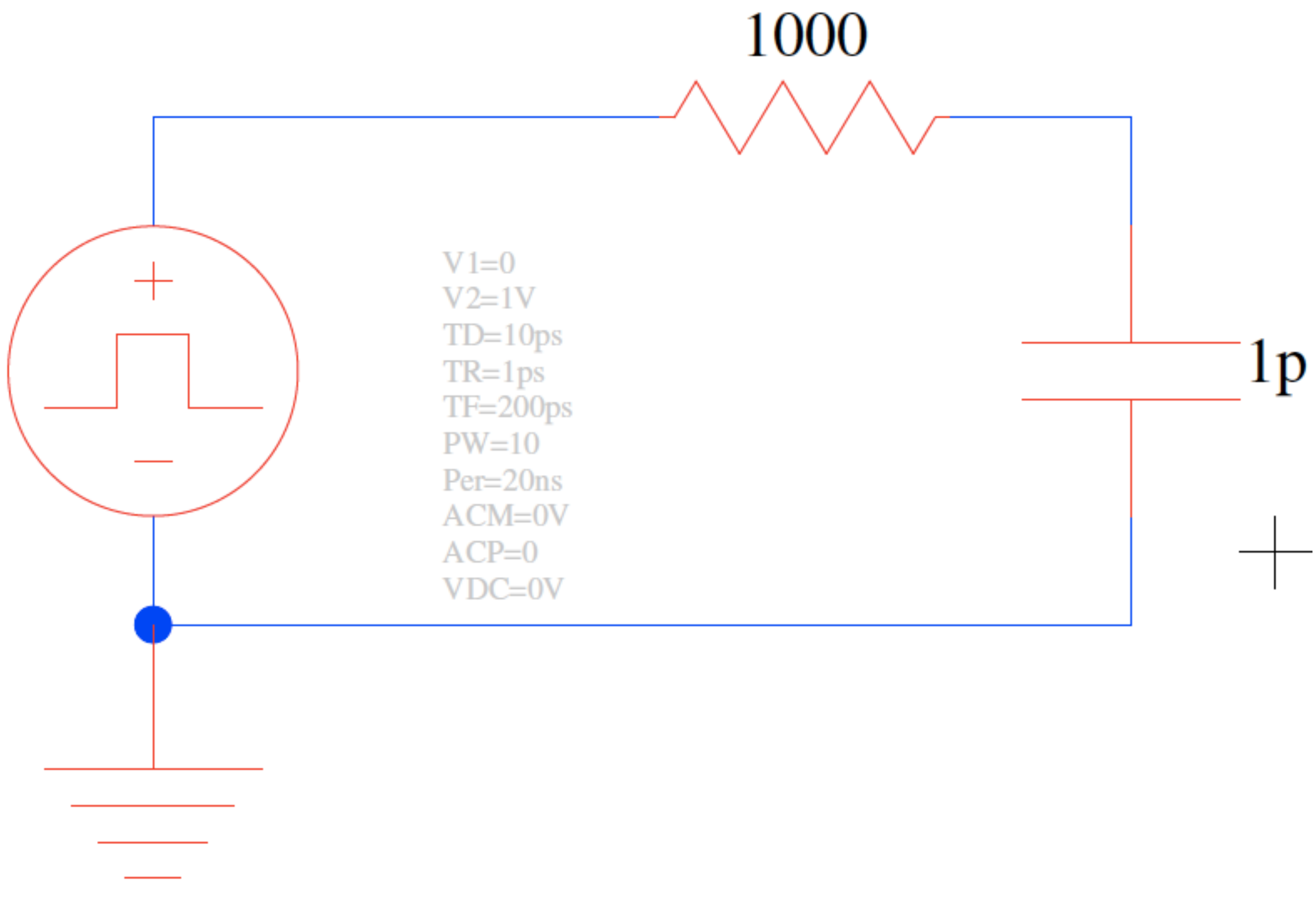
Distributed RC Wire and Elmore Delay

# Today

- Estimate delay in RC Network
  - Elmore delay calculation
- Apply to wire delay
- Apply to pass transistor circuits
- Apply to CMOS gates

# Distributed RC (setup)

# What is response? (Preclass 1)



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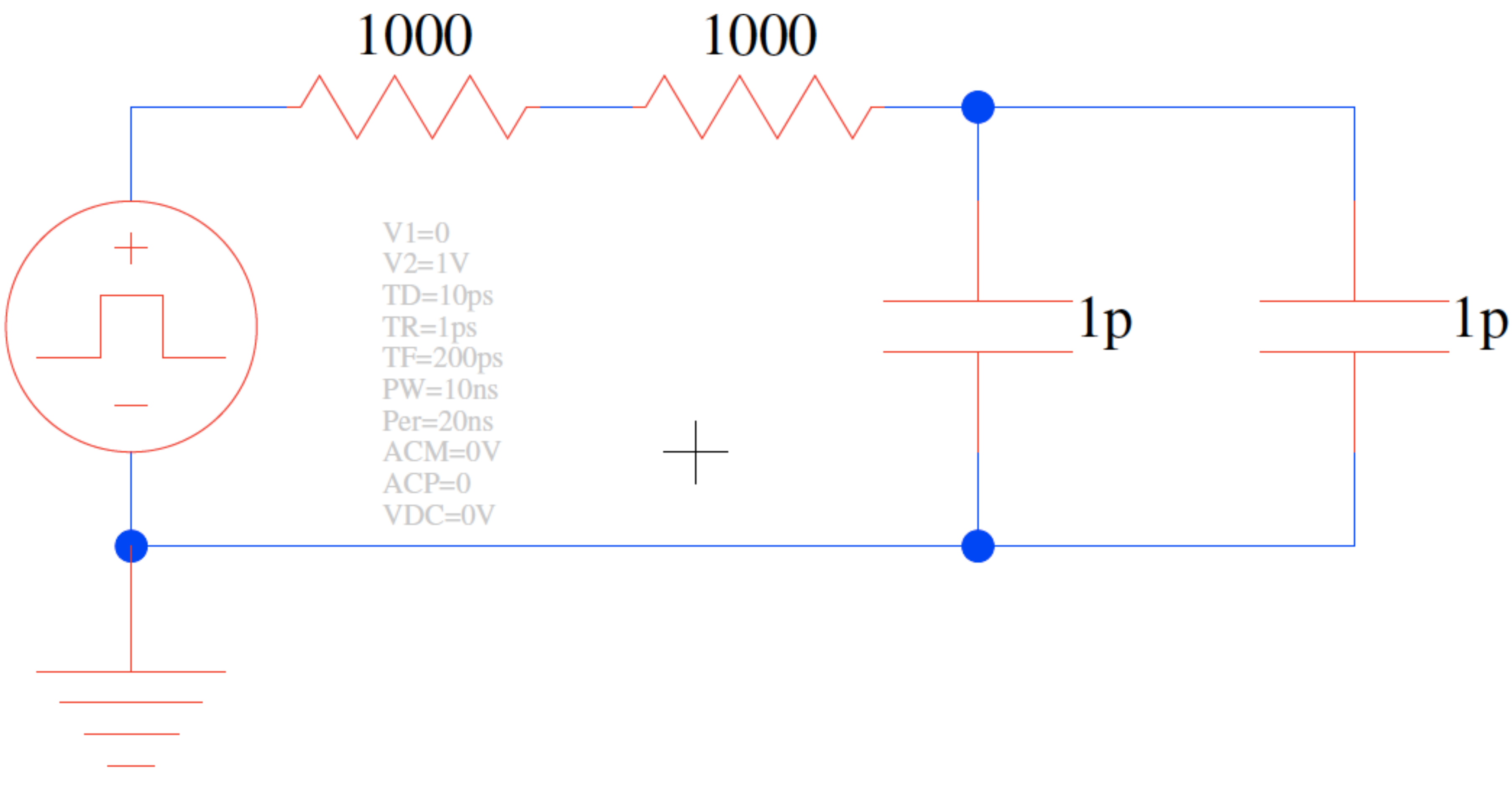

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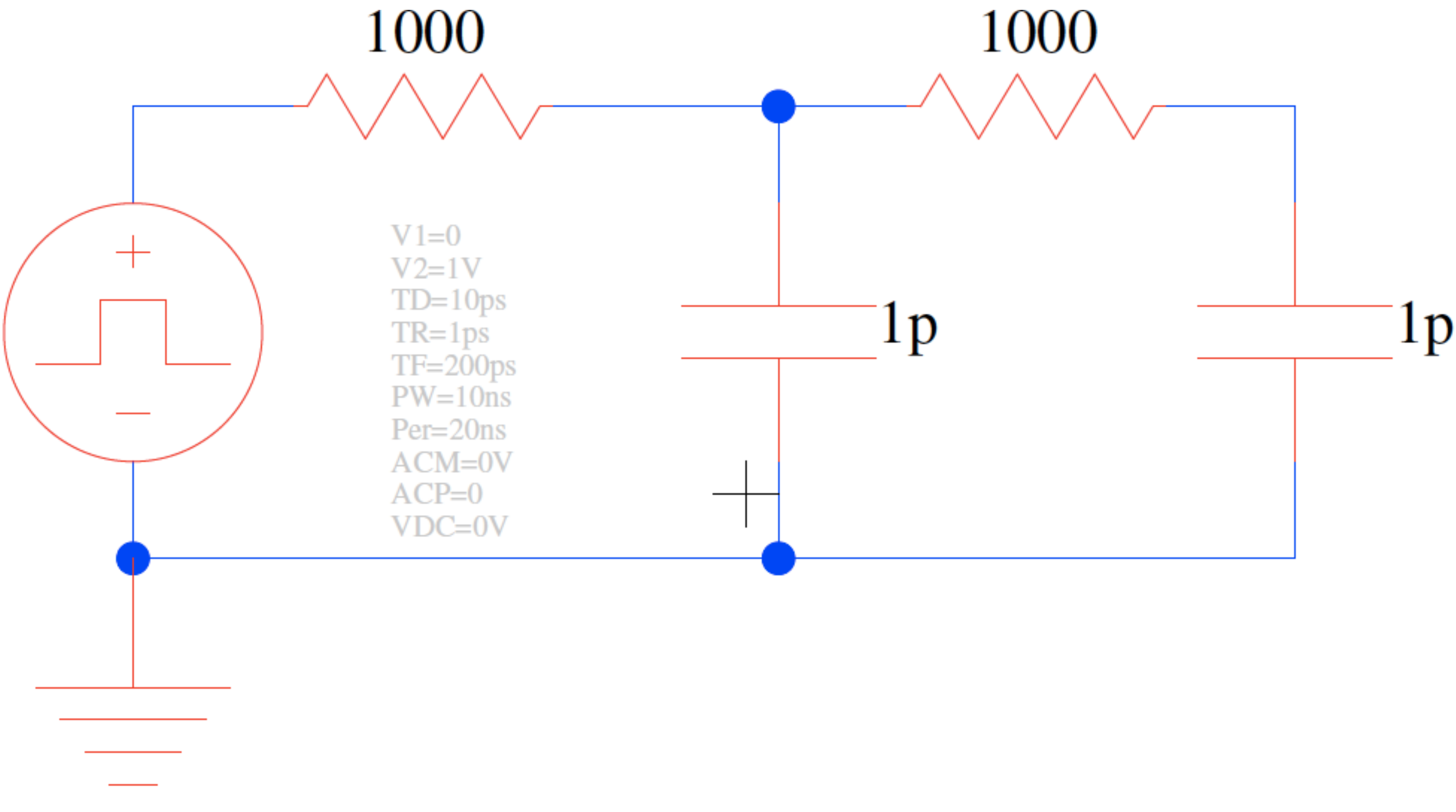
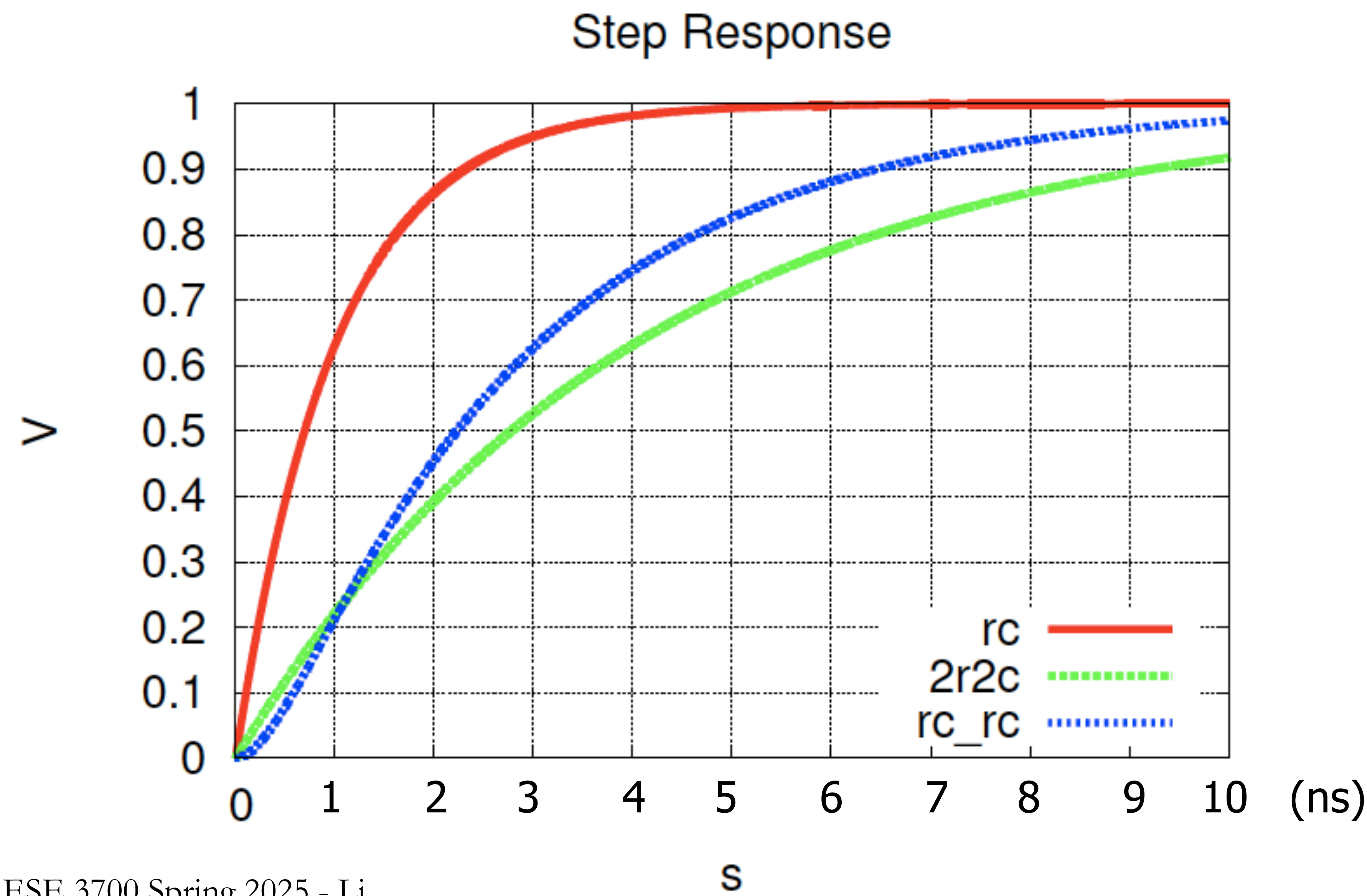


# SPICE Response



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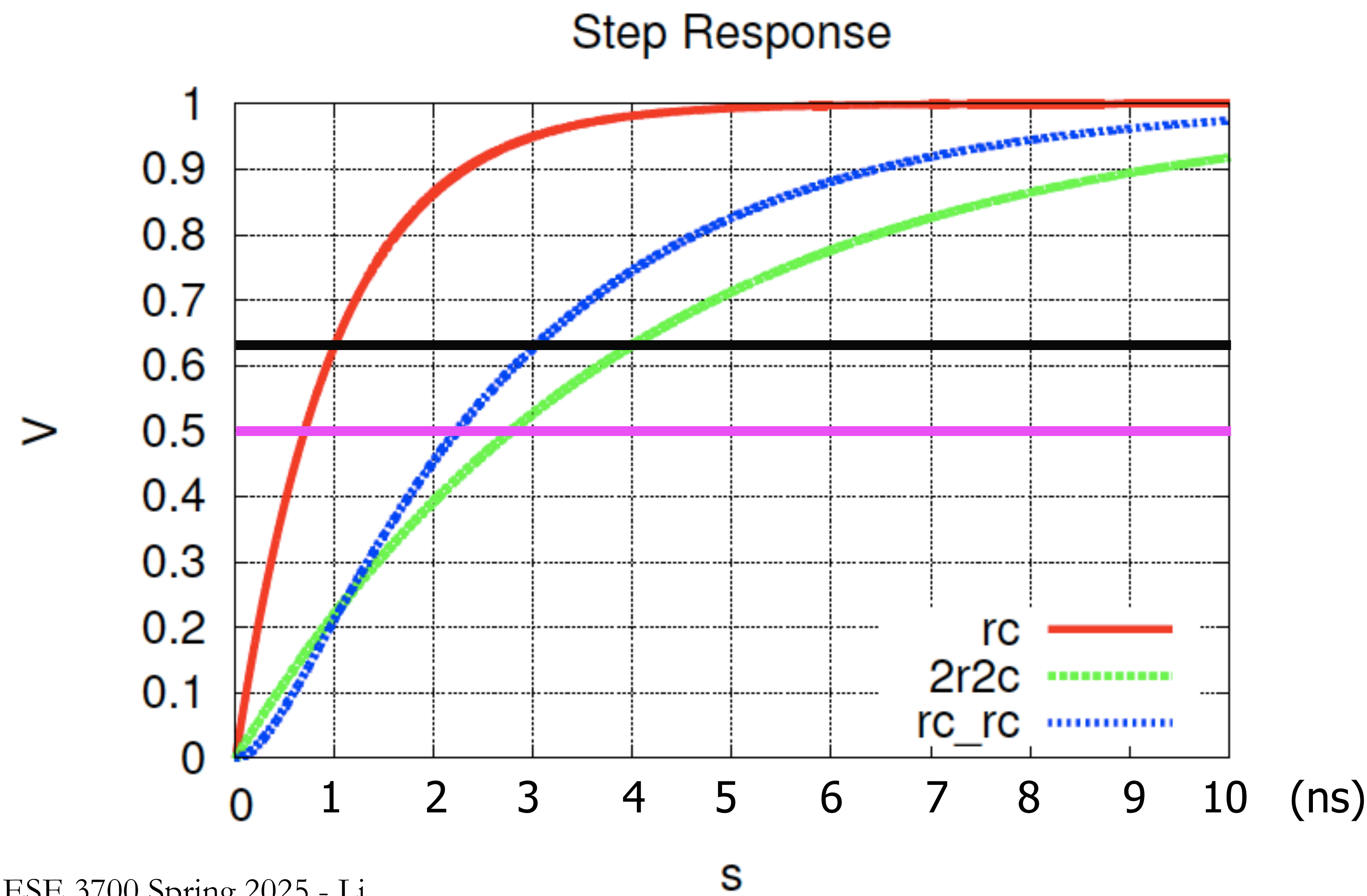
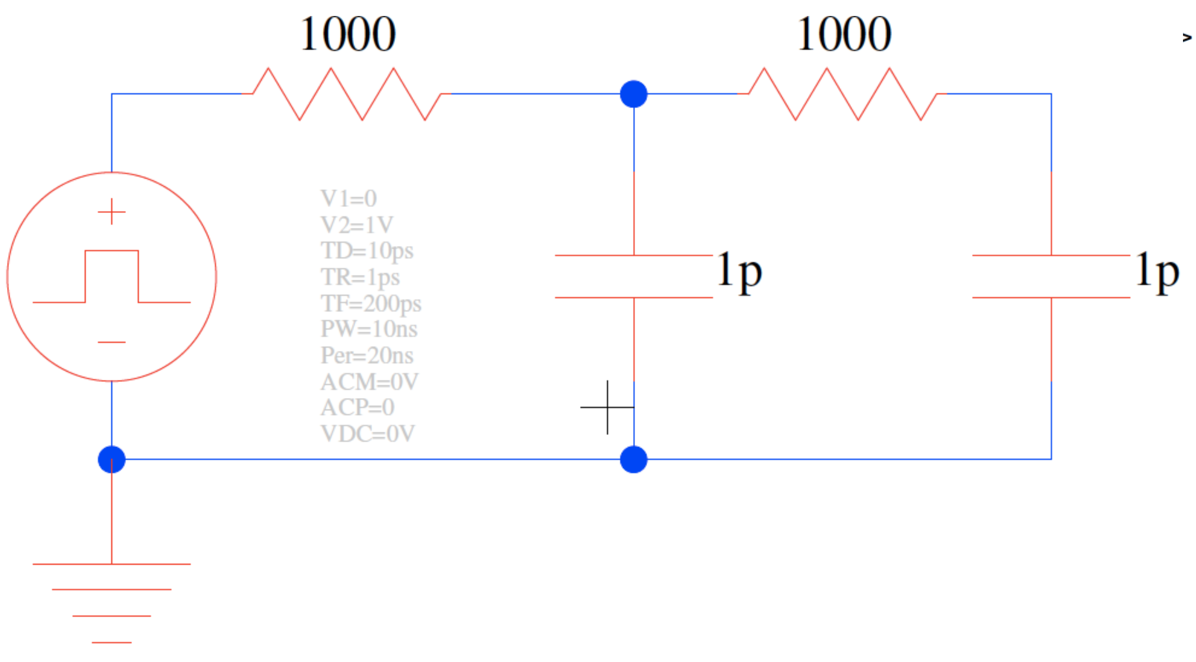
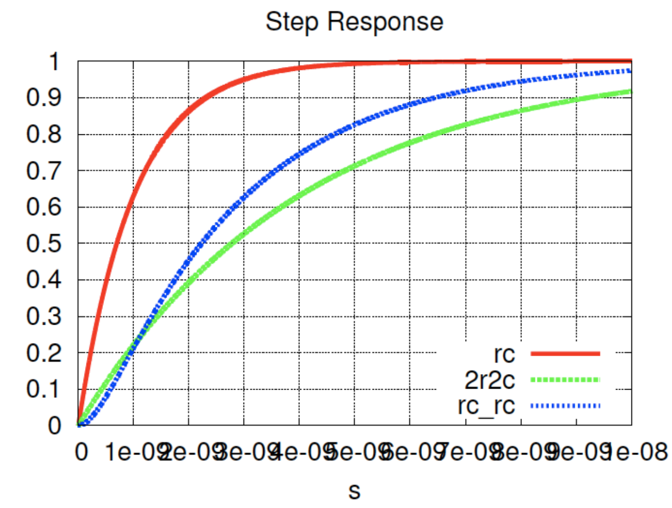


# Intuition

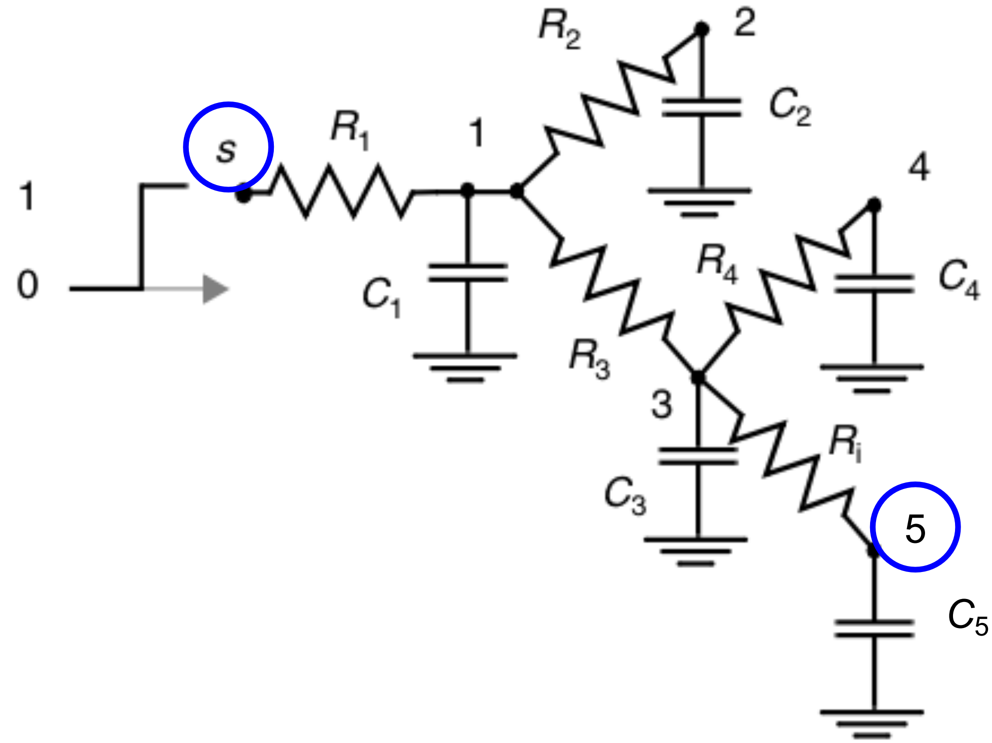
- Look at series of R's on path
  - Must move Q=V(ΣC) across each R



# Elmore Delay: Distributed RC network

- The delay from source *s* to node *5*
  - N = number of nodes in circuit



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- The delay from source *s* to node *5*
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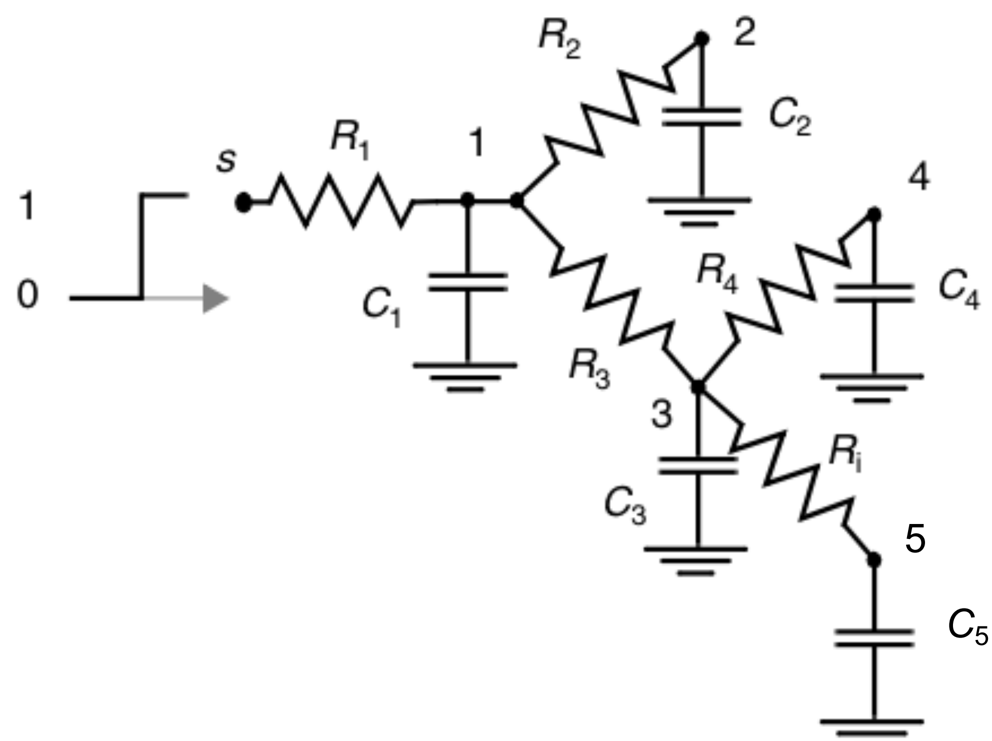
$$R_{5k} = \sum R_j \Rightarrow (R_j \in [path(s \rightarrow 5) \cap path(s \rightarrow k)])$$

$$\tau_{D5} = \sum_{k=1}^{N} C_k R_{5k}$$



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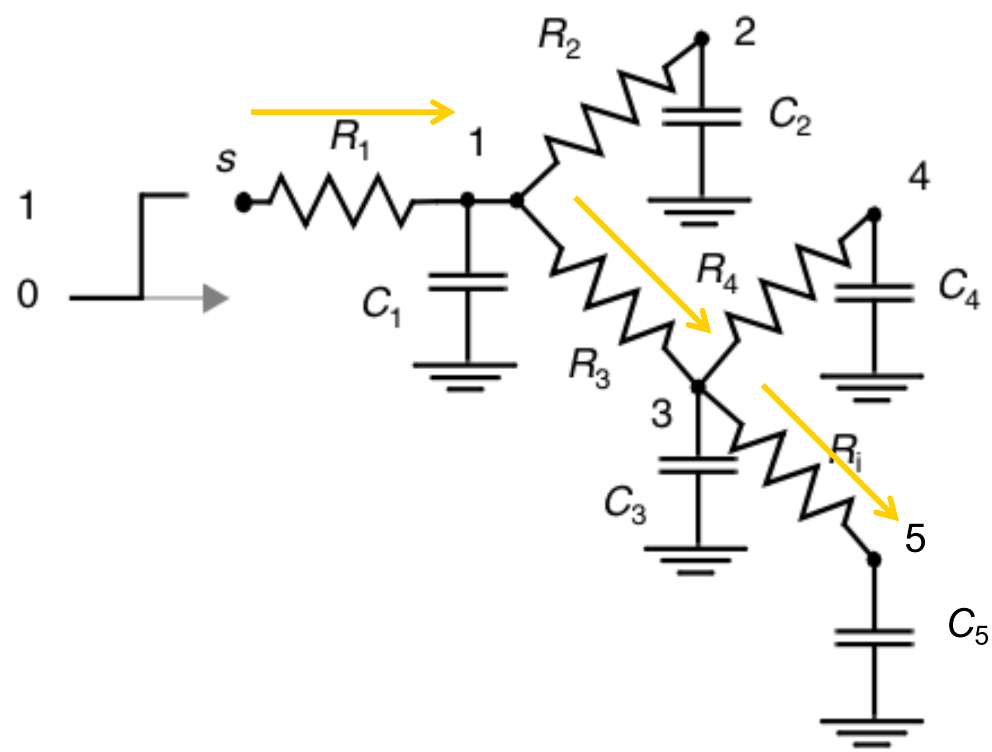

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*k=4*

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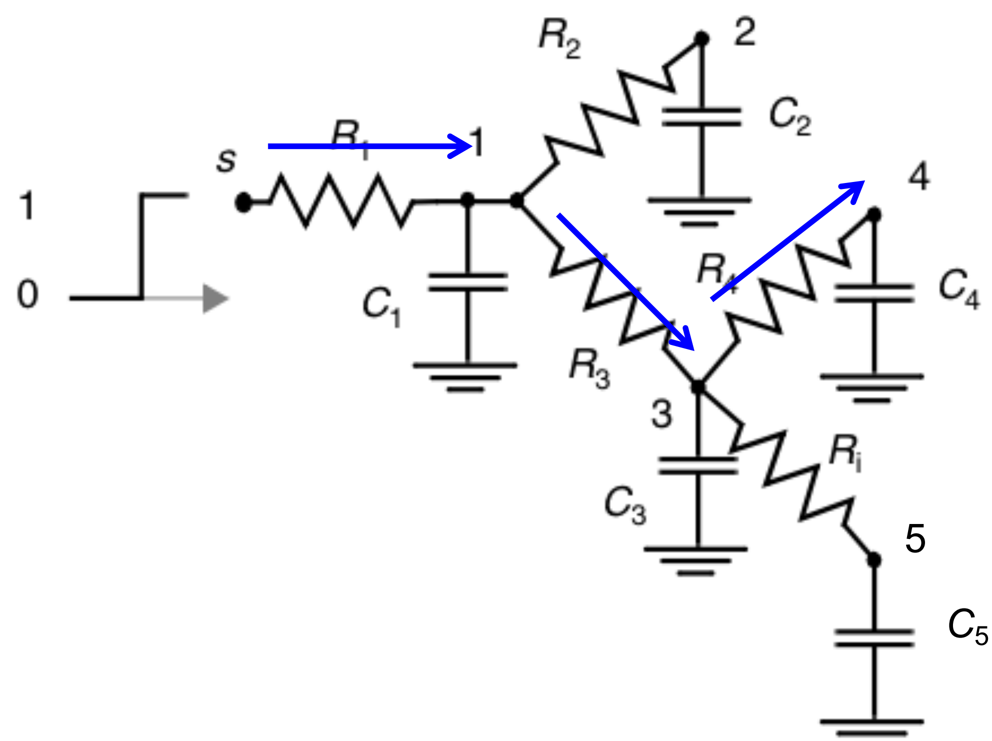

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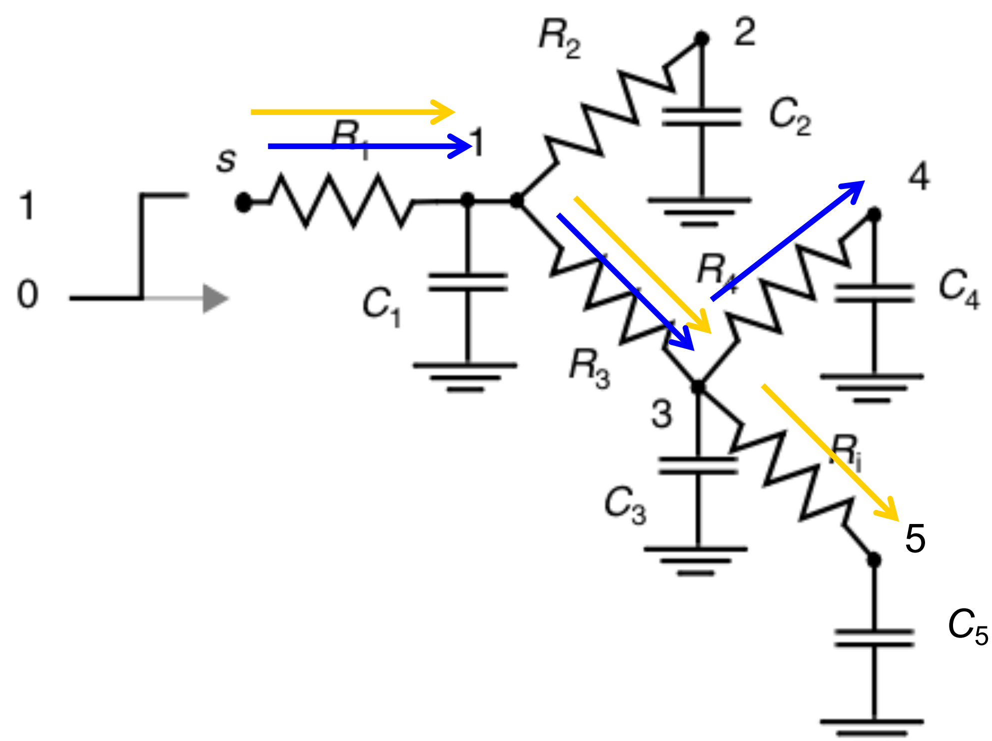

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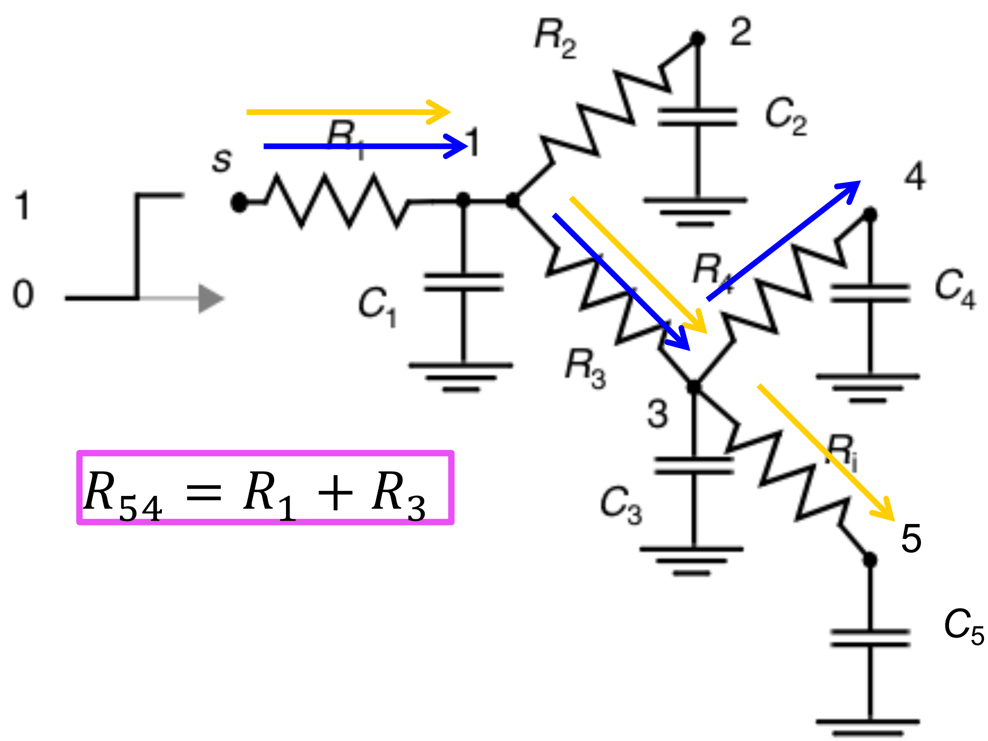


$R_{54} = R_1 + R_3$

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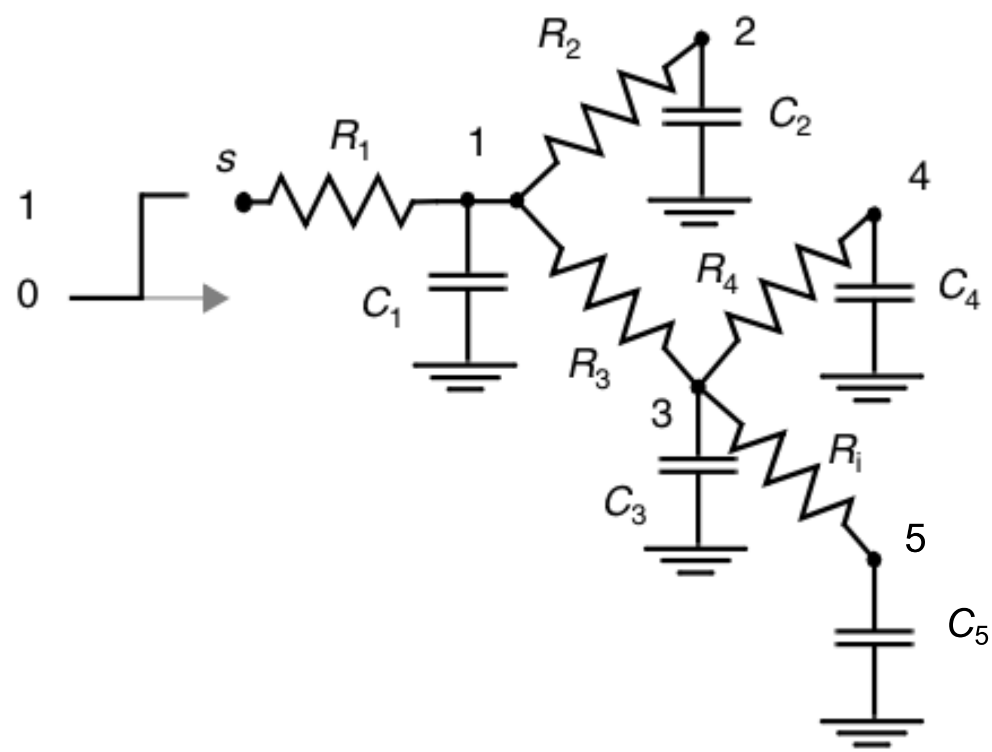

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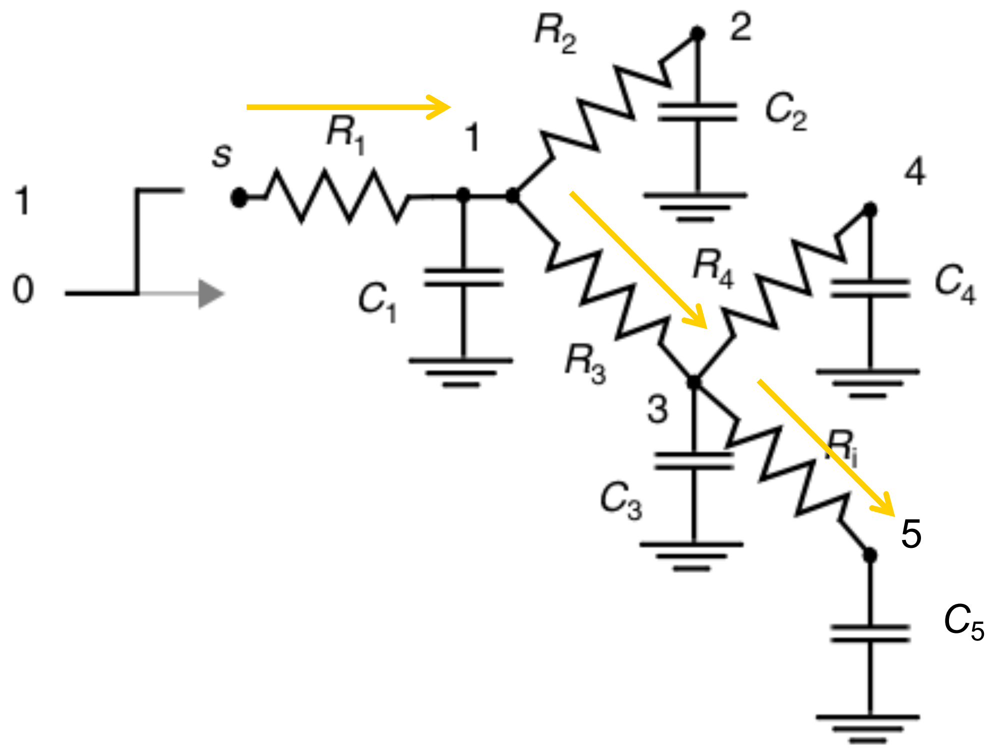
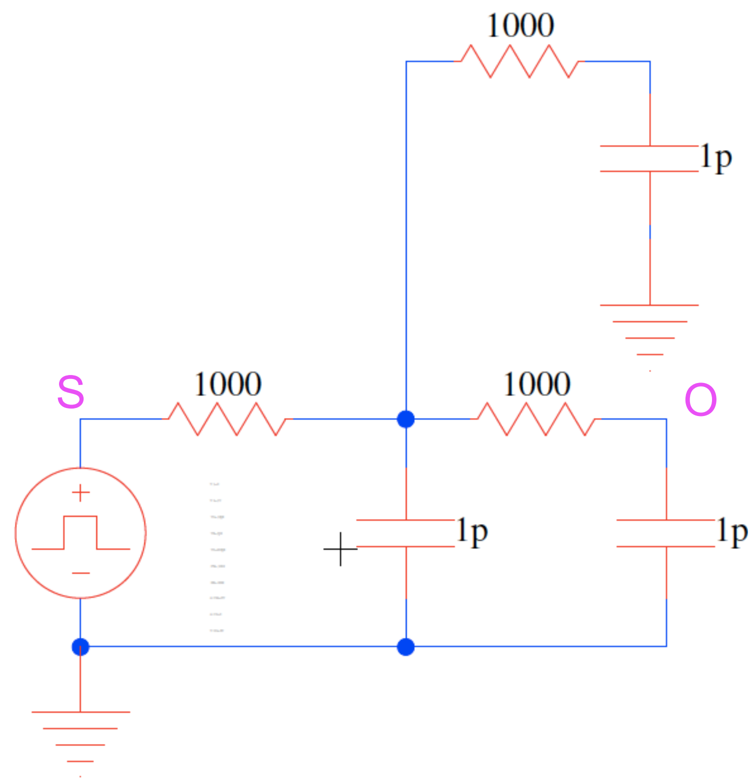


# Elmore Delay: Practice

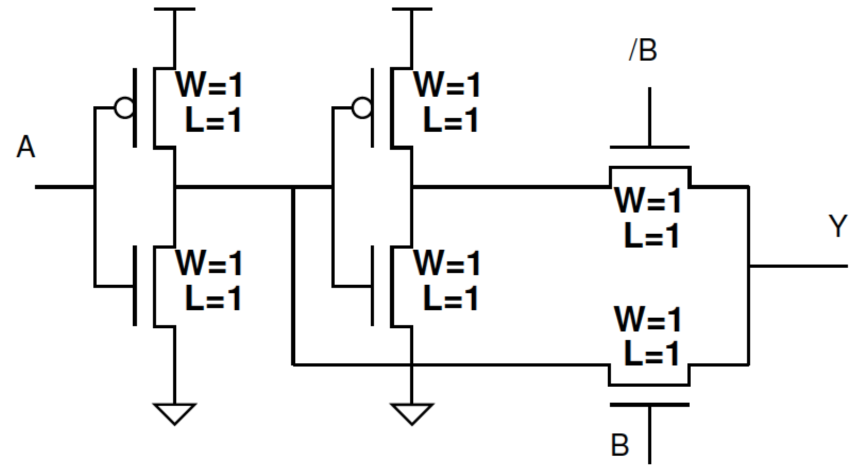
- The delay from source *s* to node *O*
  - N = number of nodes in circuit

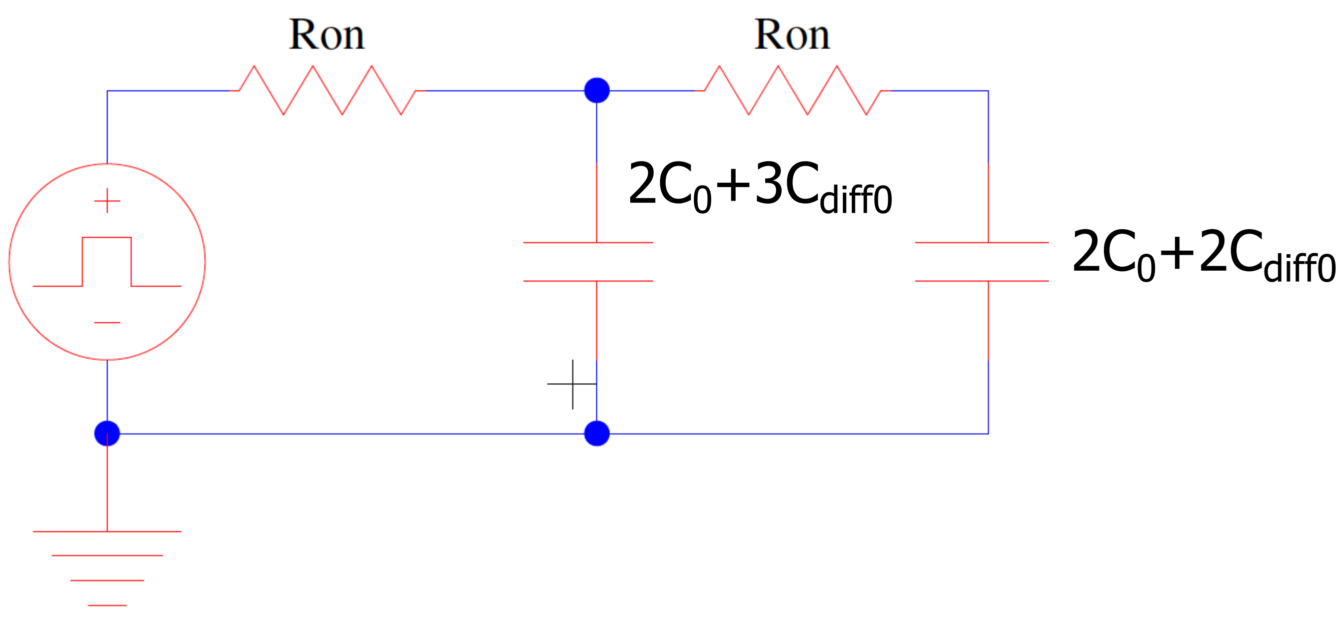
$$R_{Ok} = \sum R_j \Rightarrow (R_j \in [path(s \to O) \cap path(s \to k)])$$

$$\tau_{DO} = \sum_{k=1}^{N} C_k R_{Ok} = ?$$

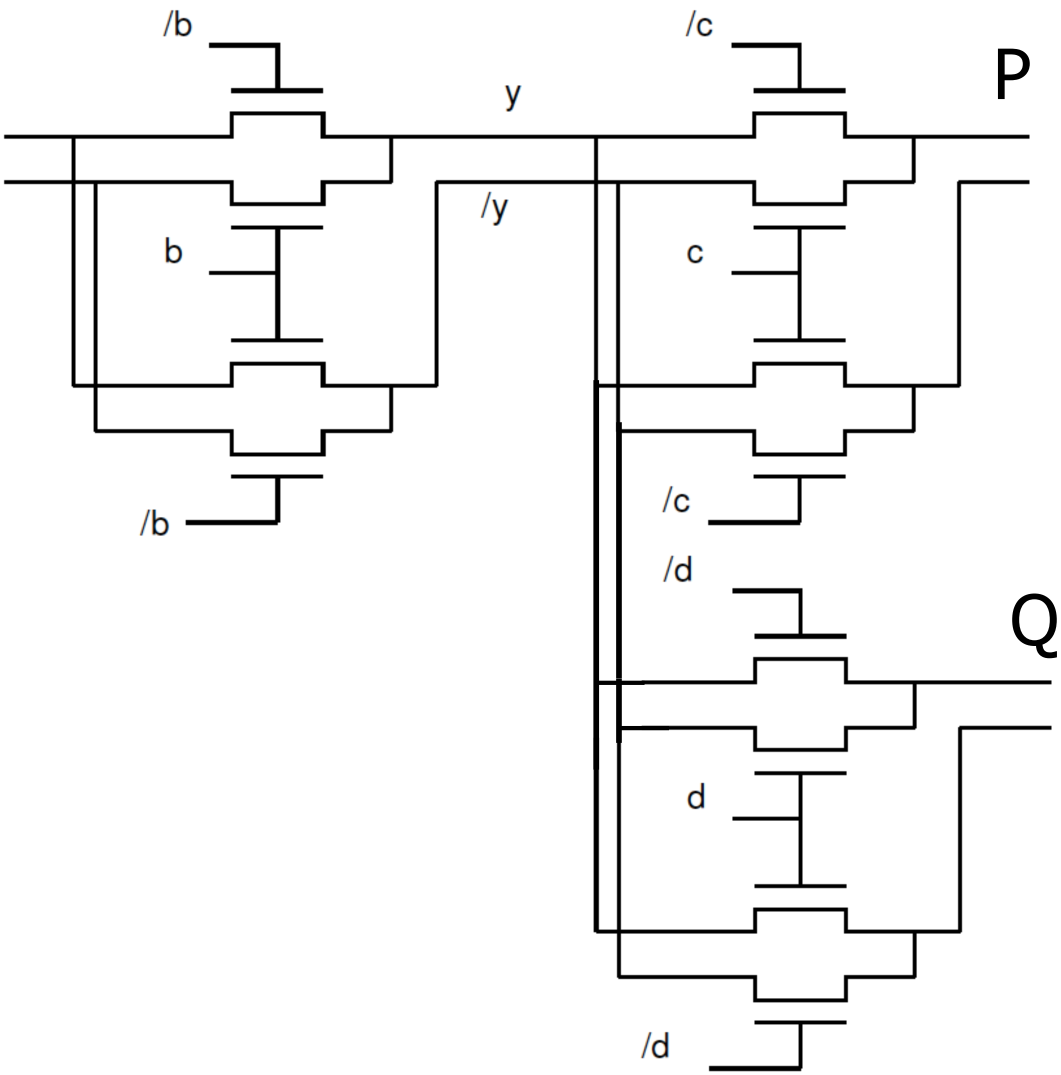


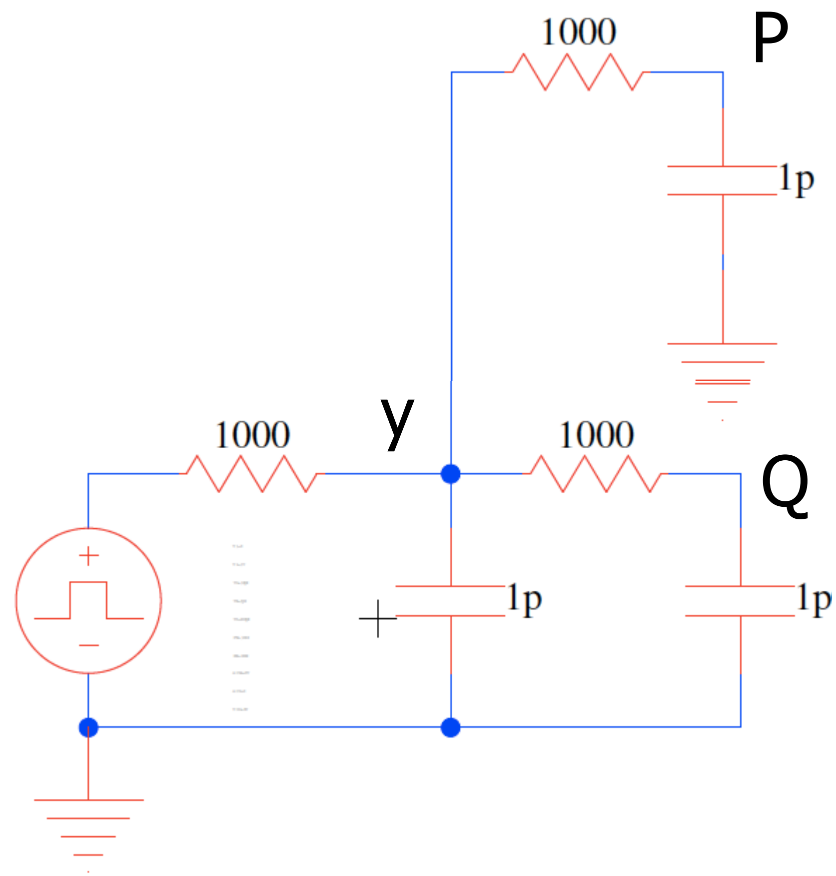
# Previously: Equivalent RC



# What is response?



# Elmore Delay: Practice (preclass 1)

- The delay from source *s* to node *i*
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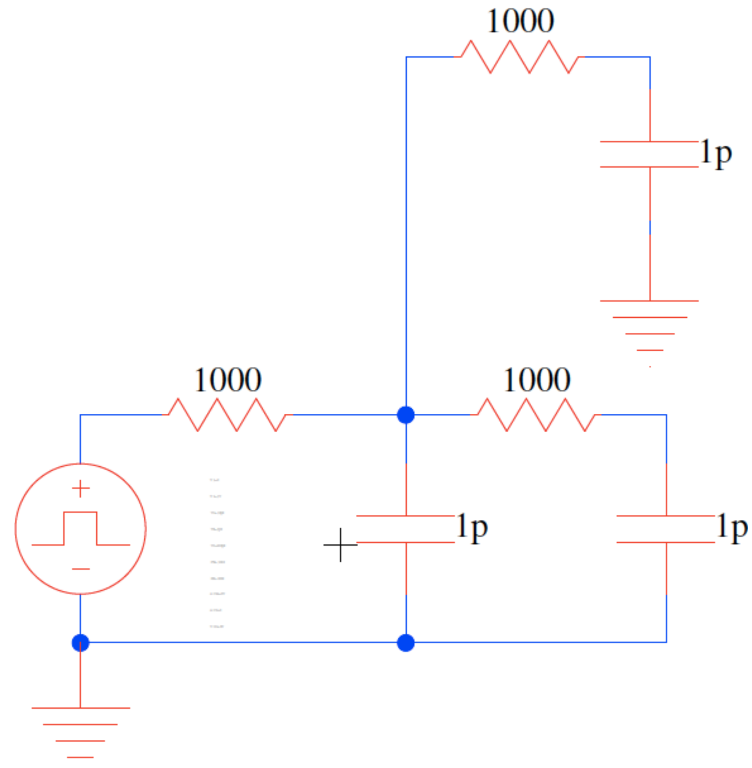
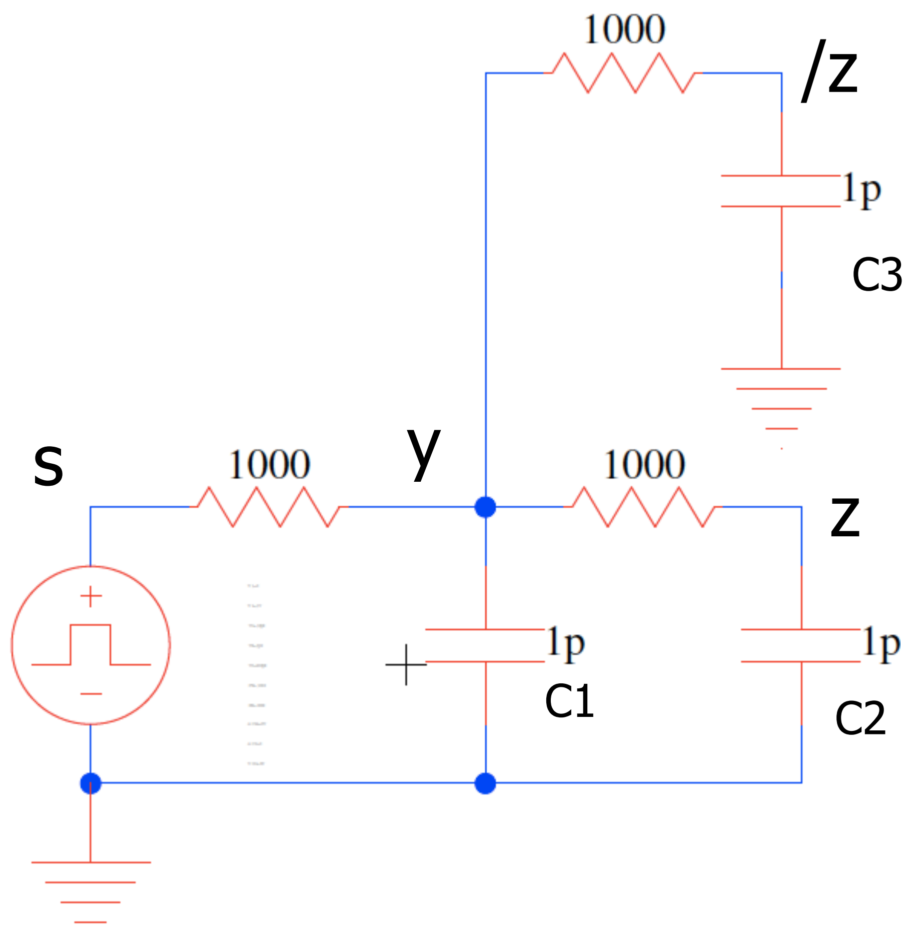
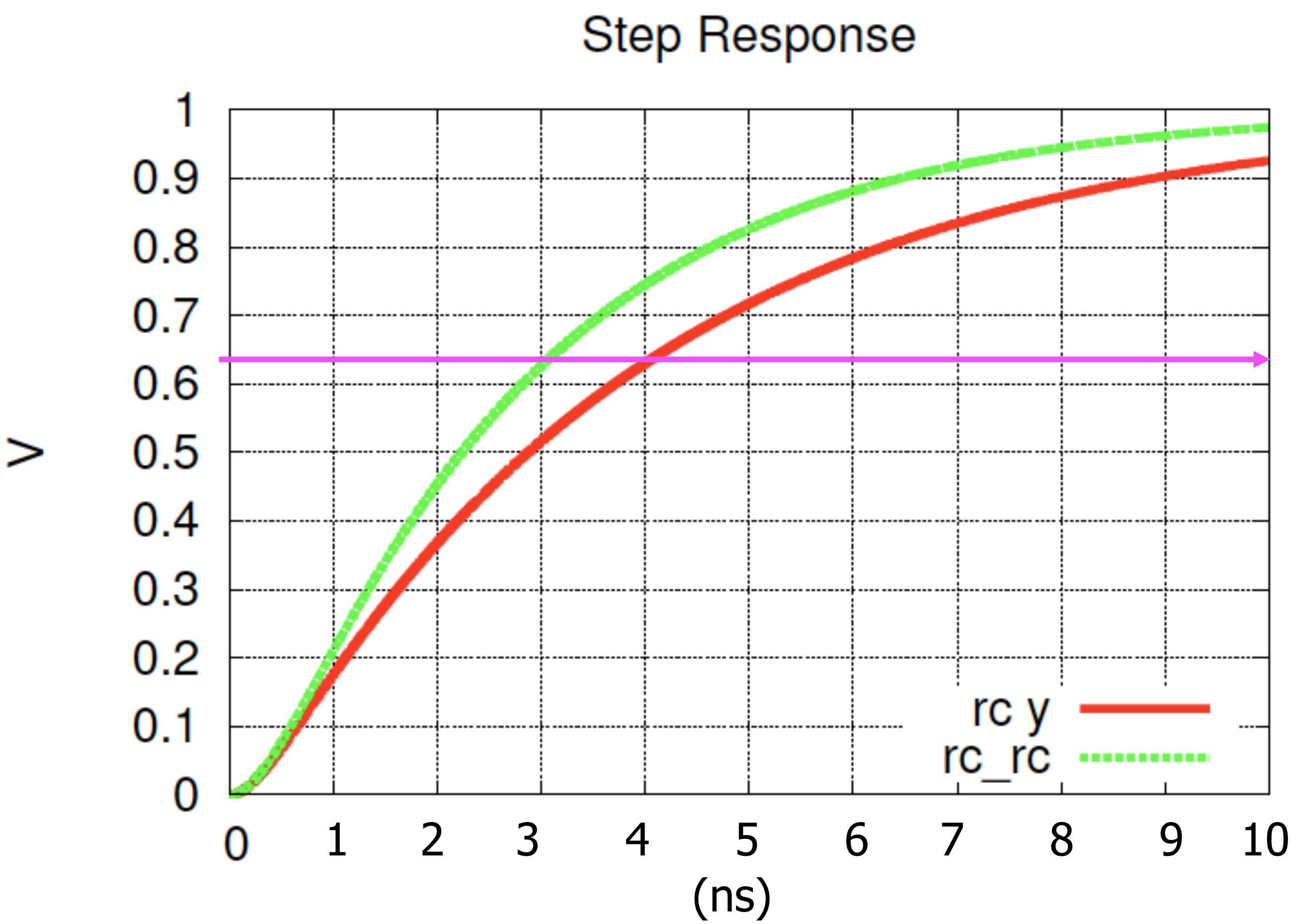


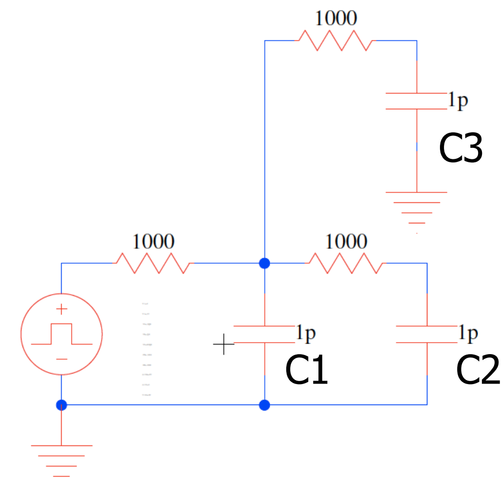
# Apply (preclass 1)

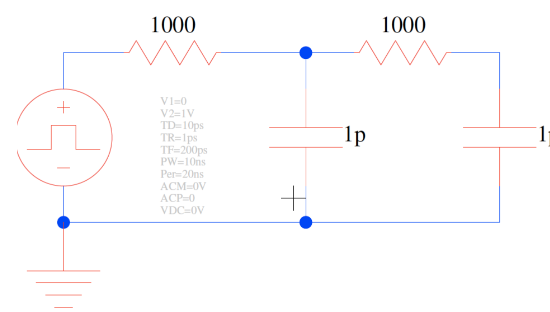
- What is Elmore delay?
  - S → Z



# SPICE Response



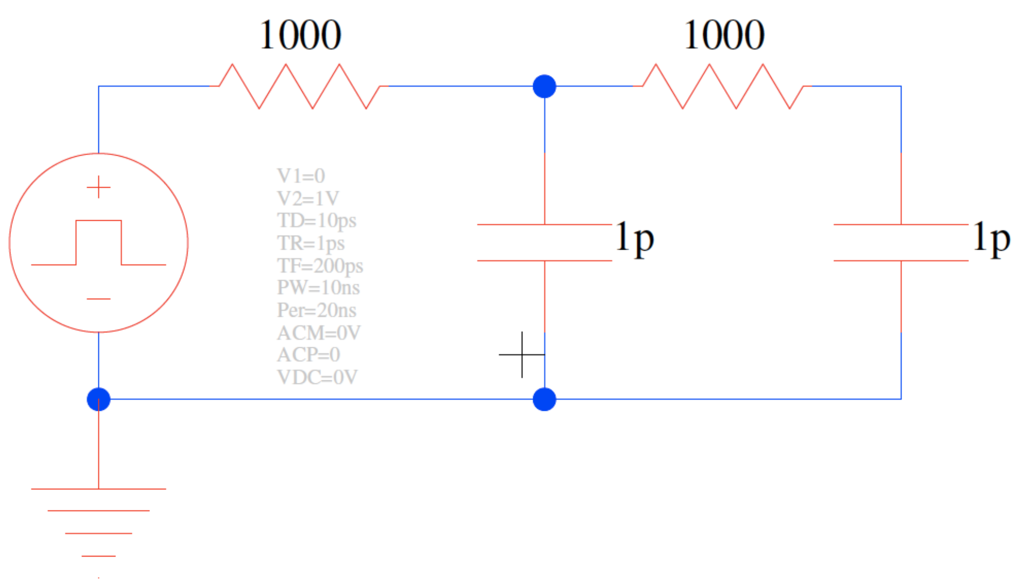



# Elmore Delay: Special Ladder Case

- For each resistor $C_k$ in path
  - Compute $R_{kk}$ = sum of all $R$s upstream of $C_k$

$$\tau_{DN} = \sum_{k=1}^{N} C_k \sum_{j=1}^{k} R_j = \sum_{k=1}^{N} C_k R_{kk}$$



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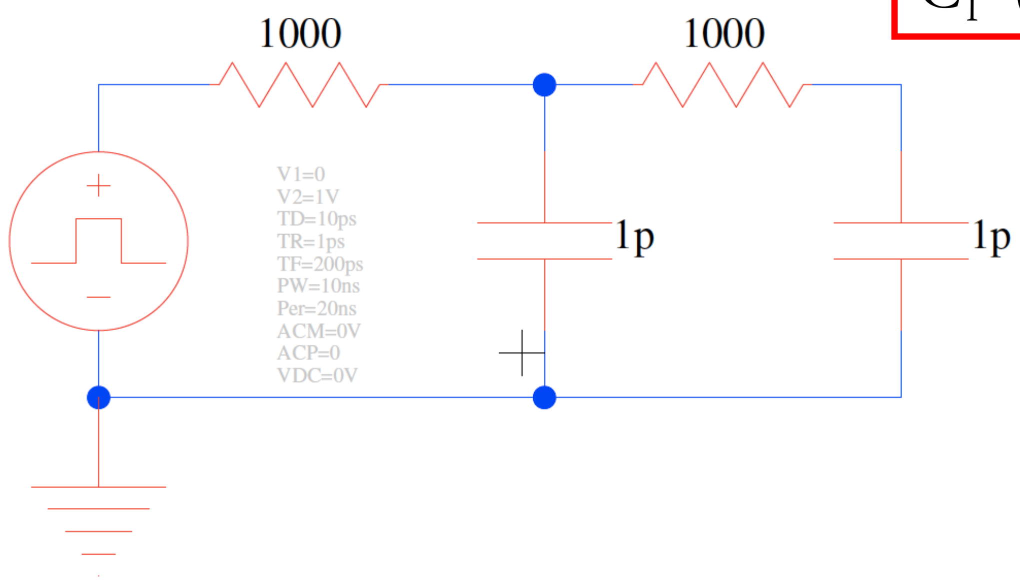
$C_1*(R_1)+C_2*(R_1+R_2)$



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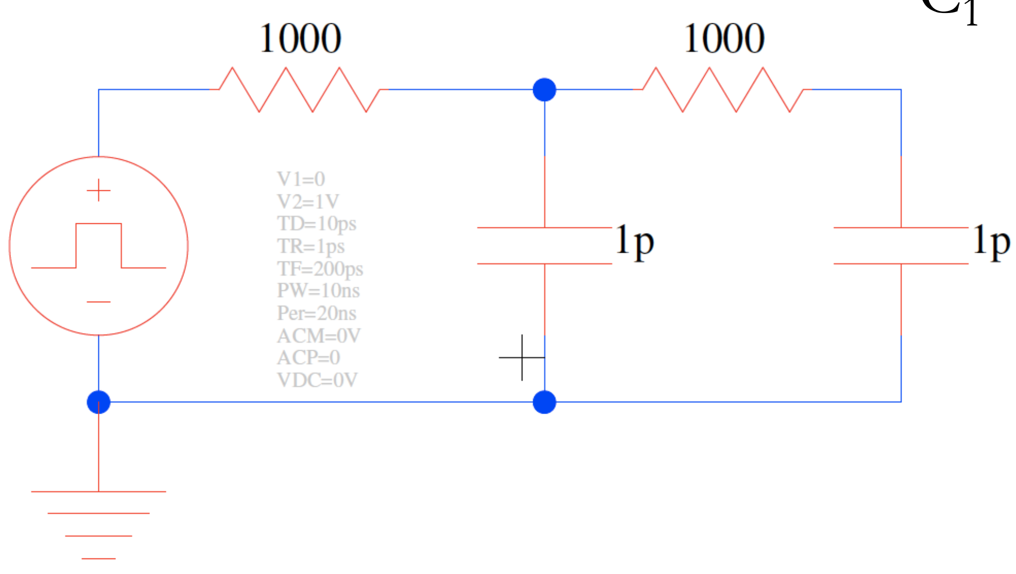


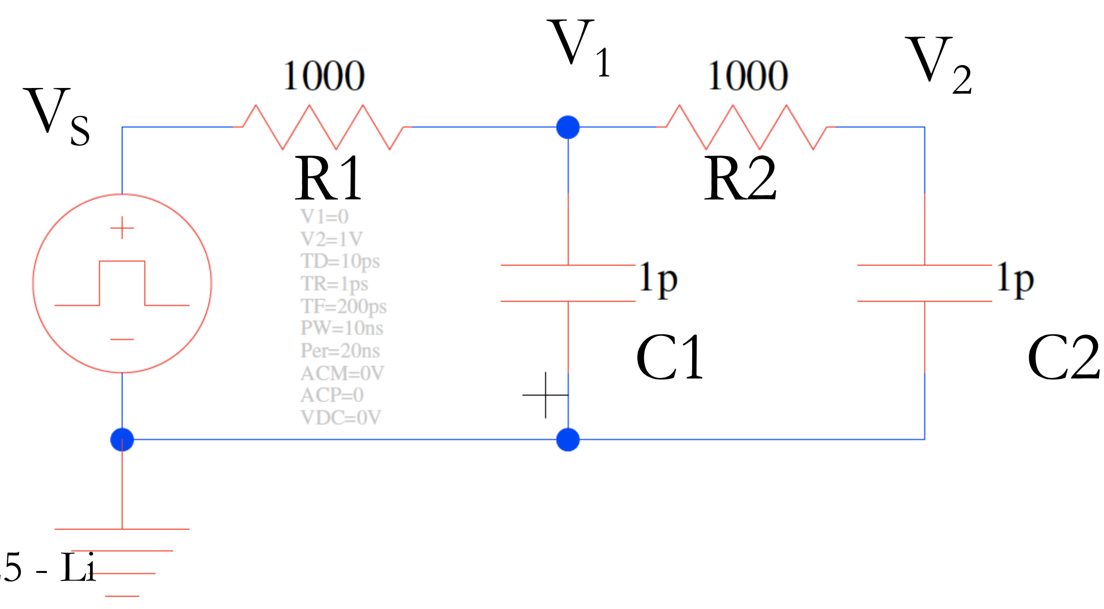

=3RC
=3ns

# Compare KCL: Setup

- Equations from KCL?

@$V_1$: $$\frac{V_S - V_1}{R_1} = \frac{V_1 - V_2}{R_2} + C_1 \frac{dV_1}{dt}$$

@$V_2$: $$\frac{V_1 - V_2}{R_2} = C_2 \frac{dV_2}{dt}$$



# Compare KCL: Math

**@$V_1$:**

$$\frac{V_S - V_1}{R_1} = \frac{V_1 - V_2}{R_2} + C_1 \frac{dV_1}{dt}$$

$$\frac{V_S}{R_1} = \frac{V_1}{R_1} + \frac{V_1}{R_2} - \frac{V_2}{R_2} + C_1 \frac{dV_1}{dt}$$

$$V_S = V_1\left(1 + \frac{R_1}{R_2}\right) - \frac{R_1 V_2}{R_2} + R_1 C_1 \frac{dV_1}{dt}$$

**@$V_2$:**

$$\frac{V_1 - V_2}{R_2} = C_2 \frac{dV_2}{dt}$$

$$\frac{V_1}{R_2} = \frac{V_2}{R_2} + C_2 \frac{dV_2}{dt}$$

$$V_1 = V_2 + R_2 C_2 \frac{dV_2}{dt}$$

$$\frac{dV_1}{dt} = \frac{dV_2}{dt} + R_2 C_2 \frac{d^2 V_2}{dt^2}$$

# Compare KCL: Math

$$V_S = V_1\left(1+\frac{R_1}{R_2}\right)-\frac{R_1}{R_2}V_2+R_1C_1\frac{dV_1}{dt}$$

$$V_S = \left(V_2+R_2C_2\frac{dV_2}{dt}\right)\left(1+\frac{R_1}{R_2}\right)-\frac{R_1}{R_2}V_2+R_1C_1\left(\frac{dV_2}{dt}+R_2C_2\frac{d^2V_2}{dt^2}\right)$$

$$V_S = V_2+\left(R_2C_2\left(1+\frac{R_1}{R_2}\right)+R_1C_1\right)\frac{dV_2}{dt}+R_1C_1R_2C_2\frac{d^2V_2}{dt^2}$$

$$V_S = V_2+(R_2C_2+R_1C_2+R_1C_1)\frac{dV_2}{dt}+R_1C_1R_2C_2\frac{d^2V_2}{dt^2}$$

# Compare KCL: Math

$$R_1 = R_2 = R$$

$$C_1 = C_2 = C$$

$$V_S = V_2 + (R_2C_2 + R_1C_2 + R_1C_1)\frac{dV_2}{dt} + R_1C_1R_2C_2\frac{d^2V_2}{dt^2}$$

$$V_S = V_2 + 3RC\frac{dV_2}{dt} + R^2C^2\frac{d^2V_2}{dt^2}$$

# Compare KCL: Math

$$R_1 = R_2 = R$$

$$C_1 = C_2 = C$$

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$$V_S = V_2 + 3RC\frac{dV_2}{dt} + R^2C^2\frac{d^2V_2}{dt^2}$$

$$V_2 = A(1 + e^{-\alpha t}) \rightarrow$$

$$V_S = A(1 + e^{-\alpha t}) - 3RC \cdot \alpha A e^{-\alpha t} + R^2C^2 \cdot \alpha^2 A e^{-\alpha t}$$

# Compare KCL: Math

$$V_2 = A(1 + e^{-\alpha t}) \rightarrow$$

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$$t = \infty \rightarrow V_2 = V_S = A$$

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$$t = \infty \rightarrow V_2 = V_S = A$$

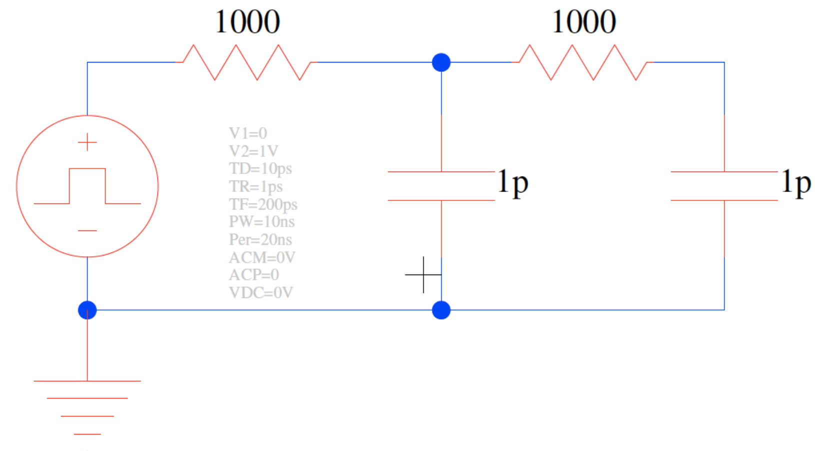
$$V_S = V_S(1 + e^{-\alpha t}) - 3RC \cdot \alpha V_S e^{-\alpha t} + R^2C^2 \cdot \alpha^2 V_S e^{-\alpha t}$$

$$0 = e^{-\alpha t} - 3RC \cdot \alpha e^{-\alpha t} + R^2C^2 \cdot \alpha^2 e^{-\alpha t}$$

$$0 = e^{-\alpha t}(1 - 3RC\alpha + R^2C^2 \cdot \alpha^2)$$

$$0 = 1 - 3RC\alpha + R^2C^2 \cdot \alpha^2$$

# Compare KCL: Math



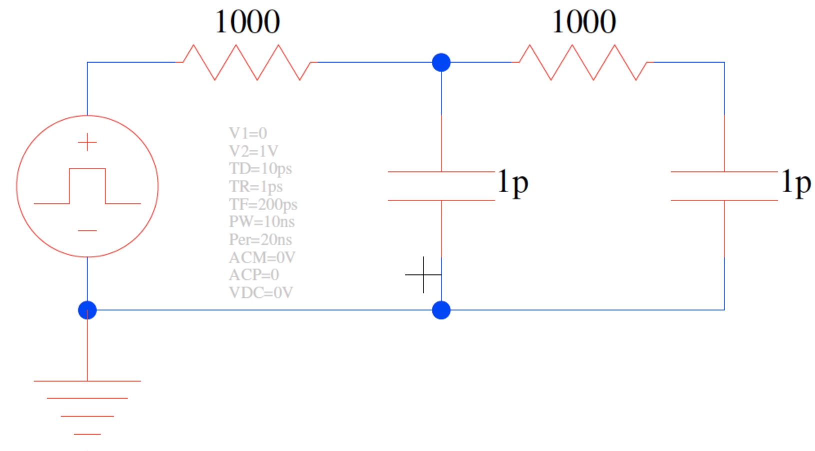

$$V_2 = V_S(1 + e^{-\alpha t})$$

$$0 = 1 - 3RC\alpha + R^2C^2 \cdot \alpha^2$$

$$\alpha = \frac{3RC \pm \sqrt{9(RC)^2 - 4(RC)^2}}{2(RC)^2} = \frac{3RC \pm \sqrt{5}RC}{2(RC)^2} = \frac{3 \pm \sqrt{5}}{2RC}$$

$$\rightarrow \tau = \frac{2}{3 \pm \sqrt{5}} RC \approx 2.6RC$$

# Compare KCL: Math

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=3RC
=3ns

# Elmore Delay: Distributed RC network

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  - N = number of nodes in circuit

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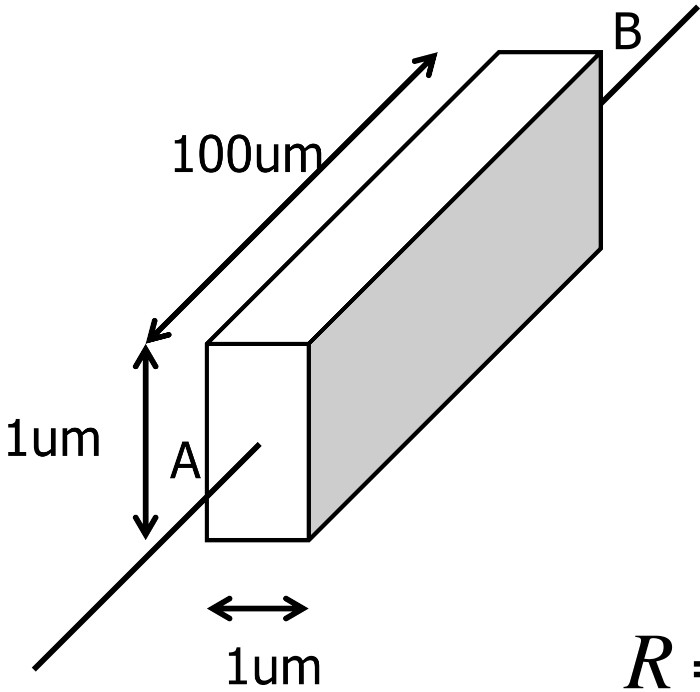
$$\tau_{Di} = \sum_{k=1}^{N} C_k R_{ik}$$

- Special ladder case

$$\tau_{DN} = \sum_{k=1}^{N} C_k \sum_{j=1}^{k} R_j = \sum_{k=1}^{N} C_k R_{kk}$$

# Wire Delay

# Wire Resistance

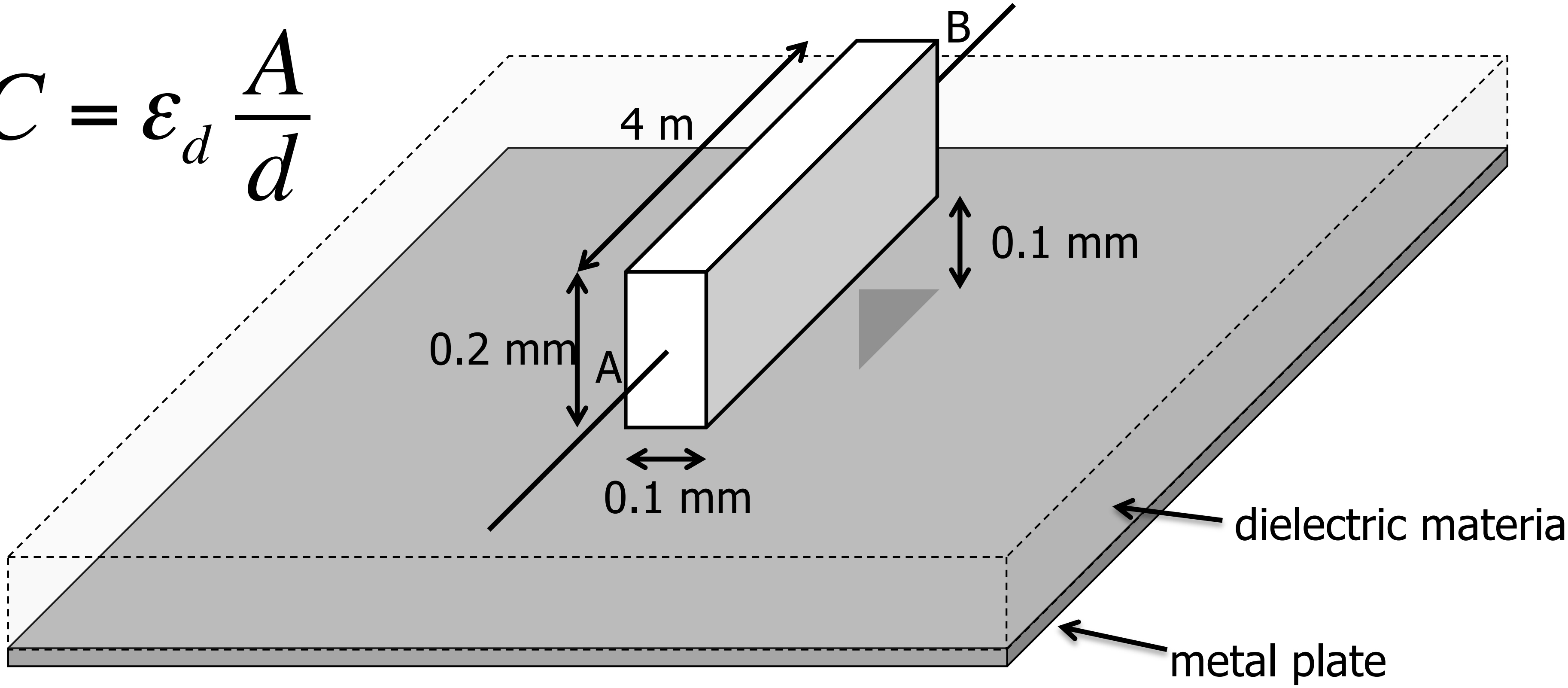



$$R = \frac{\rho L}{A}$$

$$R = \frac{10^{-7}\Omega \cdot m \cdot 100\mu m}{1\mu m \cdot 1\mu m} = 10\Omega$$

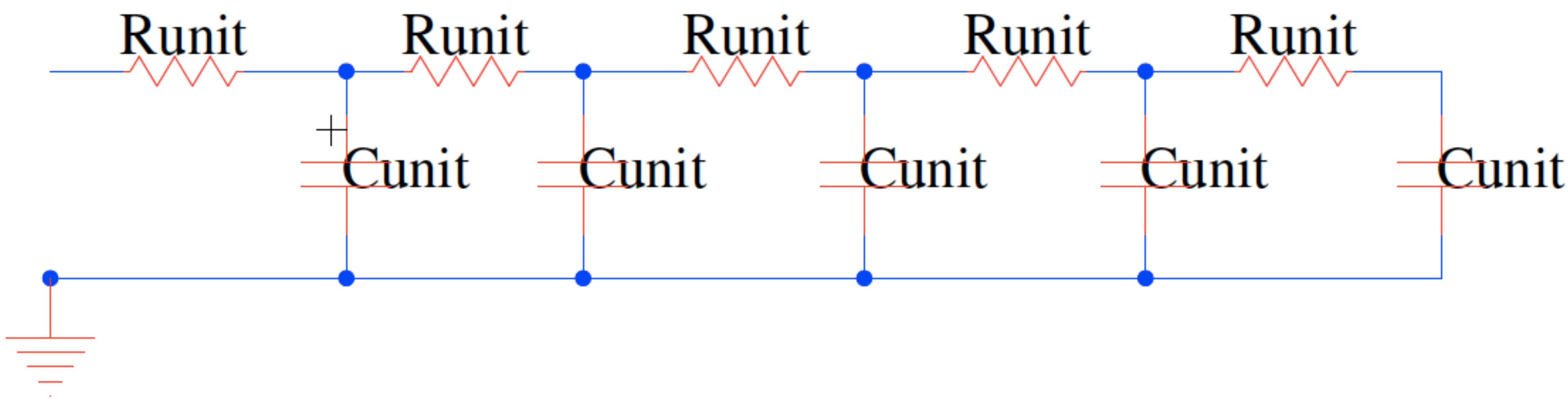
# Wire Capacitance

$$C = \varepsilon_d \frac{A}{d}$$



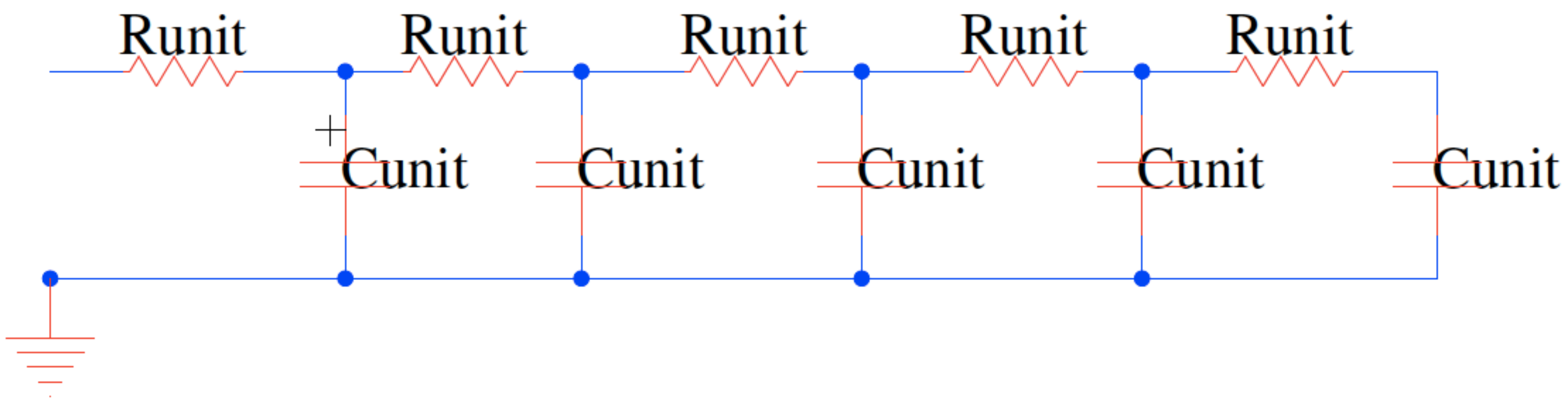
# Wire as Distributed RC Ladder

- Measure wire length in units
  - Say λ
  - Each λ unit has $C_{unit}$, $R_{unit}$
    - Unit capacitance and resistance of wire of length λ



# Wire Delay

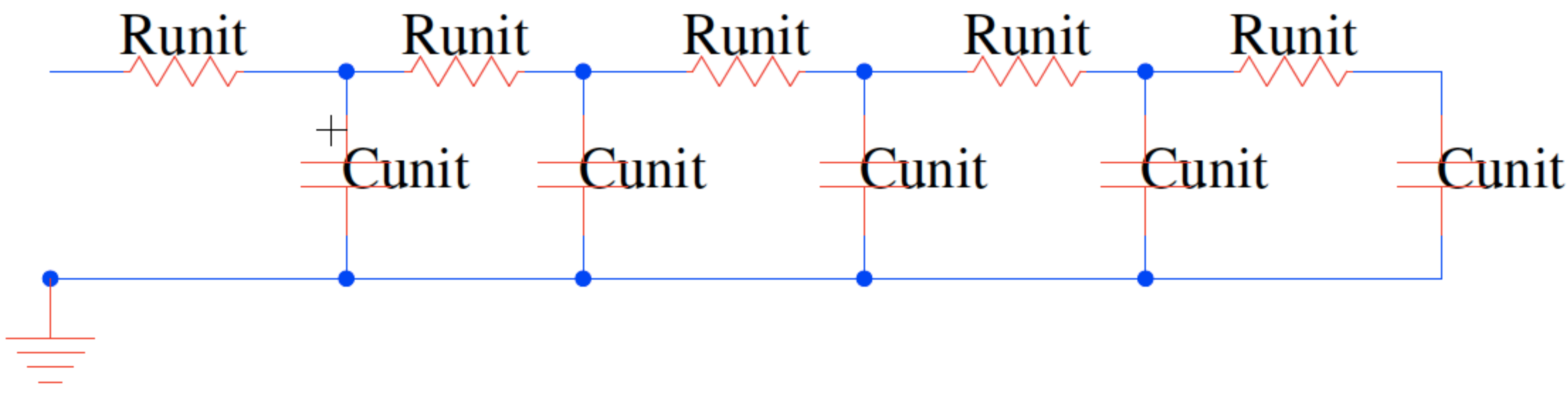
- Delay of Wire N units long:



# Wire Delay

- Delay of Wire N units long:

  $R_{unit}C_{unit}$

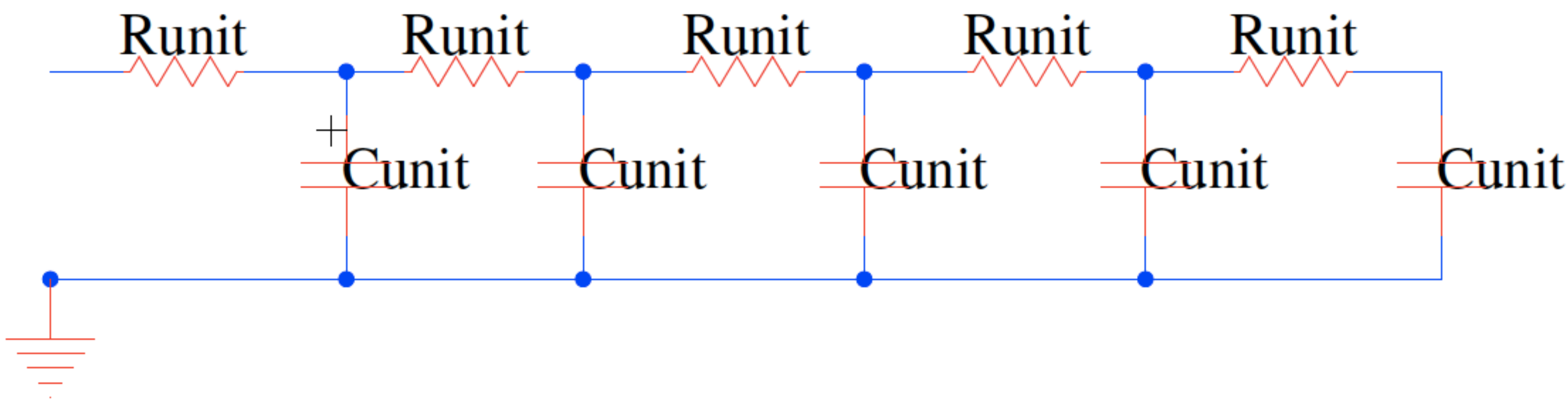


# Wire Delay

- Delay of Wire N units long:

  $R_{unit}C_{unit}$

  $+ 2R_{unit}C_{unit}$



# Wire Delay

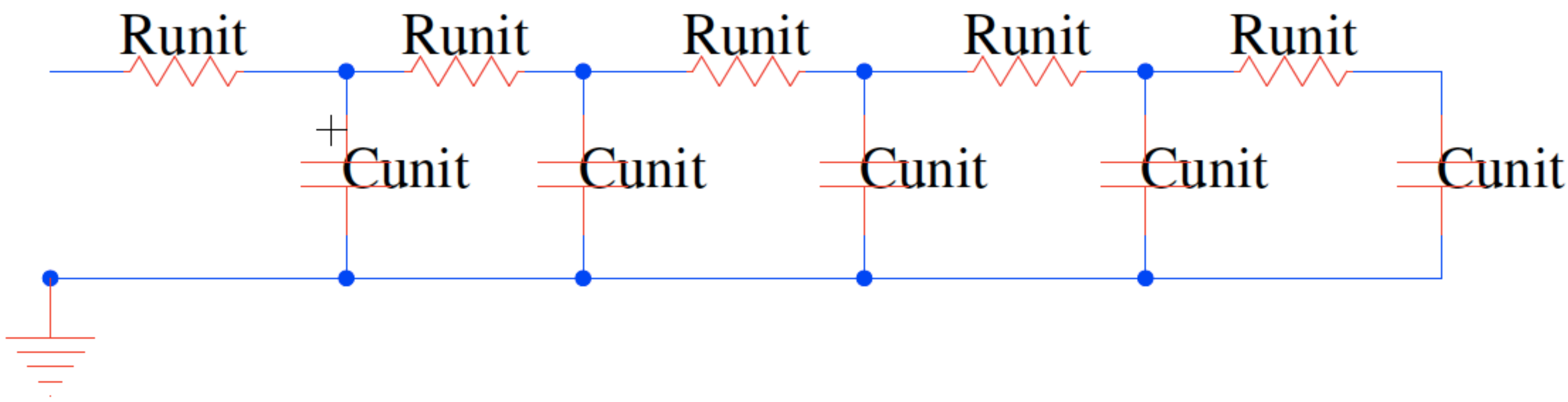
- Delay of Wire N units long:

  $R_{unit}C_{unit}$

  $+ 2R_{unit}C_{unit}$

  $+ 3R_{unit}*C_{unit} + \ldots +NR_{unit}*C_{unit}$

  $=(R_{unit}*C_{unit})*(N+(N-1)+(N-2)+\ldots.1)$



# Sum of integers (preclass 2)

- What's the sum of the integer 1 to N?

$$\sum_{k=0}^{N} k =$$

# Sum of integers

- What's the sum of the integer 1 to N?

$$\sum_{k=0}^{N} k = \frac{N(N+1)}{2} \approx \frac{N^2}{2}$$

# Wire Delay (preclass 3)

- Delay of Wire N units long:

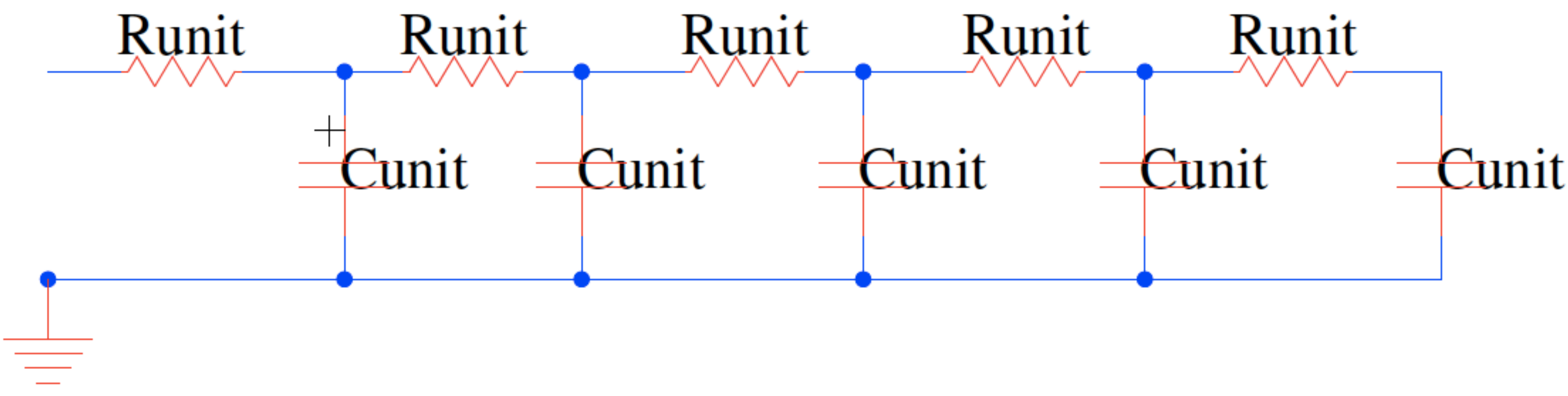
  $R_{unit}*(N*C_{unit})$

  $+ R_{unit}((N-1)*C_{unit}$

  $+ R_{unit}*(N-2)*C_{unit} + \ldots + R_{unit}*C_{unit}$

  $=(R_{unit}*C_{unit})*(N+N-1+N-2+\ldots.1)$

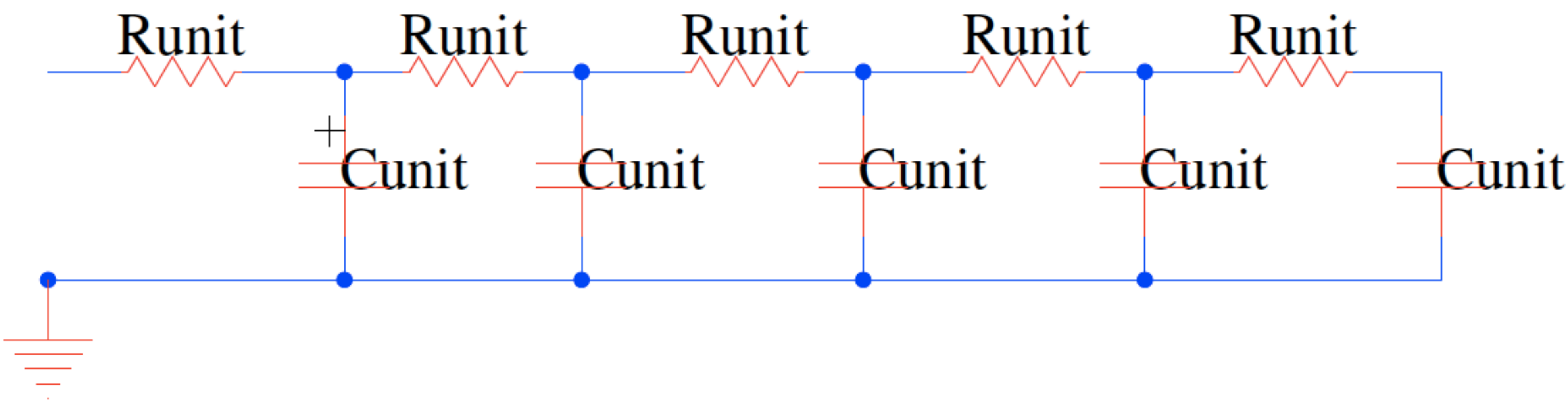
  $\sim=R_{unit}C_{unit}*N^2/2$



# Lumped RC Wire? (preclass 4)

- What would the delay be if we treated the wire as lumped R and C?

$$R_{wire} = N \times R_{unit} \qquad C_{wire} = N \times C_{unit}$$



# Wire Delay

- $R_{wire} = N*R_{unit}$
- $C_{wire}=N*C_{unit}$
- Lumped RC wire delay = $R_{unit}*C_{unit}*N^2$
- Distributed RC Wire delay = $R_{unit}*C_{unit}*N^2/2$

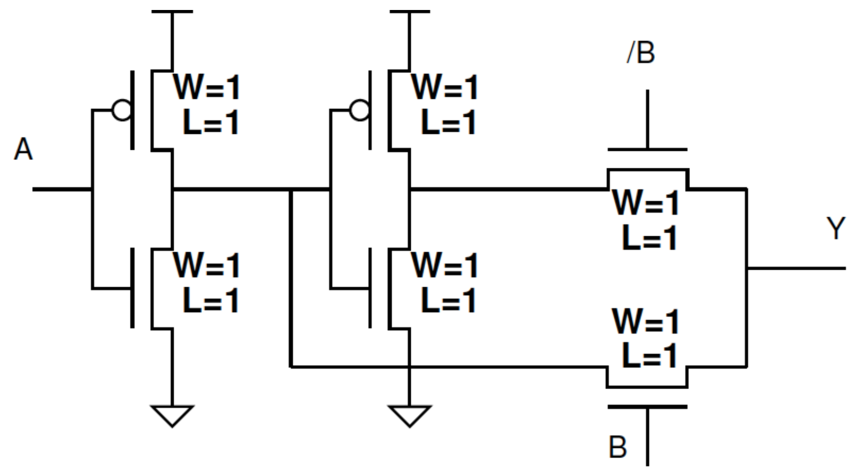
- Distributed has half the delay of lumped RC product
- Delay is quadratic in length of wire in both cases

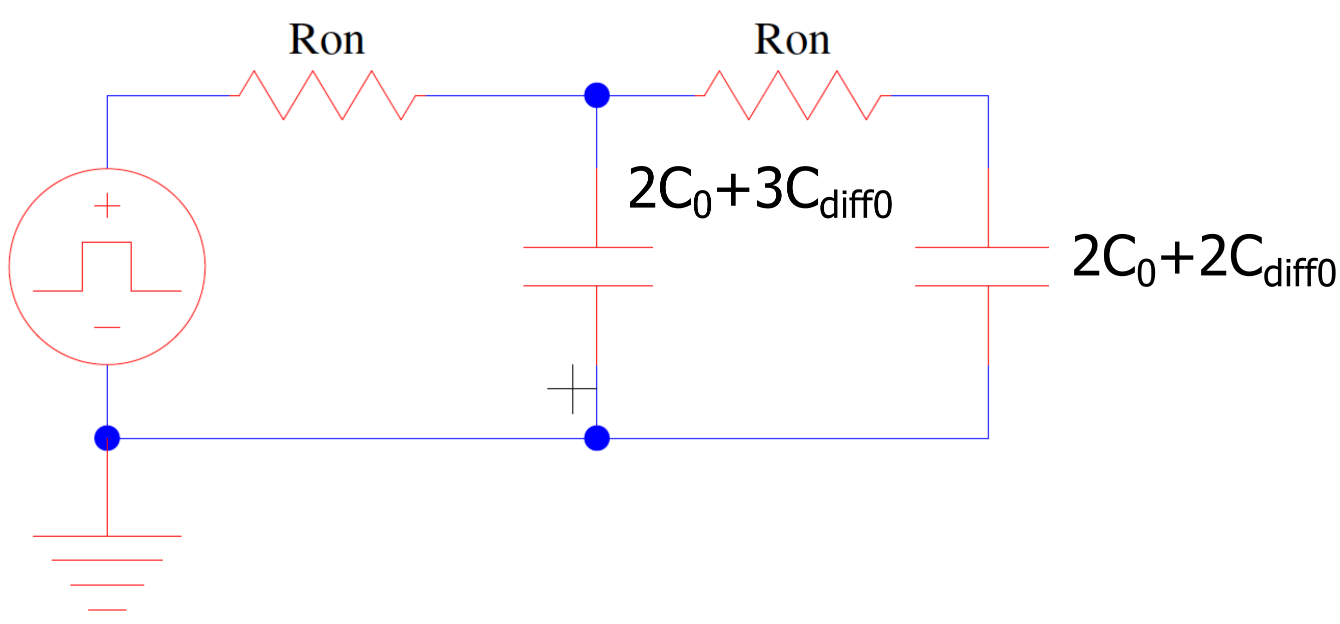
# Apply to Gates

Pass Transistor and CMOS

# Pass Transistor XOR

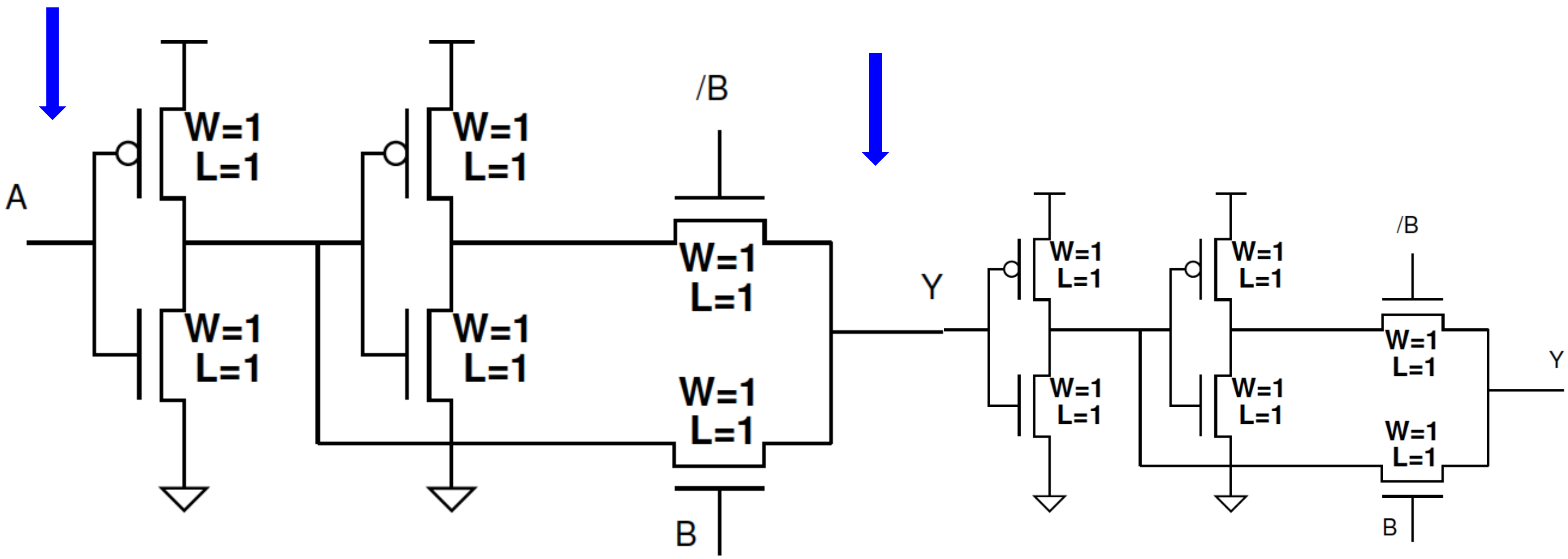
- Delay when A=B=1?



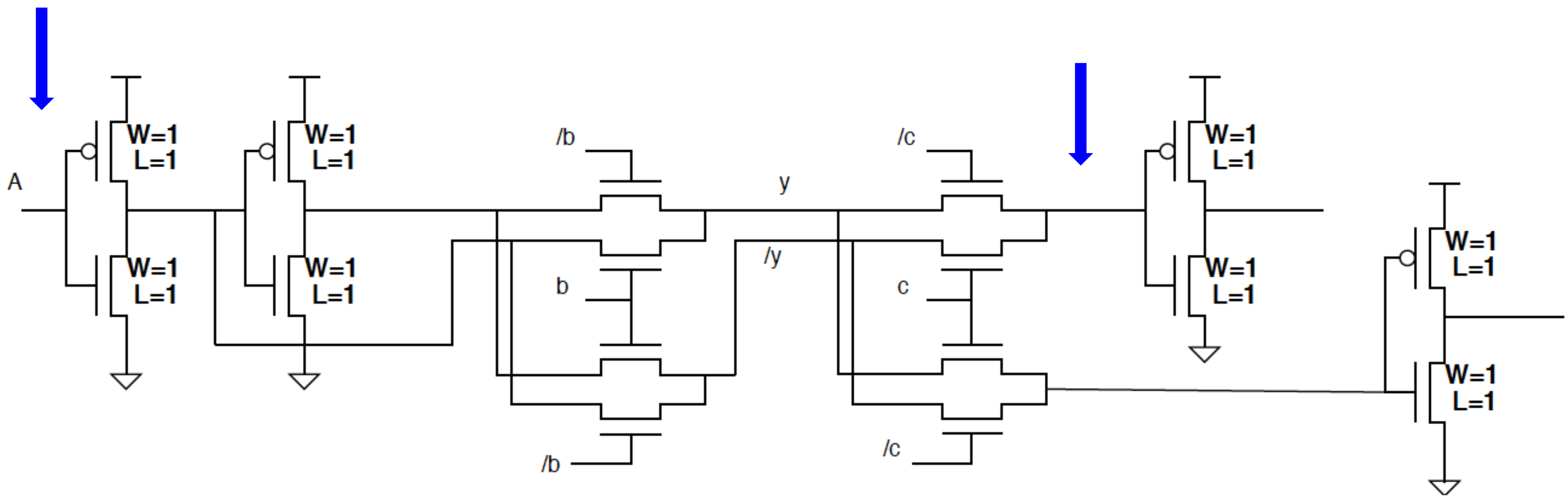
# Pass Transistor XOR (preclass 5)

- Delay when A=1, B=0?
  - Start with equivalent RC circuit



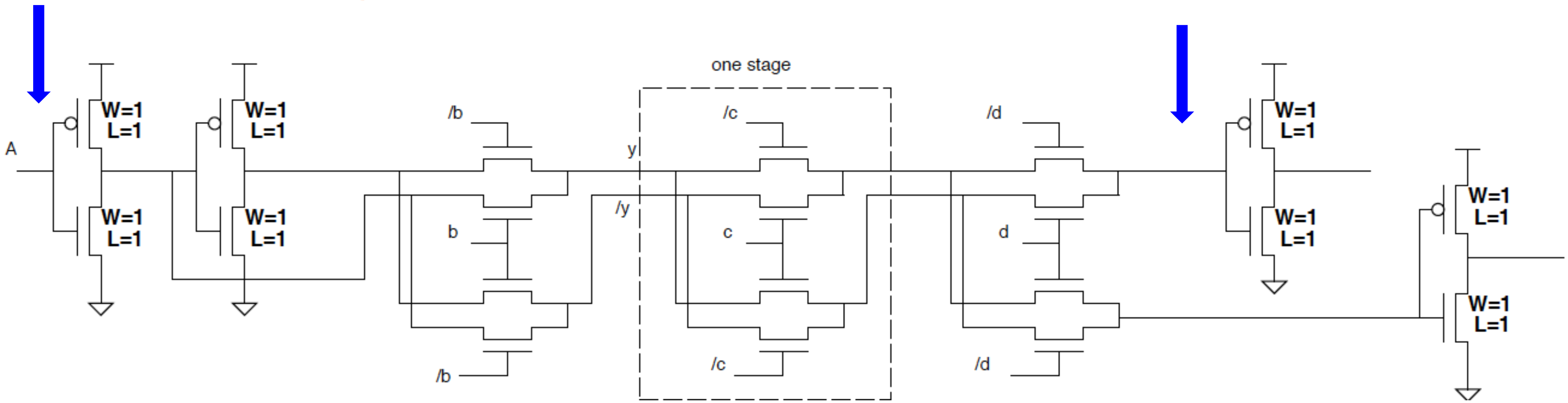
# Unbuffered (preclass 6)

- Circuit → Delay?
  - 2 stages



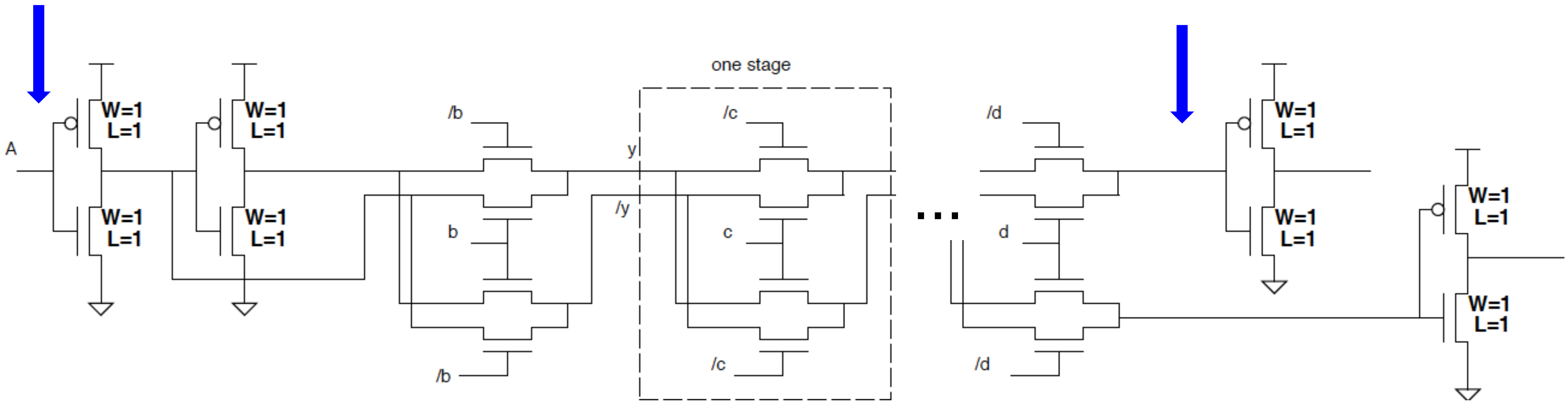
# Unbuffered (preclass 7)

- Circuit → Delay?
  - 3 stages



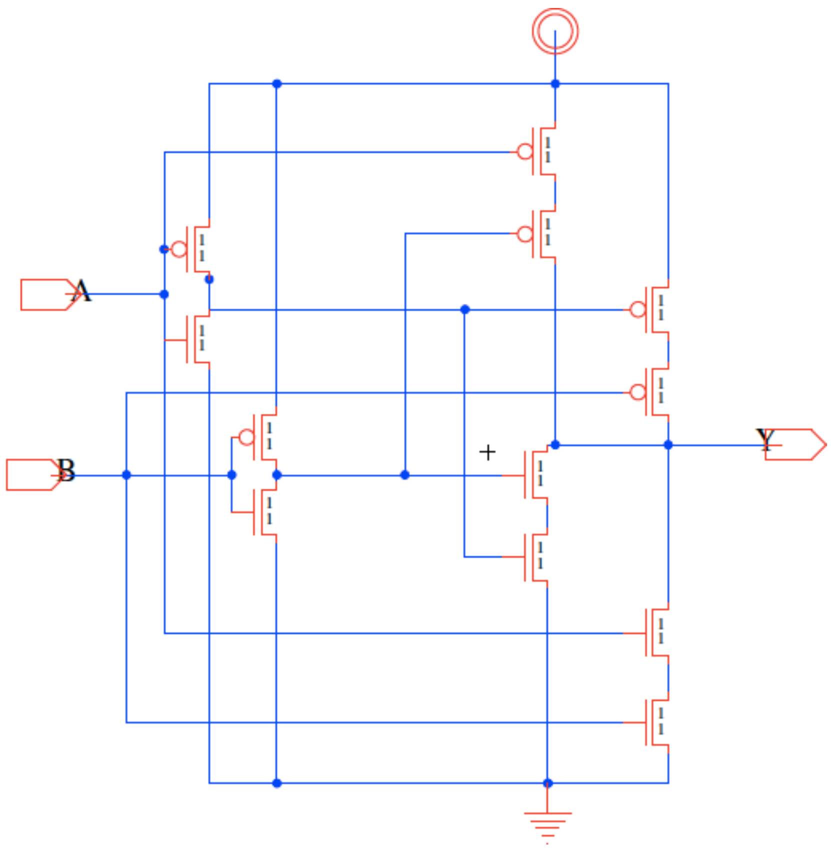
# Unbuffered (preclass 8)

- Delay as a function of number of stages, k?

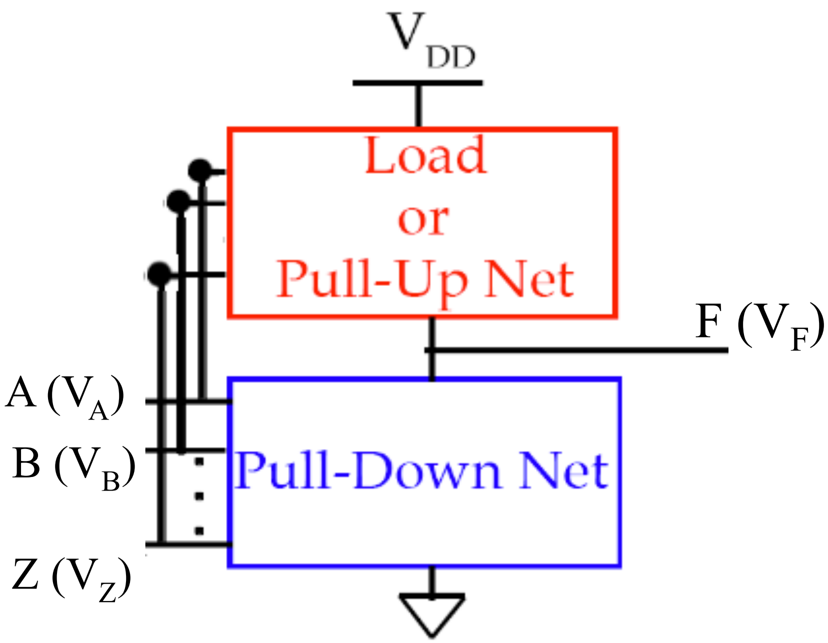


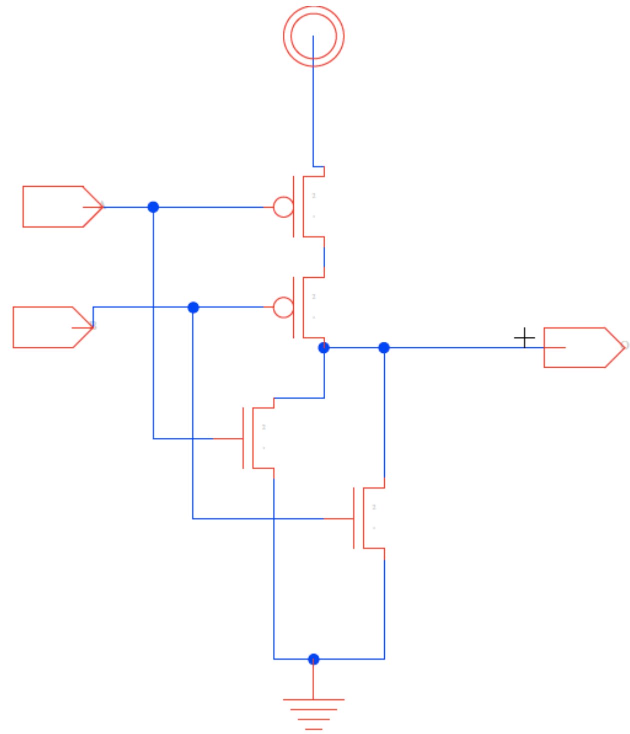
# CMOS XOR (preclass 9)

- Delay with Cdiff>0?



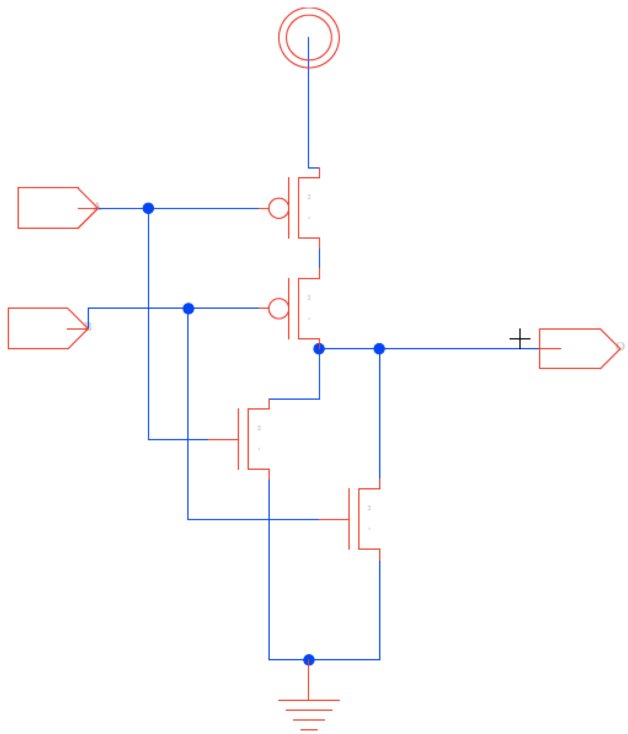
# Review: Two-Input NOR Gate (NOR2)

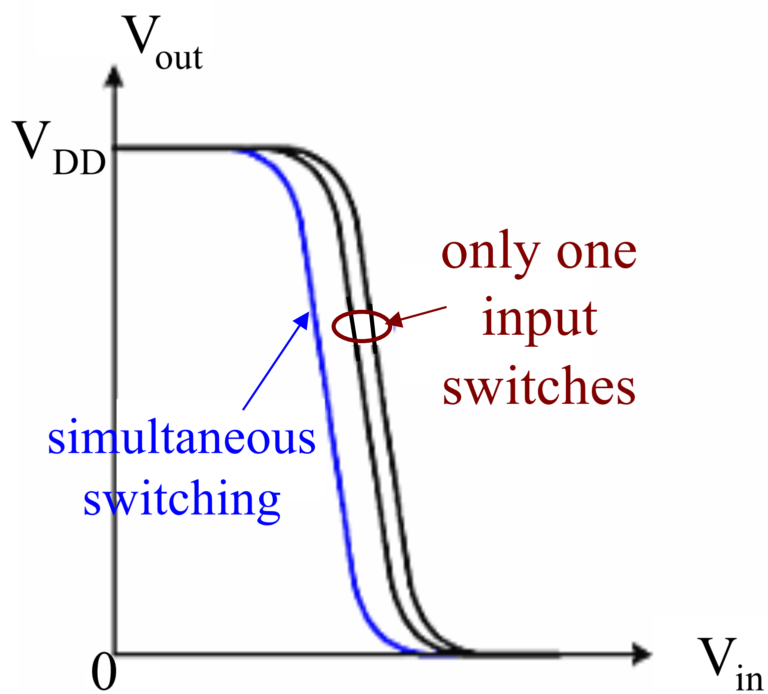


- Worst case delay of NOR2?
  - Minimum size, Loaded with itself

# NOR2 Transfer Curve






3 VTC Cases

$V_1 = 0$ V; $V_2 = 0 \rightarrow V_{DD}$

$V_1 = 0 \rightarrow V_{DD}$; $V_2 = 0$

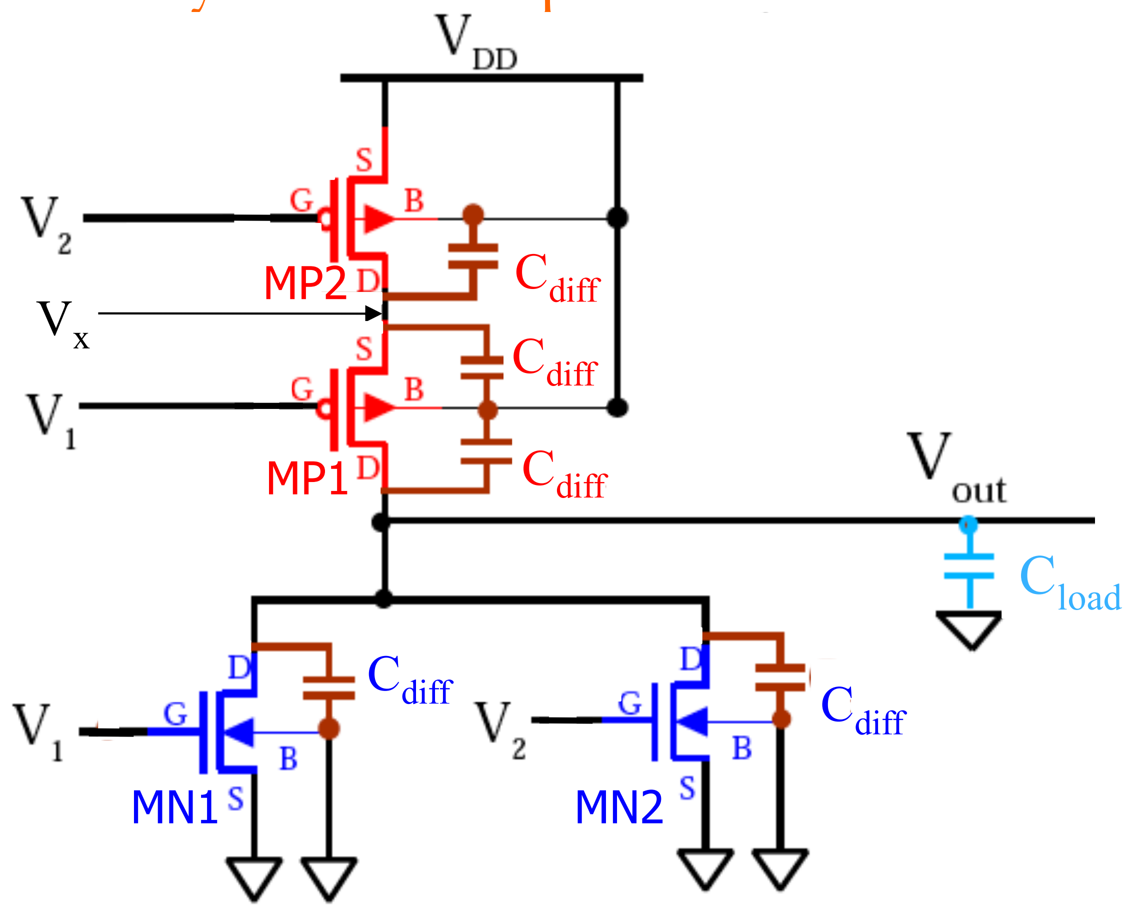
$V_1$ and $V_2 = 0 \rightarrow V_{DD}$ simultaneously

Switching Threshold Voltage:

$V_1 = V_2 = V_{out} = V_t$

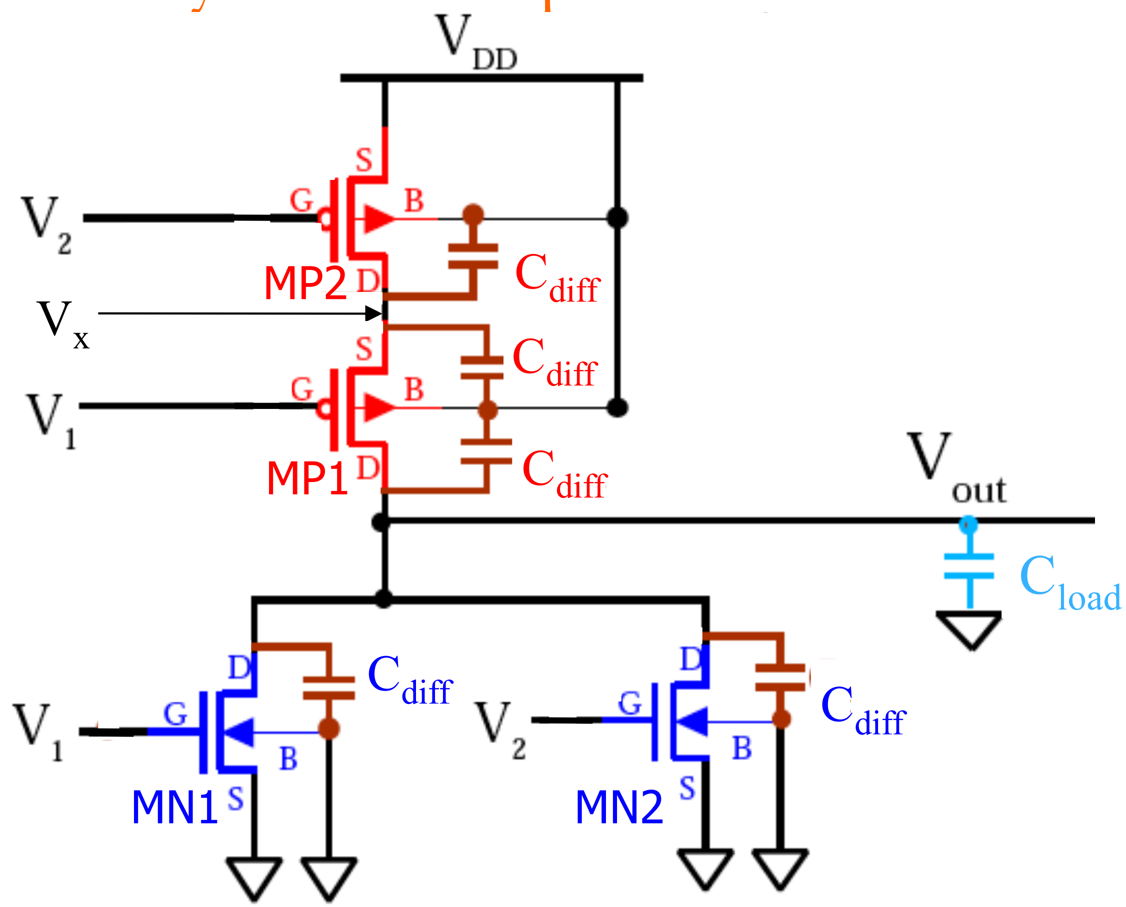
# NOR2 Delay (preclass 10)

- Worst case delay for Pull-up?



# NOR2 Delay

- Worst case delay for Pull-up?




Worst Case for Pull-up → $V_1 = 0$, $V_2 = V_{DD}$ -> 0 @t=0 & $V_x \approx V_{out} = 0$ -> $V_{DD}$

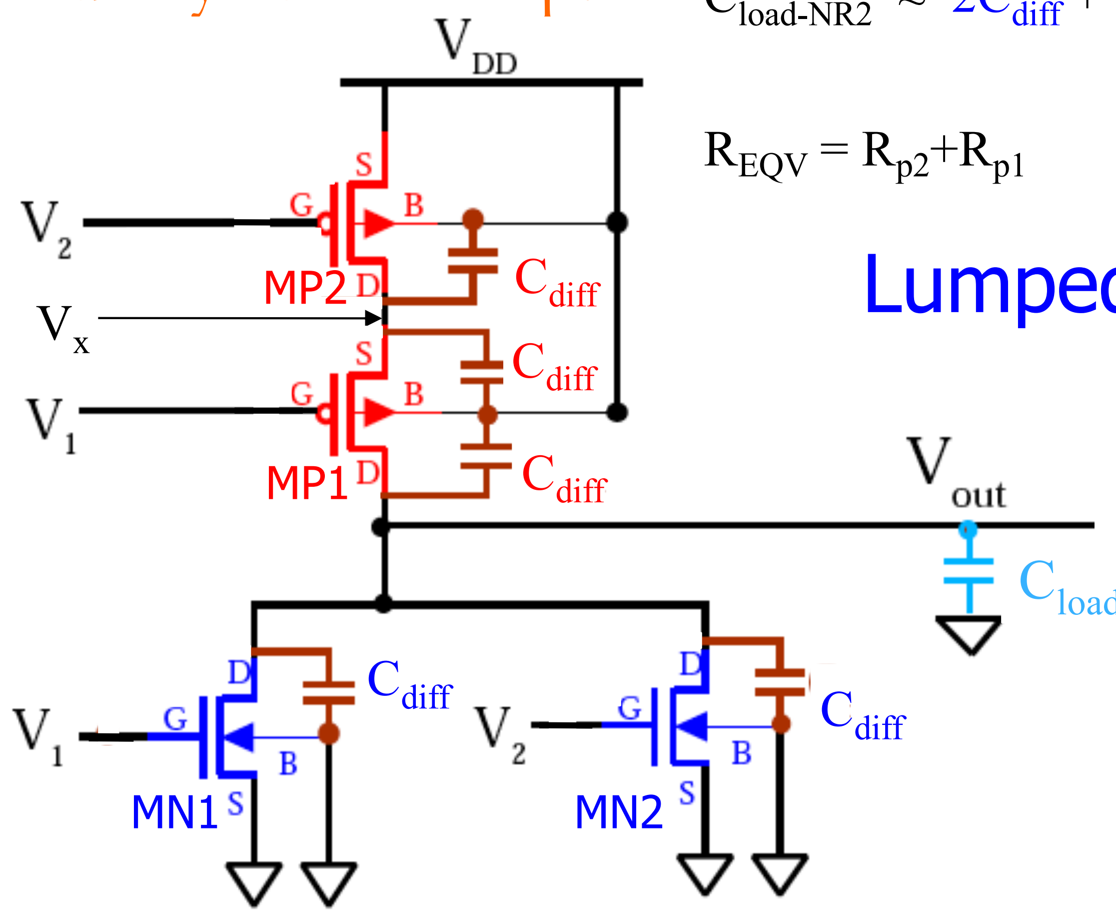
# NOR2 Delay

- Worst case delay for Pull-up?

$C_{load\text{-}NR2} \approx 2C_{diff} + 3C_{diff} + C_{load}$

$R_{EQV} = R_{p2}+R_{p1}$

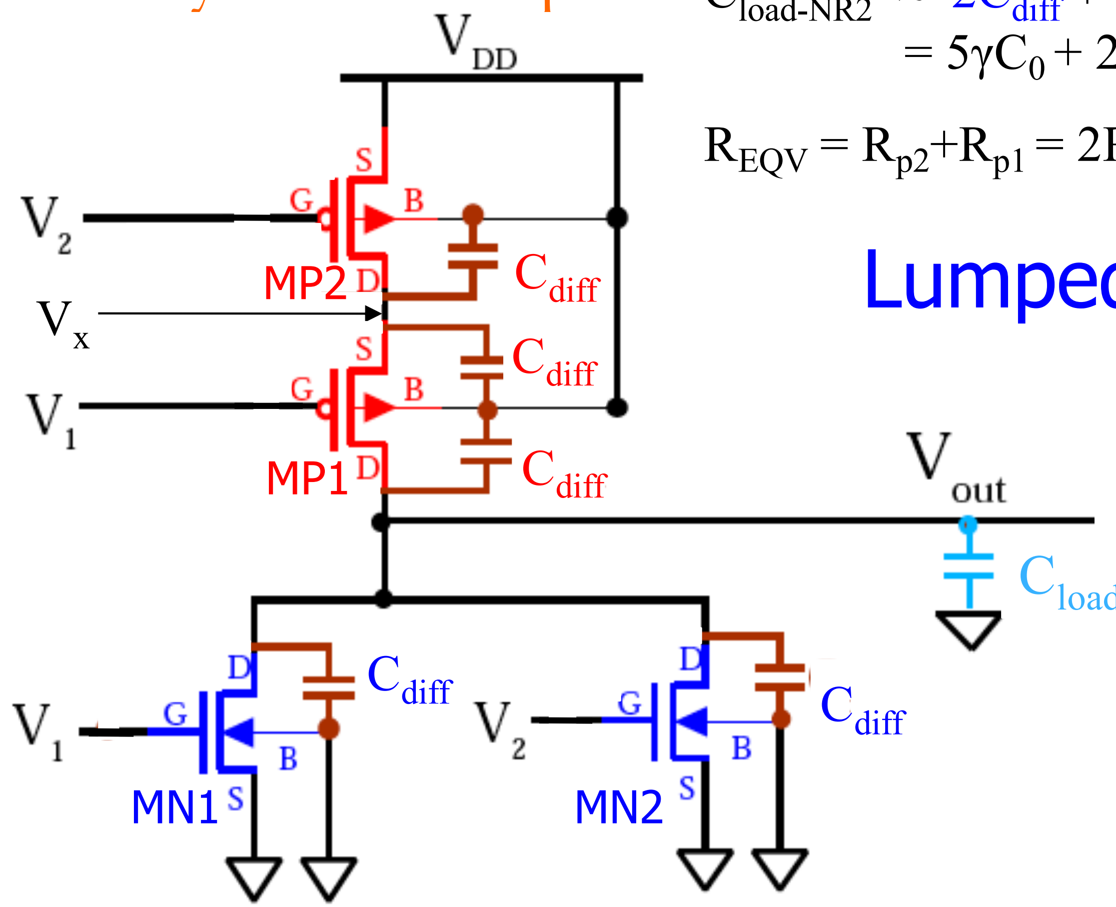
Lumped Model




Worst Case for Pull-up → $V_1 = 0$, $V_2 = V_{DD}$-> 0 @t=0 & $V_x \approx V_{out} = 0$ -> $V_{DD}$

# NOR2 Delay

- Worst case delay for Pull-up?




$C_{load\text{-}NR2} \approx 2C_{diff} + 3C_{diff} + C_{load}$
$= 5\gamma C_0 + 2C_0$

$R_{EQV} = R_{p2} + R_{p1} = 2R_0$

Lumped Model

Worst Case for Pull-up → $V_1 = 0$, $V_2 = V_{DD}$ -> 0 @t=0 & $V_x \approx V_{out} = 0$ -> $V_{DD}$

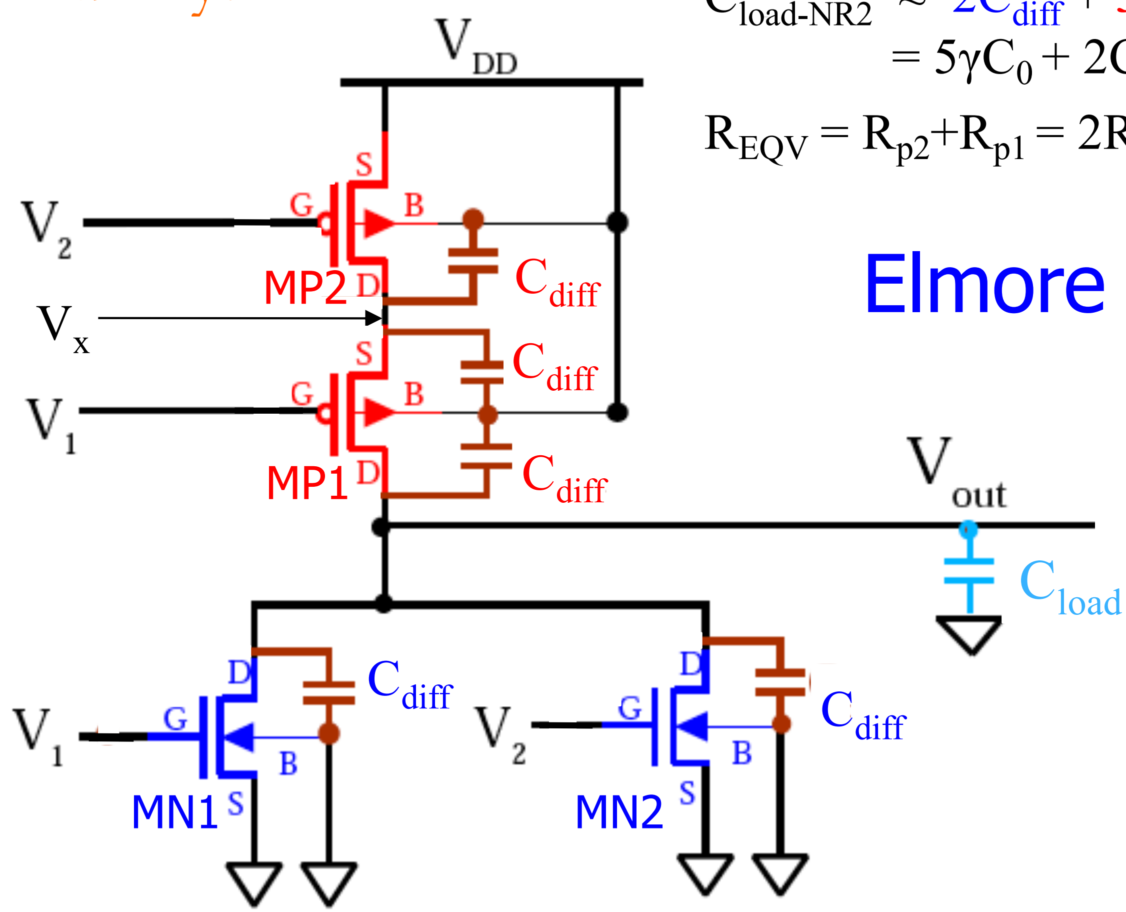
delay = $(5\gamma C_0 + 2C_0)(2R_0) = (10\gamma + 4)\tau$

# NOR2 Delay

- Worst case delay?

$C_{load\text{-}NR2} \approx 2C_{diff} + 3C_{diff} + C_{load}$
$= 5\gamma C_0 + 2C_0$

$R_{EQV} = R_{p2} + R_{p1} = 2R_0$




Elmore Model?

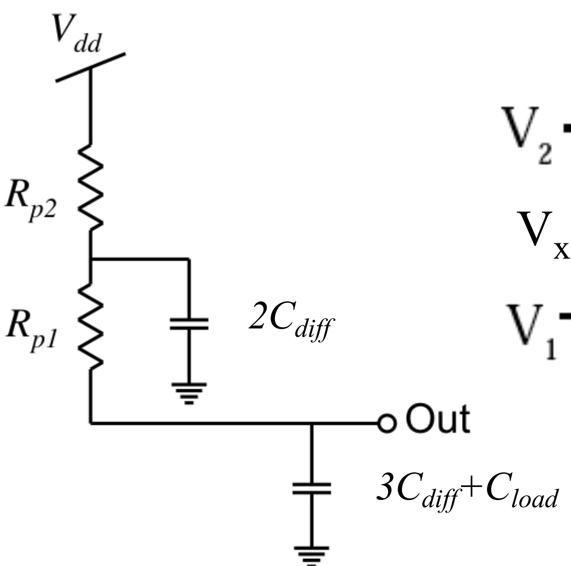
Worst Case for Pull-up → $V_1 = 0$, $V_2 = V_{DD} \to 0$ @t=0 & $V_x \approx V_{out} = 0 \to V_{DD}$

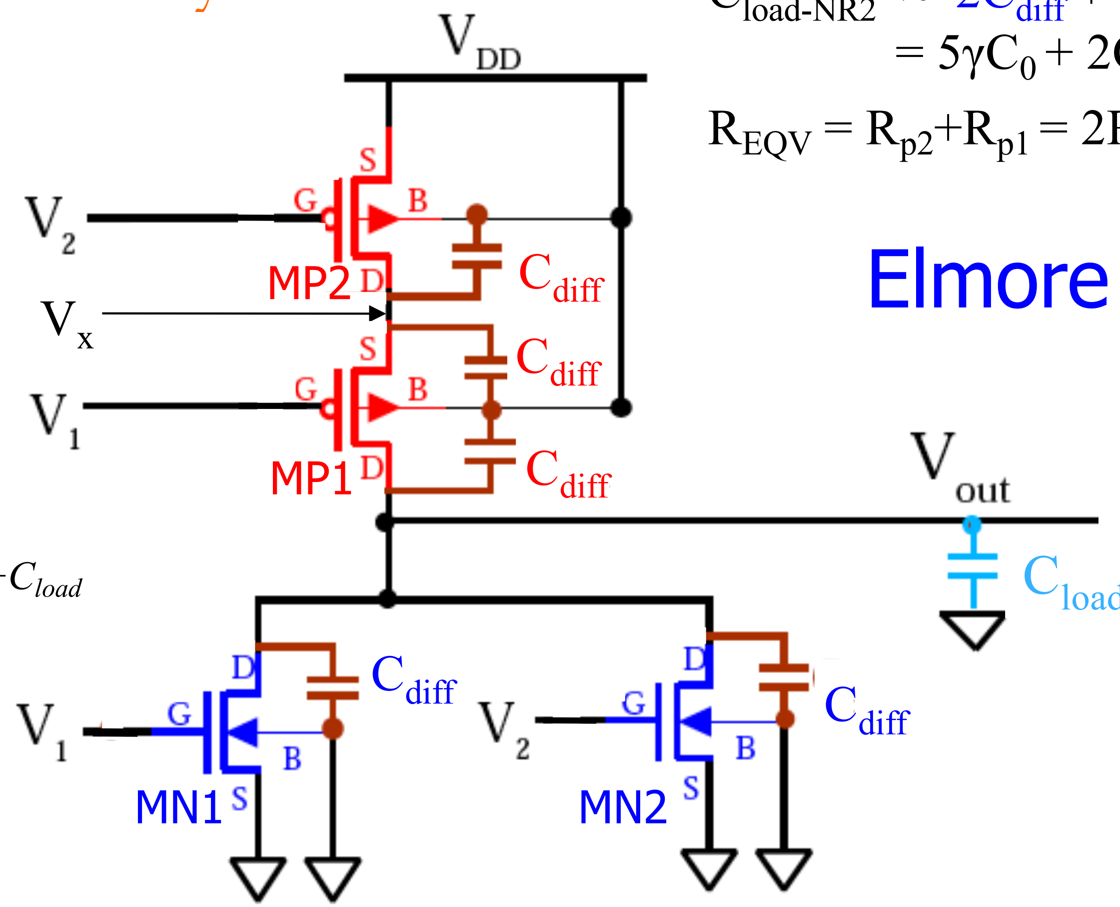
# NOR2 Delay

- Worst case delay?

$C_{load\text{-}NR2} \approx 2C_{diff} + 3C_{diff} + C_{load}$
$= 5\gamma C_0 + 2C_0$

$R_{EQV} = R_{p2}+R_{p1} = 2R_0$




Elmore Model?

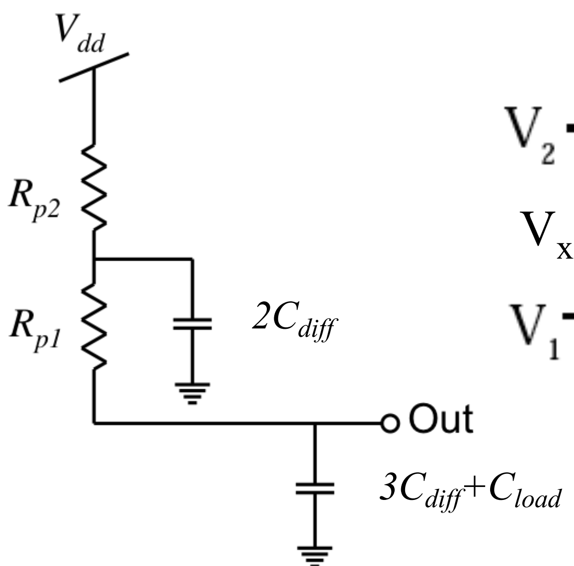
Worst Case for Pull-up → $V_1 = 0$, $V_2 = V_{DD}$-> 0 @t=0 & $V_x \approx V_{out} = 0$ -> $V_{DD}$

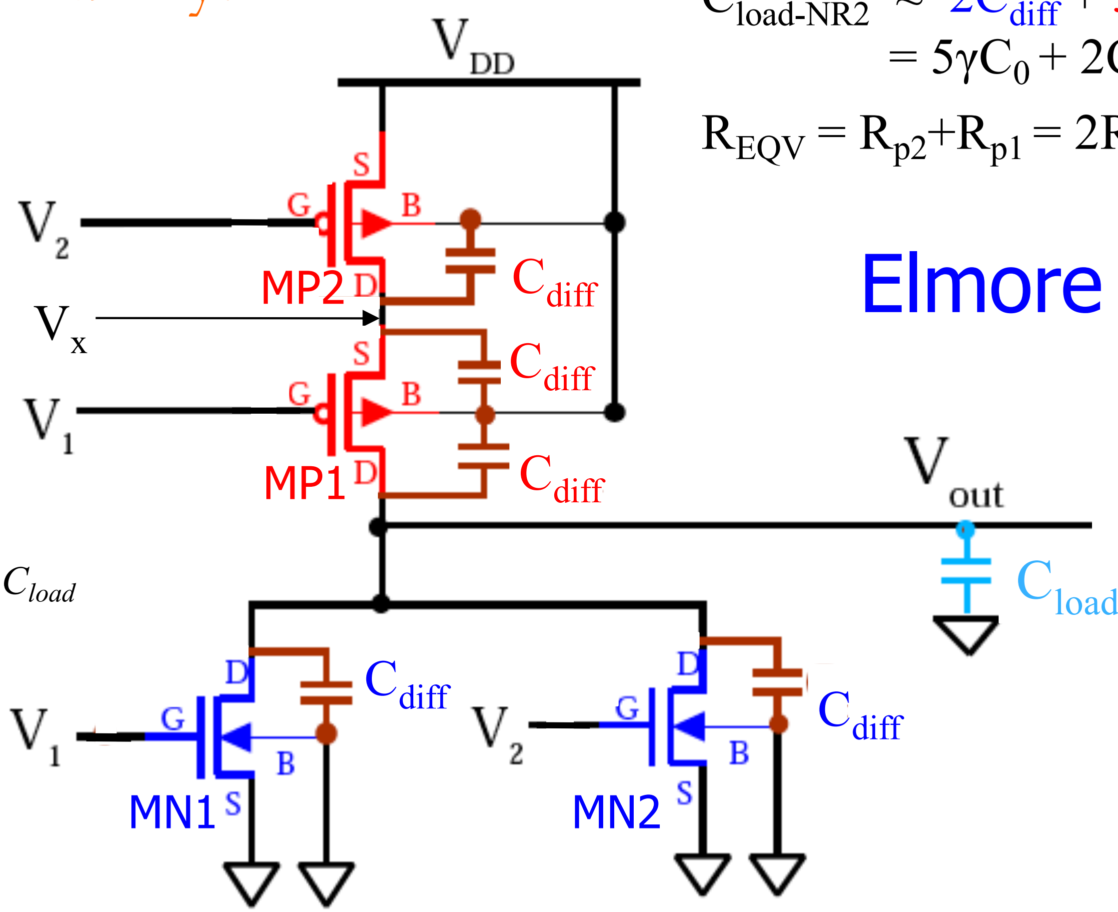
# NOR2 Delay

- Worst case delay?

$C_{load\text{-}NR2} \approx 2C_{diff} + 3C_{diff} + C_{load}$
$= 5\gamma C_0 + 2C_0$

$R_{EQV} = R_{p2}+R_{p1} = 2R_0$



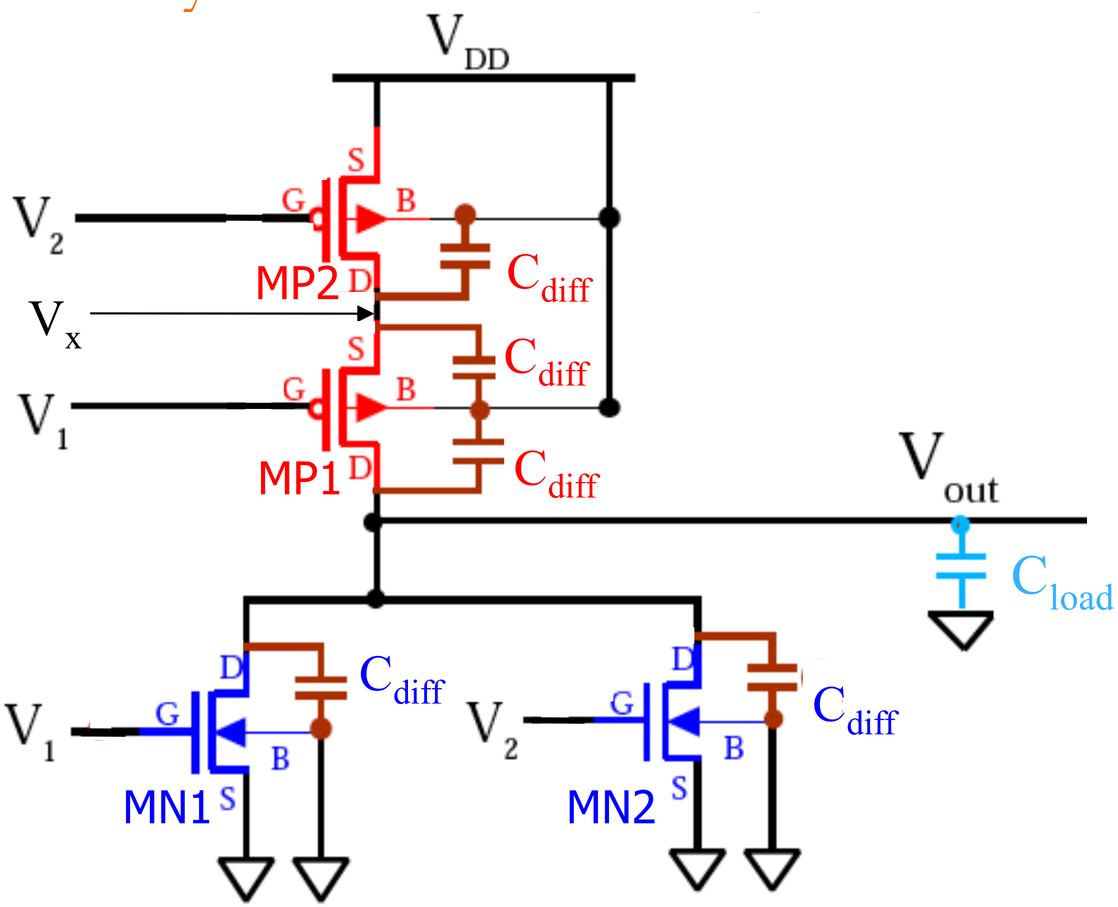

Elmore Model?

Worst Case for Pull-up → $V_1 = 0$, $V_2 = V_{DD}$ -> 0 @t=0 & $V_x \approx V_{out} = 0$ -> $V_{DD}$

$\text{delay} = (2C_{diff})(R_{p2})+(3C_{diff}+C_{load})(R_{p1}+R_{p2})=(8\gamma+4)\tau$

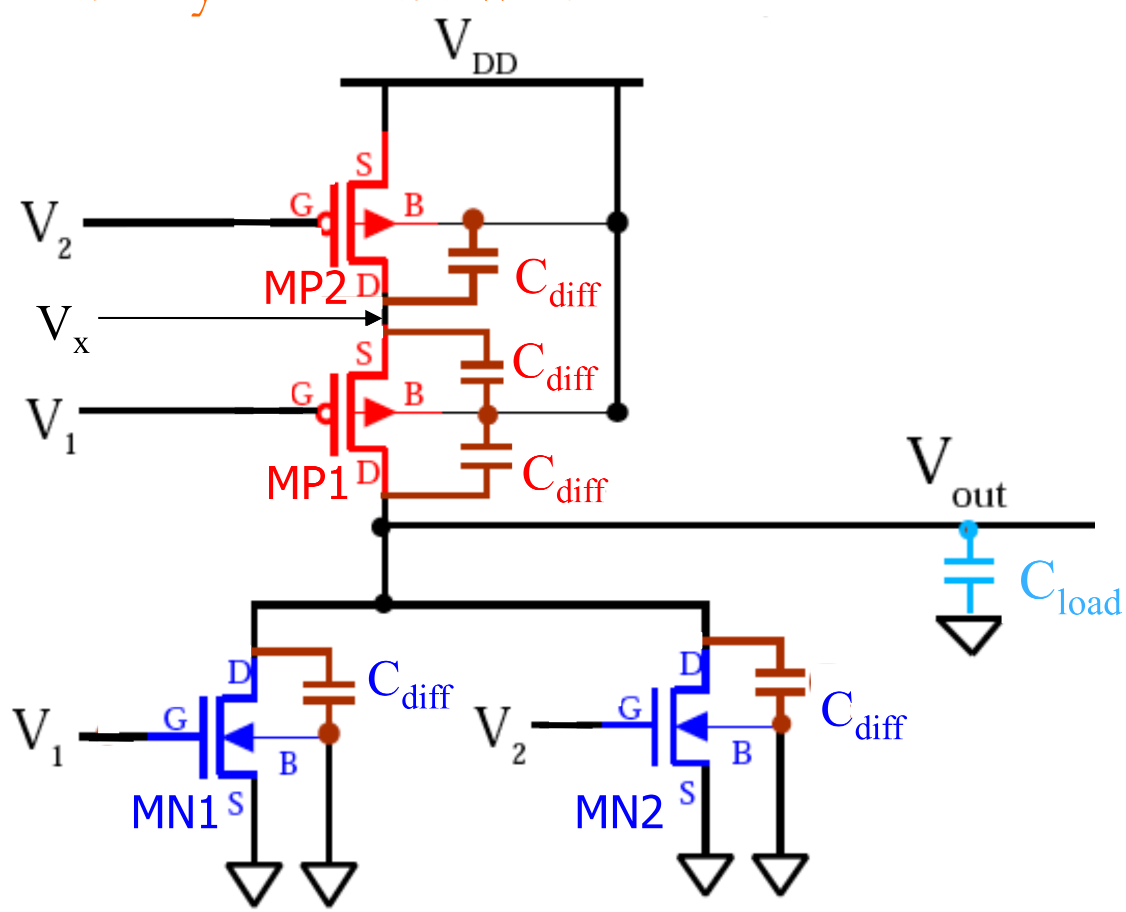
# NOR2 Delay

- Worst case delay Pull-down?



# NOR2 Delay

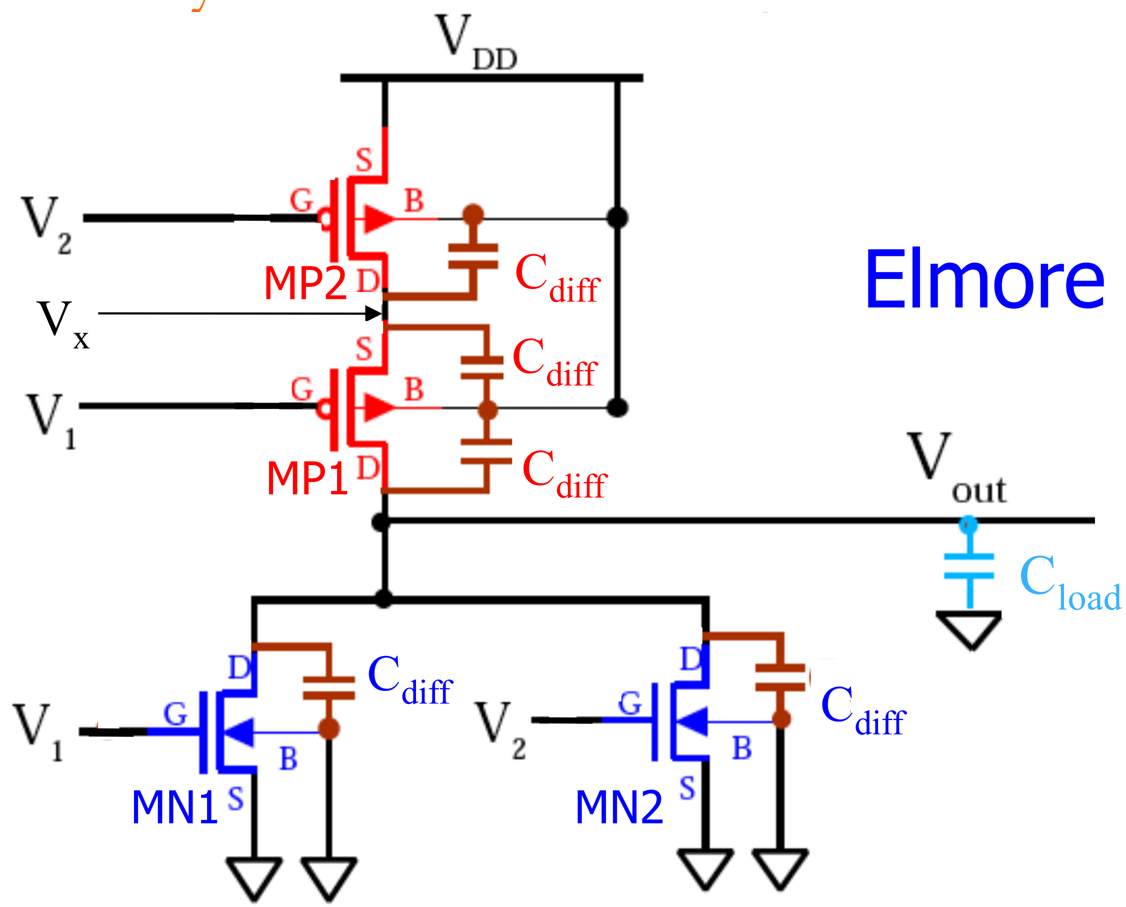
- Worst case delay Pull-down?




Worst case for Pull-down → $V_1 = 0$, $V_2 = 0$ ->$V_{DD}$ @t=0 & $V_x \approx V_{out}=V_{DD}$-> 0

# NOR2 Delay

- Worst case delay Pull-down?

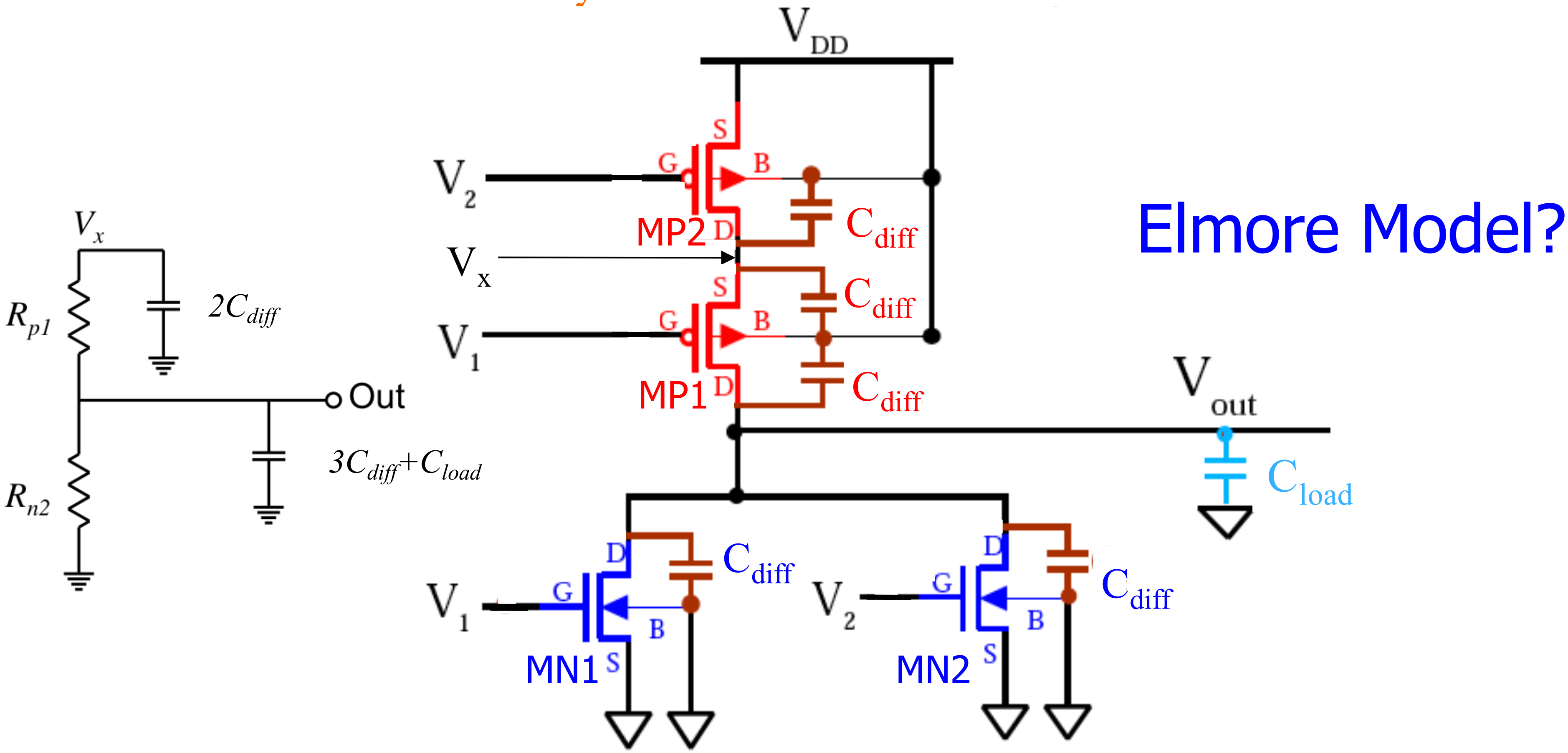



Elmore Model?

Worst case for Pull-down → $V_1 = 0$, $V_2 = 0 \rightarrow V_{DD}$ @t=0 & $V_x \approx V_{out} = V_{DD} \rightarrow 0$

# NOR2 Delay

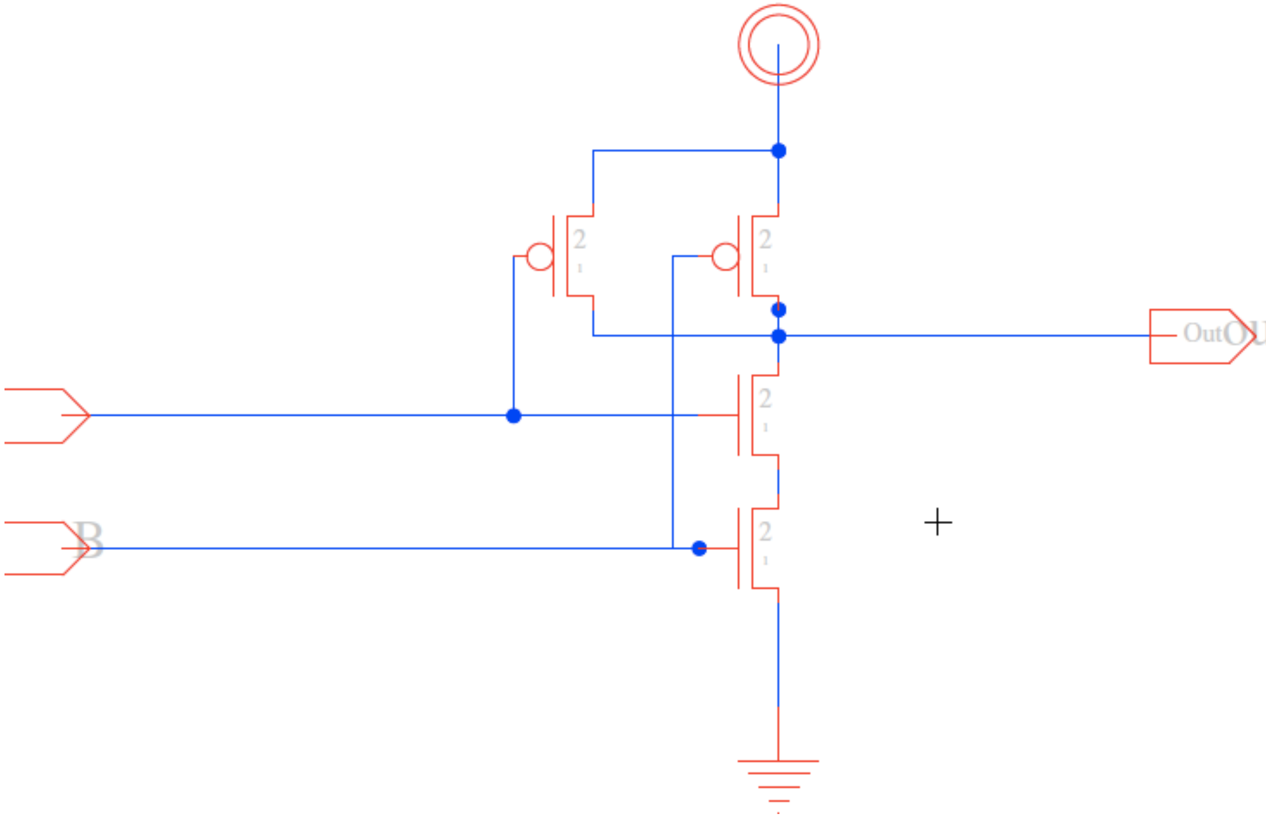
- Worst case delay Pull-down?



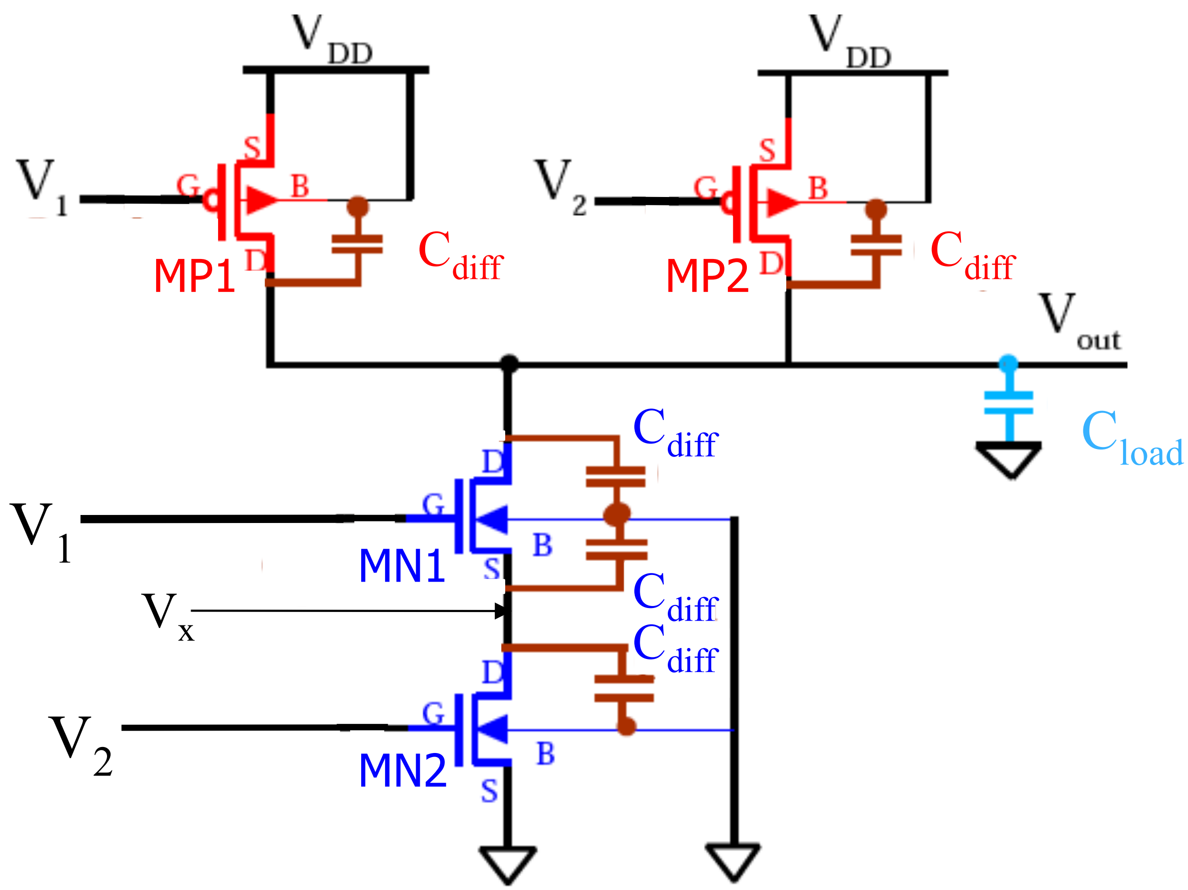

Worst case for Pull-down → $V_1 = 0$, $V_2 = 0 \to V_{DD}$ @t=0 & $V_x \approx V_{out} = V_{DD} \to 0$

$$\text{delay} = (2C_{diff})(R_{p1}+R_{n2})+(3C_{diff}+C_{load})(R_{n2})=(7\gamma+2)\tau$$

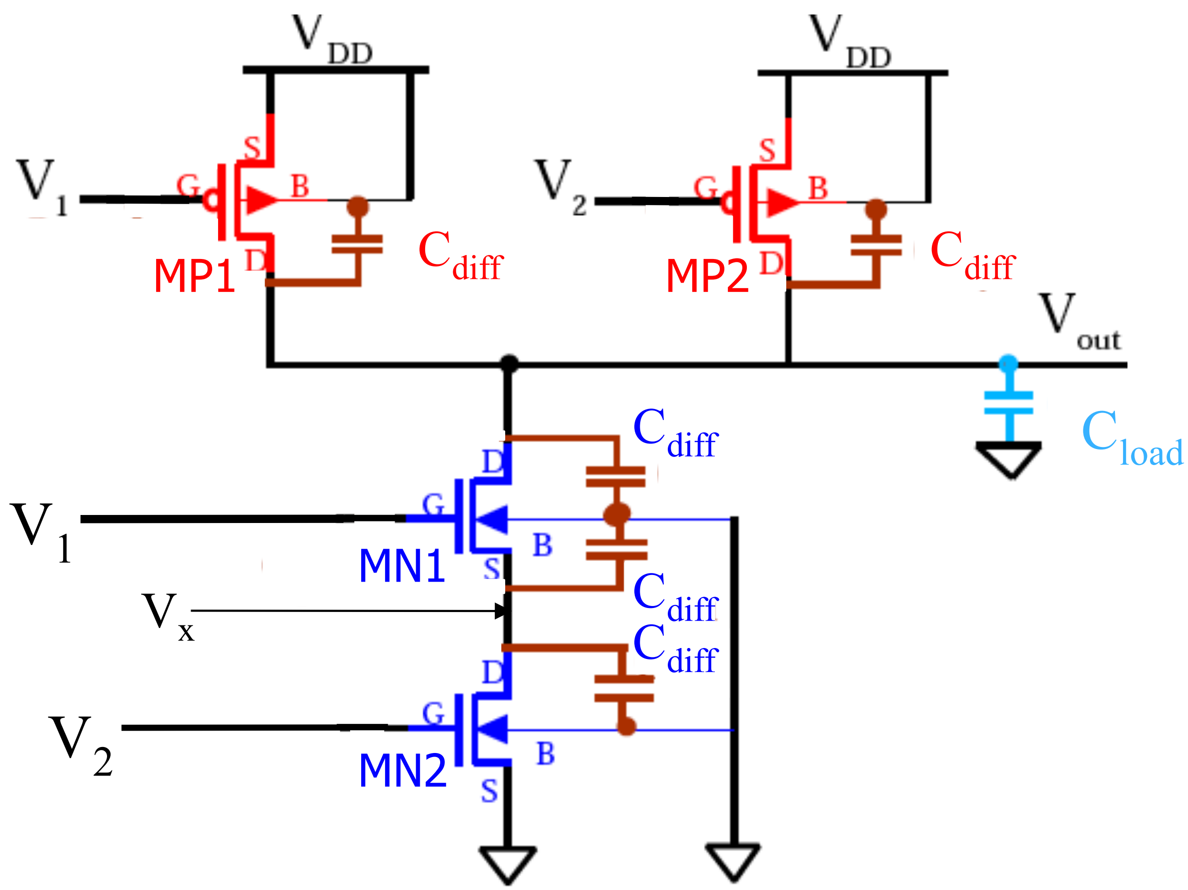
# NAND2 Delay (preclass 11)



# Parasitic Caps for NAND2 (worst case)

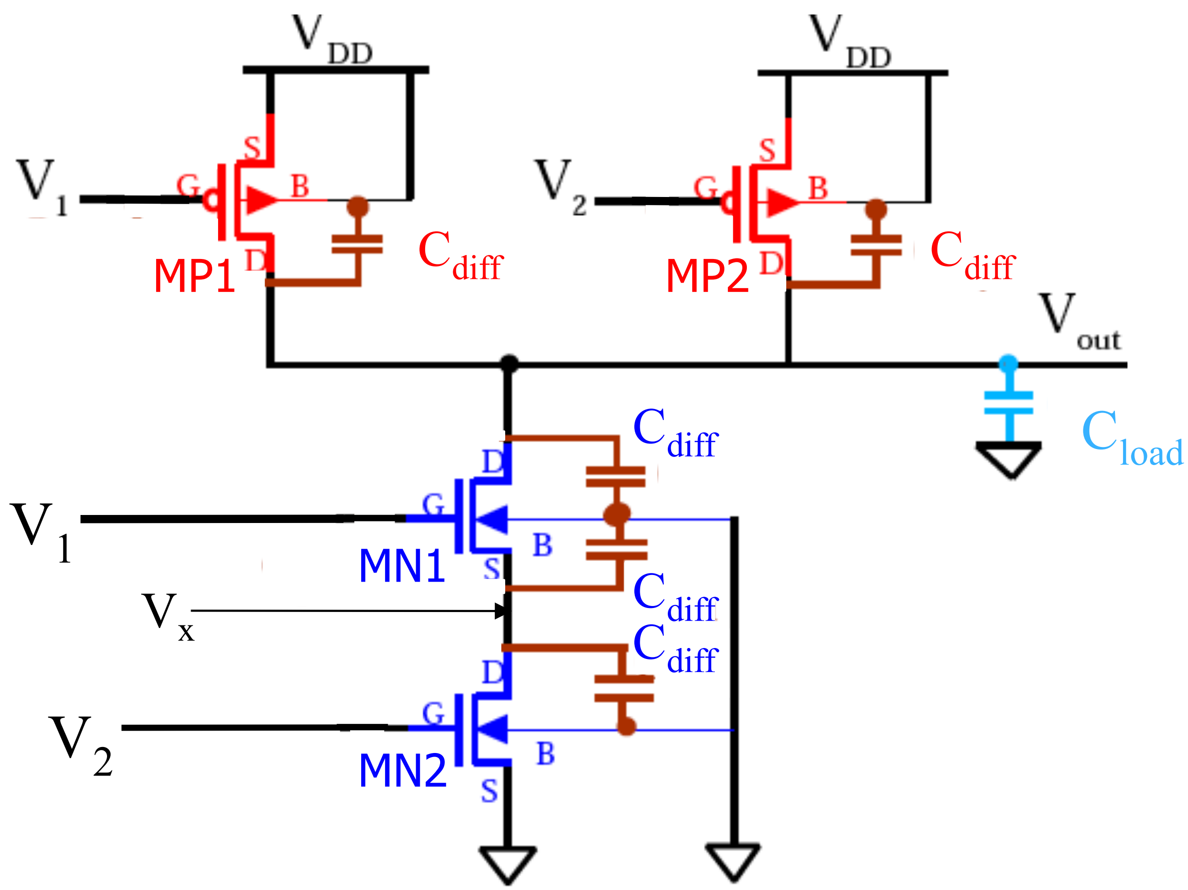


# Parasitic Caps for NAND2 (worst case)



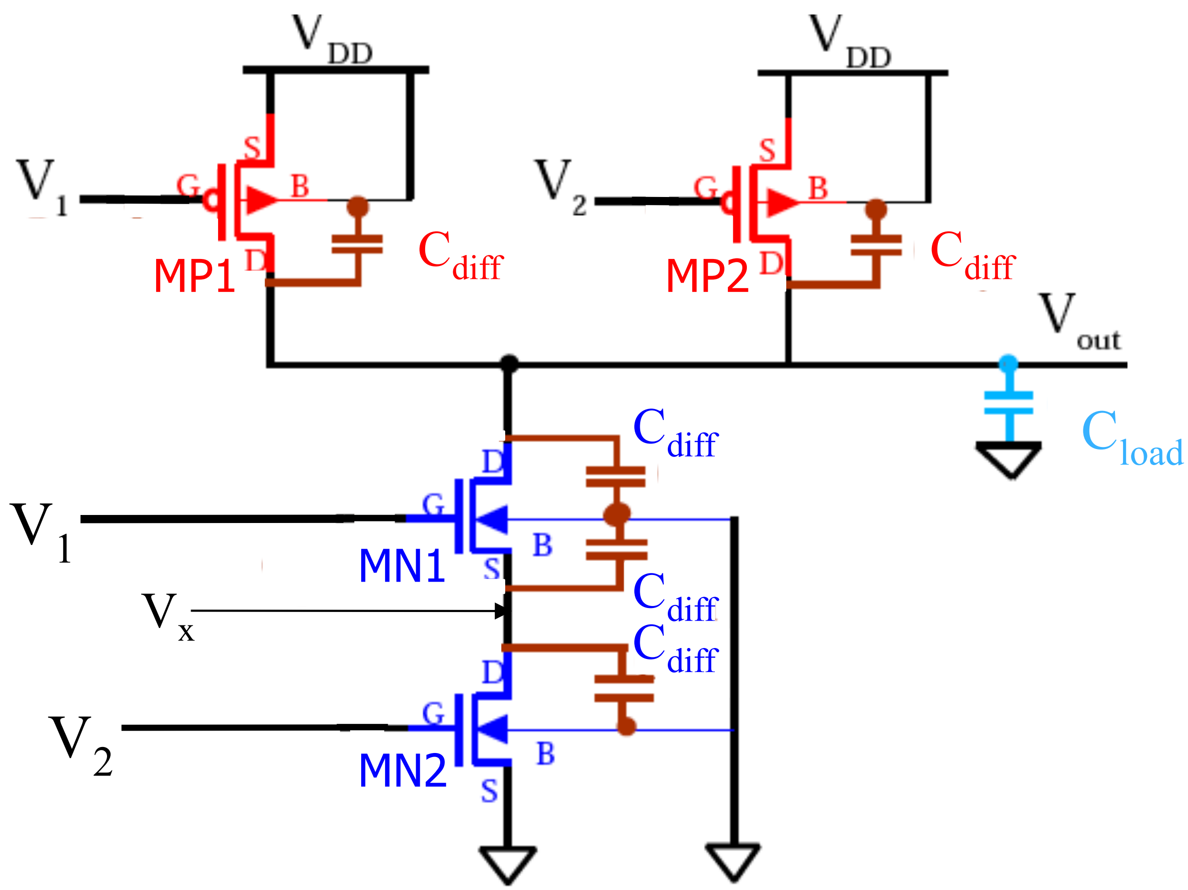

Worst case for Pull-up →

# Parasitic Caps for NAND2 (worst case)



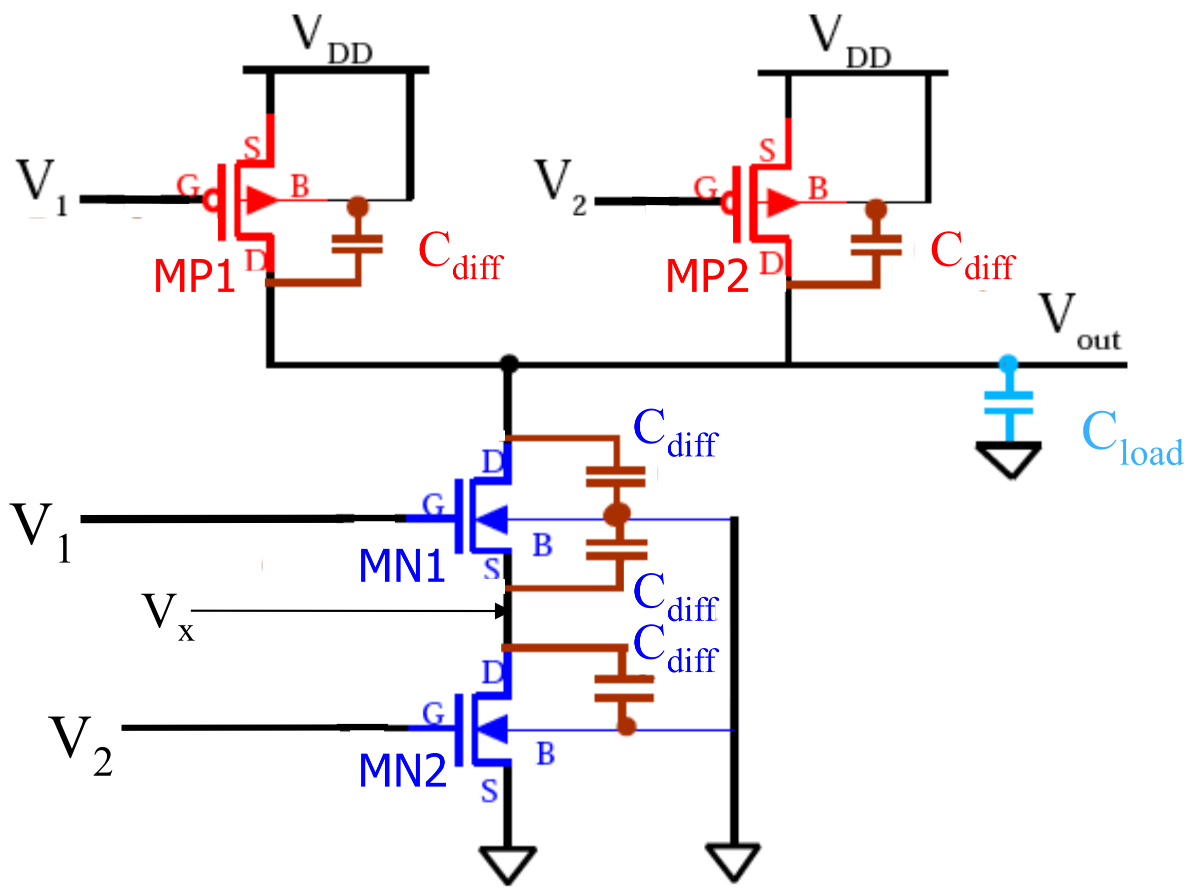

Worst case for Pull-up → $V_1 = V_{DD}$, $V_2 = V_{DD}$-> 0 @t=0 & $V_x \approx V_{out}$= 0 ->$V_{DD}$

# Parasitic Caps for NAND2 (worst case)




Worst case for pull down →

# Parasitic Caps for NAND2 (worst case)




Worst case for pull down → $V_1 = V_{DD}$, $V_2 = 0 \rightarrow V_{DD}$ @t=0 & $V_x \approx V_{out} = V_{DD} \rightarrow 0$

# Logic Types

- CMOS Gates
  - Dual pull-down and pull-up networks, only one enabled at a time
  - Performance of gate is strong function of the fanin of gate
    - Techniques to improve performance include sizing, input reordering, and buffering (staging)
- Ratioed Gates
  - Have active pull-down (-up) network connected to load device
  - Reduced gate complexity at expense of static power asymmetric transfer function
    - Techniques to improve performance include sizing to improve noise margins and reduce static power
- Pass Gates
  - Implement logic gate as switch network for area and often delay win
  - Long cascades of switches result in quadratic increase in delay
  - Also suffer from reduced noise margins ($V_T$ drop)
    - Use level-restoring buffers to improve noise margins
- Dynamic logic … next time

# Idea

- Elmore delay calculation allows us to estimate delay for distributed RC network
  - Necessary for pass transistors
- Wires are distributed RC
  - Half delay lumped calculation
  - Still quadratic in length

# Admin

- Project 1 out now
  - Design 8-bit ripple-carry adder
    - You already know how to do this
    - Refresh yourself on binary addition of 2 bits
  - Work individually
  - Full Report due F 3/28
  - **Can't pass the class if you don't turn in projects**
    - **Can't do project last minute**

# Acknowledgement

- Prof. André DeHon (University of Pennsylvania)
- Prof. Tania Khanna (University of Pennsylvania)