

ESE3700: Circuit-Level Modeling, Design, and Optimization for Digital Systems

Lec 14: March 26, 2025

Driving Large Capacitive Loads





Today

- ❑ Back to CMOS today
- ❑ How do we drive a large capacitive load?
 - Stages and buffer sizing
 - Minimum delay

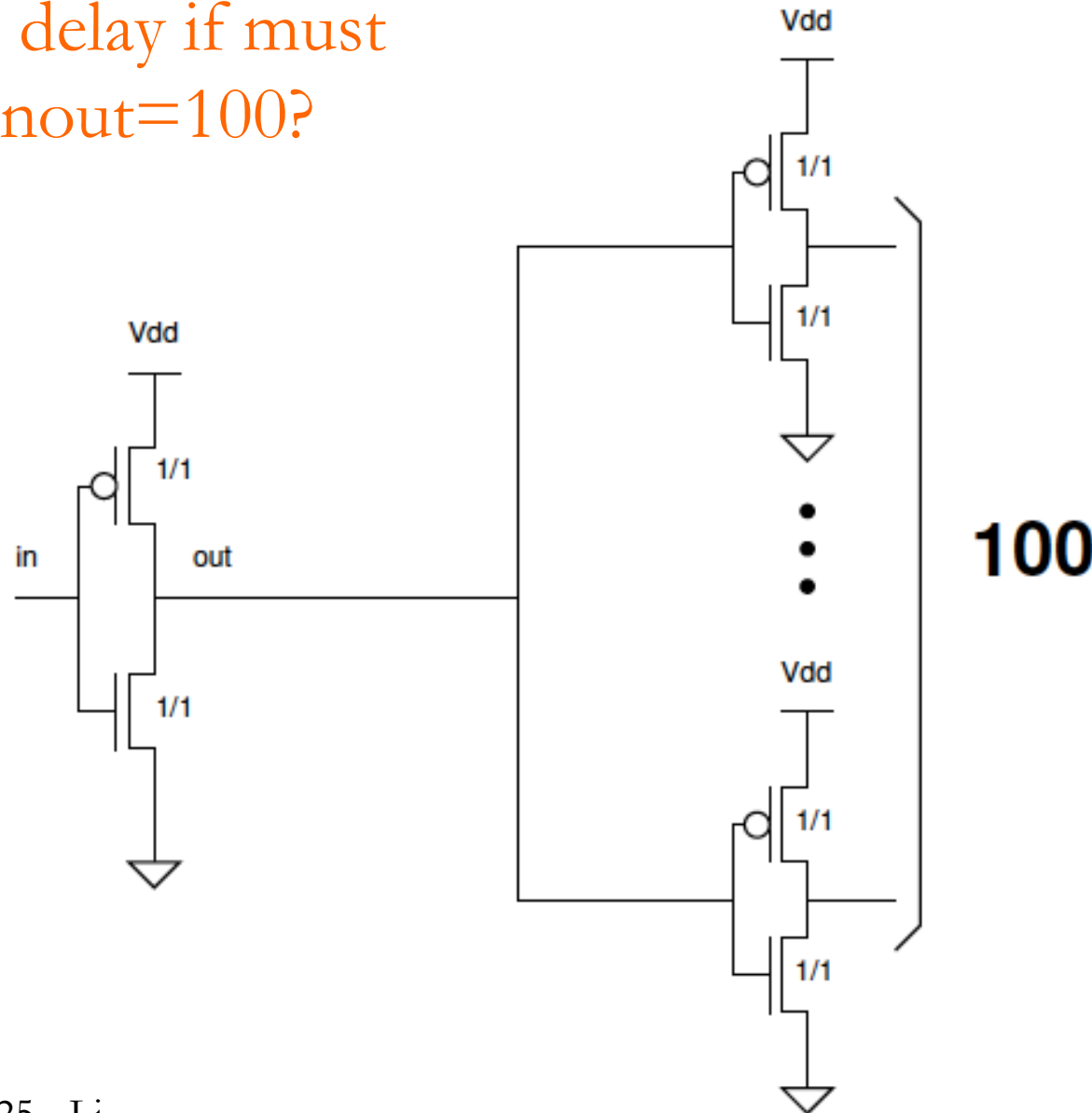


Message

- ❑ To drive large loads
 - Scale buffers geometrically
 - Exponential scale up in buffer size
- ❑ Scale factor: 3—4 typically
 - One origin of FO4 target
- ❑ Drains contribute capacitance too (C_{diff})
- ❑ Can formulate sizing to optimize

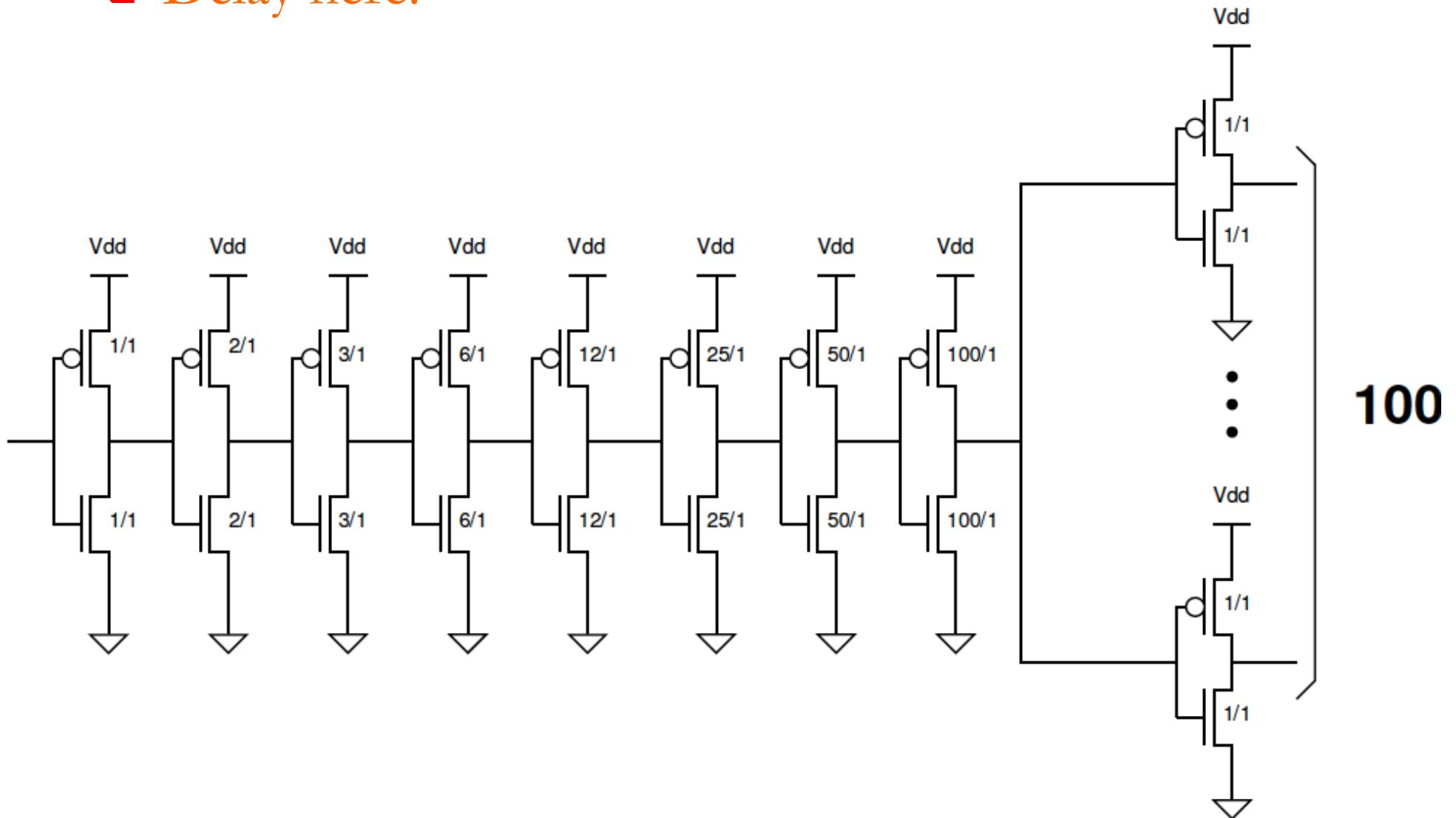
Call back: Large Fanout Delay

- What is delay if must drive fanout=100?



Call back: ...and Again

□ Delay here?

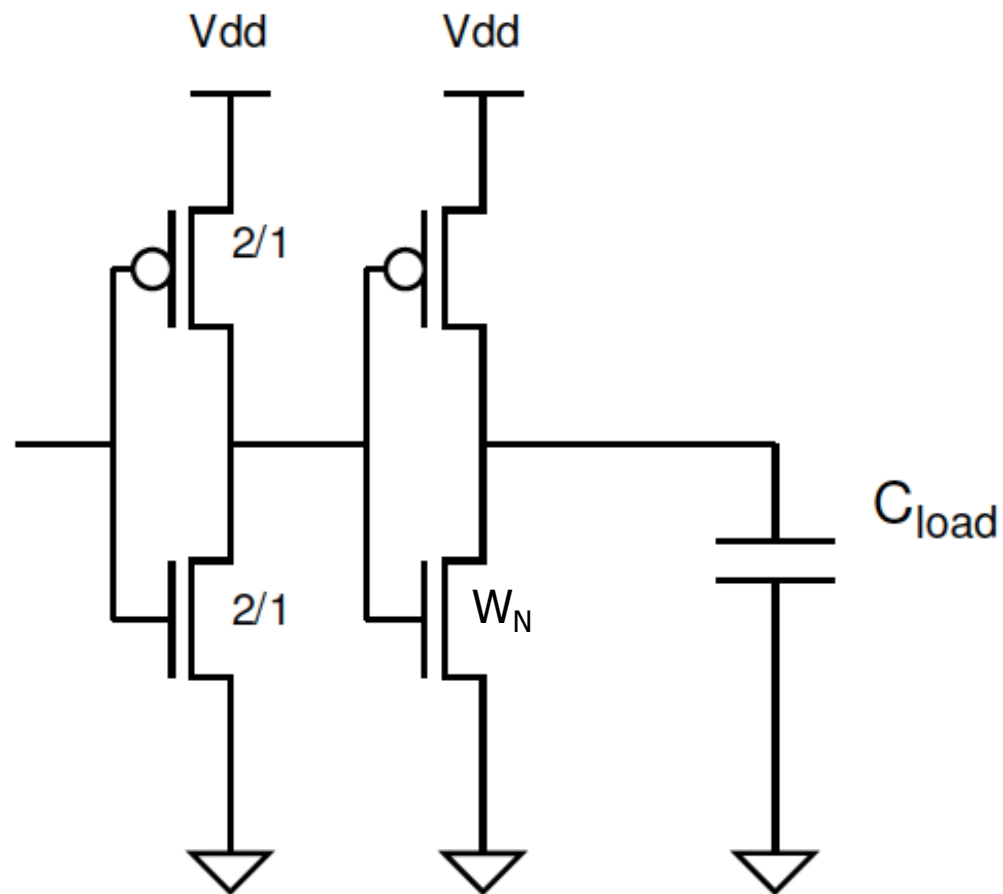


Start $C_{\text{diff}}=0$



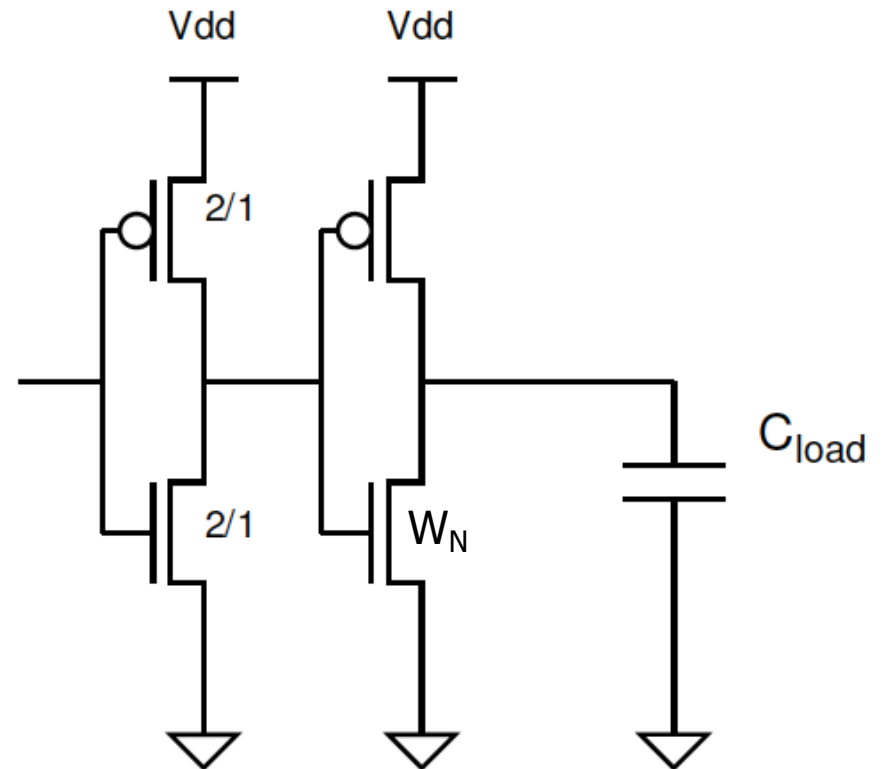
One Stage (Preclass 1)

- How do we size to minimize delay?



One Stage (Preclass 1)

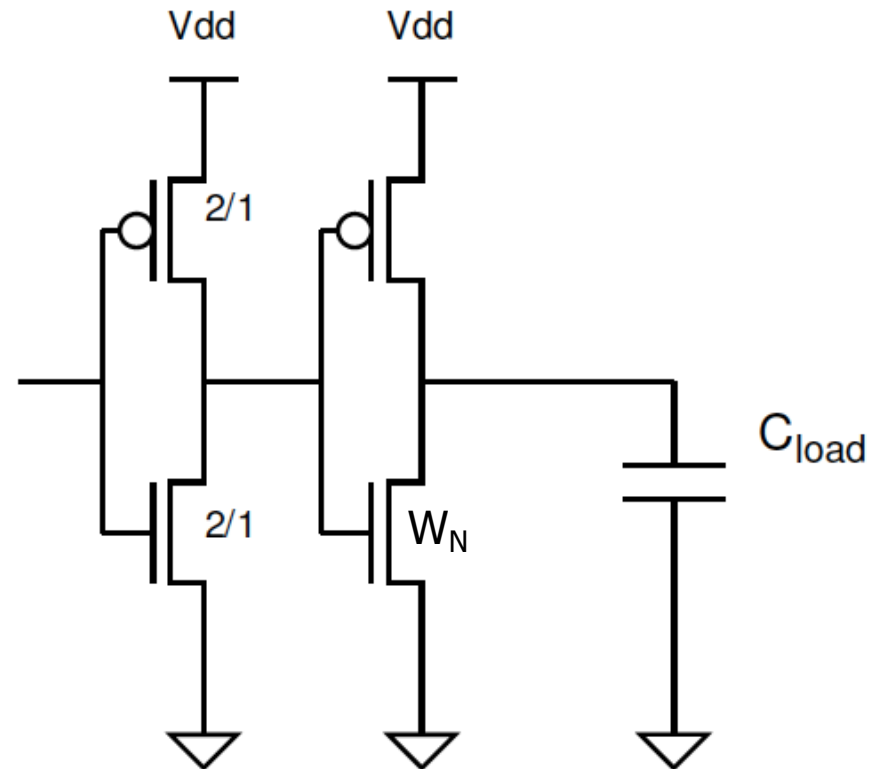
□ Delay equation?



One Stage (Preclass 1)

□ Delay equation?

$$\text{delay} = \frac{R_0}{2} 2W_N \cdot C_0 + \frac{R_0}{W_N} \cdot C_{\text{load}}$$



Minimize (Preclass 1)

$$\text{delay} = R_0 W_N \cdot C_0 + \frac{R_0}{W_N} \cdot C_{load}$$

- Differentiate and set to zero

$$R_0 C_0 - \frac{R_0}{W_N^2} \cdot C_{load} = 0$$

- What's W_N ?



Minimize

$$\text{delay} = R_0 W_N \cdot C_0 + \frac{R_0}{W_N} \cdot C_{load}$$

- Differentiate and set to zero

$$R_0 C_0 - \frac{R_0}{W_N^2} \cdot C_{load} = 0$$

- What's W_N ?

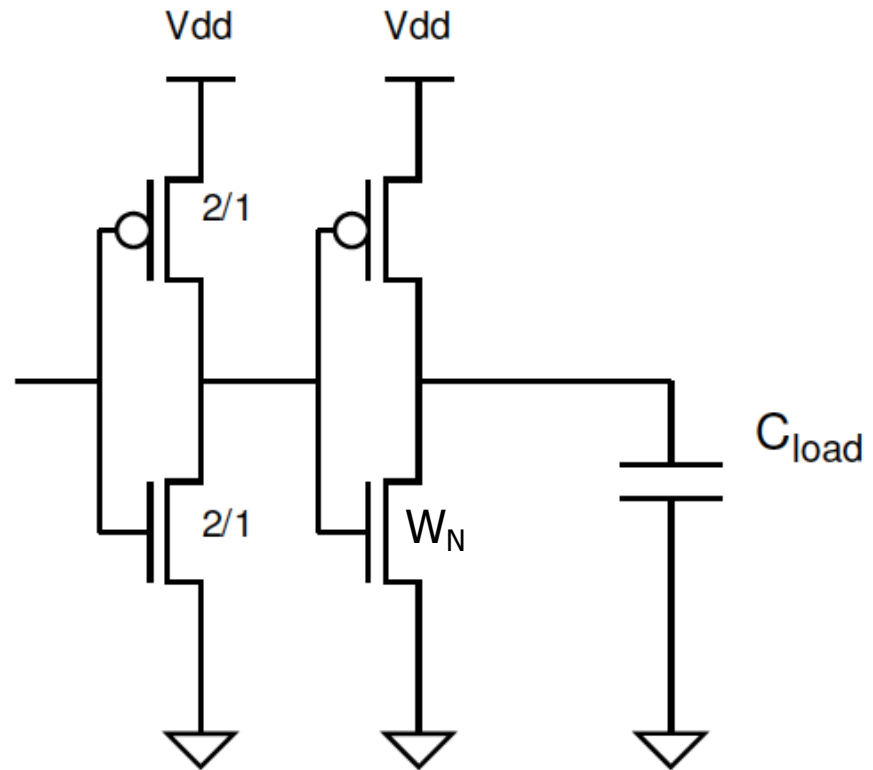
$$W_N^2 = \frac{C_{load}}{C_0}$$

$$W_N = \sqrt{\frac{C_{load}}{C_0}}$$

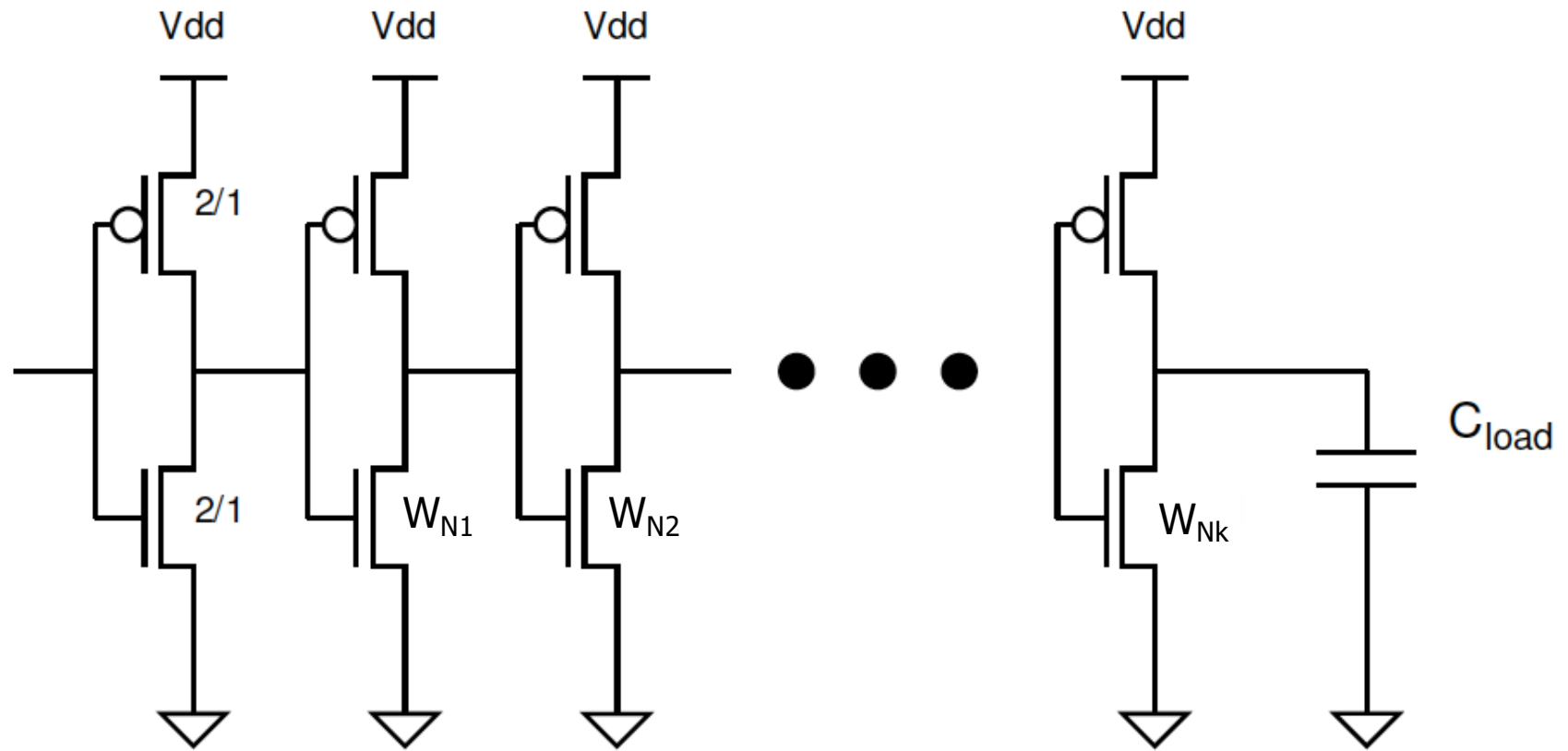
Concrete?

□ What is W_N for $C_{load} = 4 \times 10^4 C_0$?

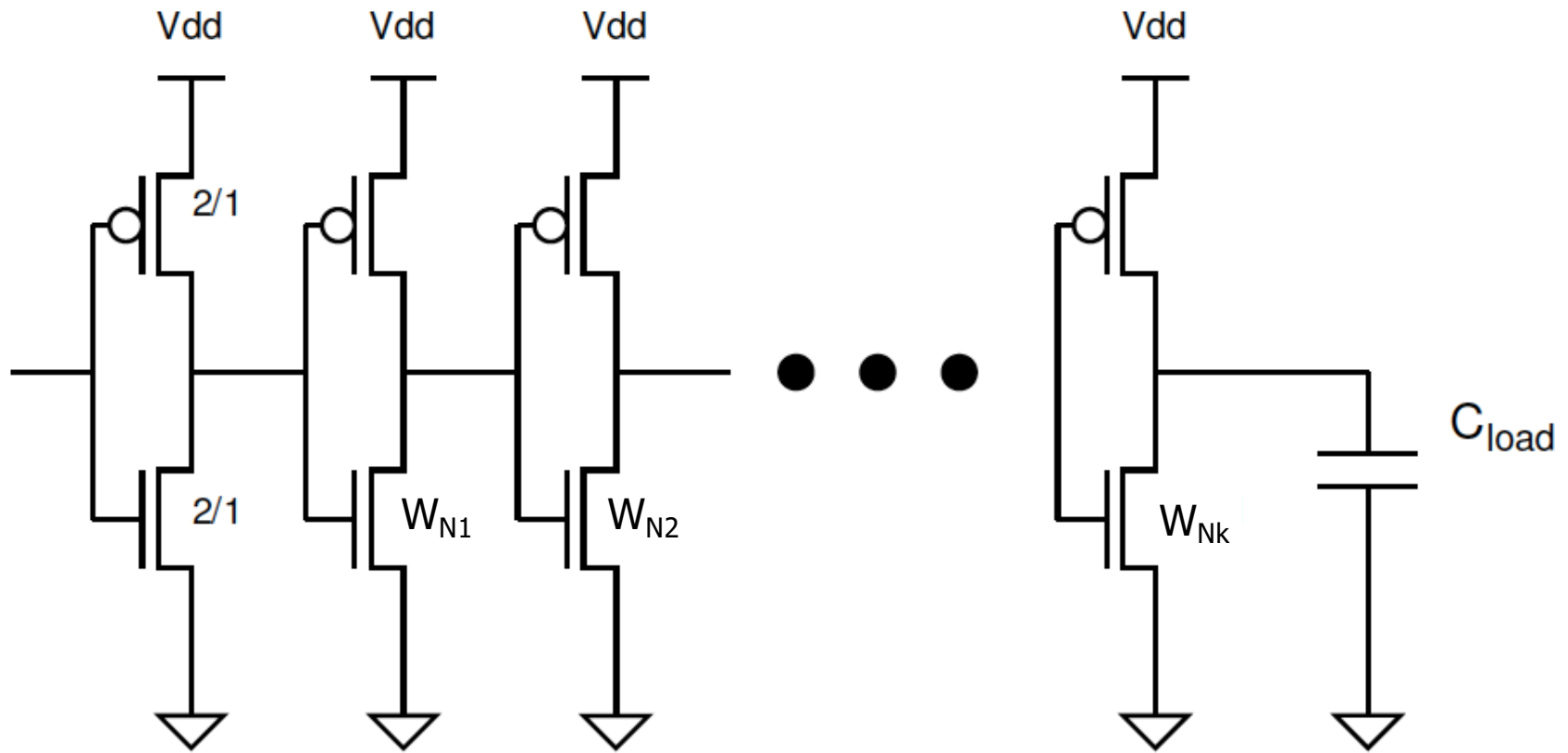
$$W_N = \sqrt{\frac{C_{load}}{C_0}}$$



k-stage (Preclass 2a)



k-stage Delay (Preclass 2a)



$$2\tau \left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} \right) + \frac{R_0}{W_{Nk}} \times C_{load}$$

Size W_{Ni} to minimize delay (Preclass 2b)

□ How do we minimize?

$$2\tau \left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} \right) + \frac{R_0}{W_{Nk}} \times C_{load}$$

Size W_{Ni} to minimize delay (Preclass 2b)

□ Take partial derivative with respect to W_{Ni} and set $= 0$

$$2\tau \left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} \right) + \frac{R_0}{W_{Nk}} \times C_{load}$$

$$2\tau \left(0 + 0 + 0 + \dots + \frac{1}{W_{N(i-1)}} - \frac{W_{N(i+1)}}{(W_{Ni})^2} + \dots + 0 \right) + 0 = 0$$

Size W_{Ni} to minimize delay (Preclass 2b)

□ Take partial derivative with respect to W_{Ni} and set $= 0$

$$2\tau \left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} \right) + \frac{R_0}{W_{Nk}} \times C_{load}$$

$$2\tau \left(0 + 0 + 0 + \dots + \frac{1}{W_{N(i-1)}} - \frac{W_{N(i+1)}}{(W_{Ni})^2} + \dots + 0 \right) + 0 = 0$$

$$\frac{1}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}^2} \rightarrow \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}}$$



Delay

□ **Conclude:** at optimal sizing, **ratio** of stages is same:

$$\frac{1}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}^2} \rightarrow \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}}$$



Delay

□ Call that ratio ρ

$$\rho = \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}}$$



Stage Delay

$$2\tau \left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} \right) + \frac{R_0}{W_{Nk}} \times C_{load}$$



Stage Delay

$$2\tau \left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} \right) + \frac{R_0}{W_{Nk}} \times C_{load}$$

$$2\tau \left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} + \frac{R_0}{2\tau \cdot W_{Nk}} \times C_{load} \right)$$

Stage Delay

$$2\tau \left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} \right) + \frac{R_0}{W_{Nk}} \times C_{load}$$

$$2\tau \left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} + \frac{R_0}{2\tau \cdot W_{Nk}} \times C_{load} \right)$$

$$\rho = \frac{W_{N1}}{2} = \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}} = \frac{C_{load}}{W_{Nk} (2C_0)}$$



Stage Delay

$$\rho = \frac{W_{N1}}{2} = \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}} = \frac{C_{load}}{W_{Nk} (2C_0)}$$

Two math simplifications: 1) to relate ρ and k , 2) total delay in terms of ρ and k



Stage Delay

$$\rho = \frac{W_{N1}}{2} = \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}} = \frac{C_{load}}{W_{Nk}(2C_0)}$$

$$\left(\frac{W_{N1}}{2}\right)\left(\frac{W_{N2}}{W_{N1}}\right)\left(\frac{W_{N3}}{W_{N2}}\right)\cdots\left(\frac{W_{Ni}}{W_{N(i-1)}}\right)\left(\frac{W_{N(i+1)}}{W_{Ni}}\right)\cdots\left(\frac{W_{Nk}}{W_{N(k-1)}}\right)\frac{C_{load}}{W_{Nk}(2C_0)} = \rho^{k+1}$$



Stage Delay

$$\rho^{k+1} = \frac{C_{load}}{4C_0} \rightarrow \rho = \sqrt[k+1]{\frac{C_{load}}{4C_0}}$$

$$\rho = \left(\frac{C_{load}}{4C_0} \right)^{\left(\frac{1}{k+1} \right)}$$

Total Delay (Preclass 2c)

$$\rho = \frac{W_{N1}}{2} = \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}} = \frac{C_{load}}{W_{Nk} (2C_0)}$$

$$2\tau \left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} + \frac{R_0}{2\tau \cdot W_{Nk}} \times C_{load} \right)$$

Total Delay (Preclass 2c)

$$\rho = \frac{W_{N1}}{2} = \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}} = \frac{C_{load}}{W_{Nk} (2C_0)}$$

$$2\tau \left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} + \frac{R_0}{2\tau \cdot W_{Nk}} \times C_{load} \right)$$

$$TotalDelay = 2\tau(k+1)\rho$$



Total Delay

$$\rho = \left(\frac{C_{load}}{4C_0} \right)^{\left(\frac{1}{k+1} \right)}$$

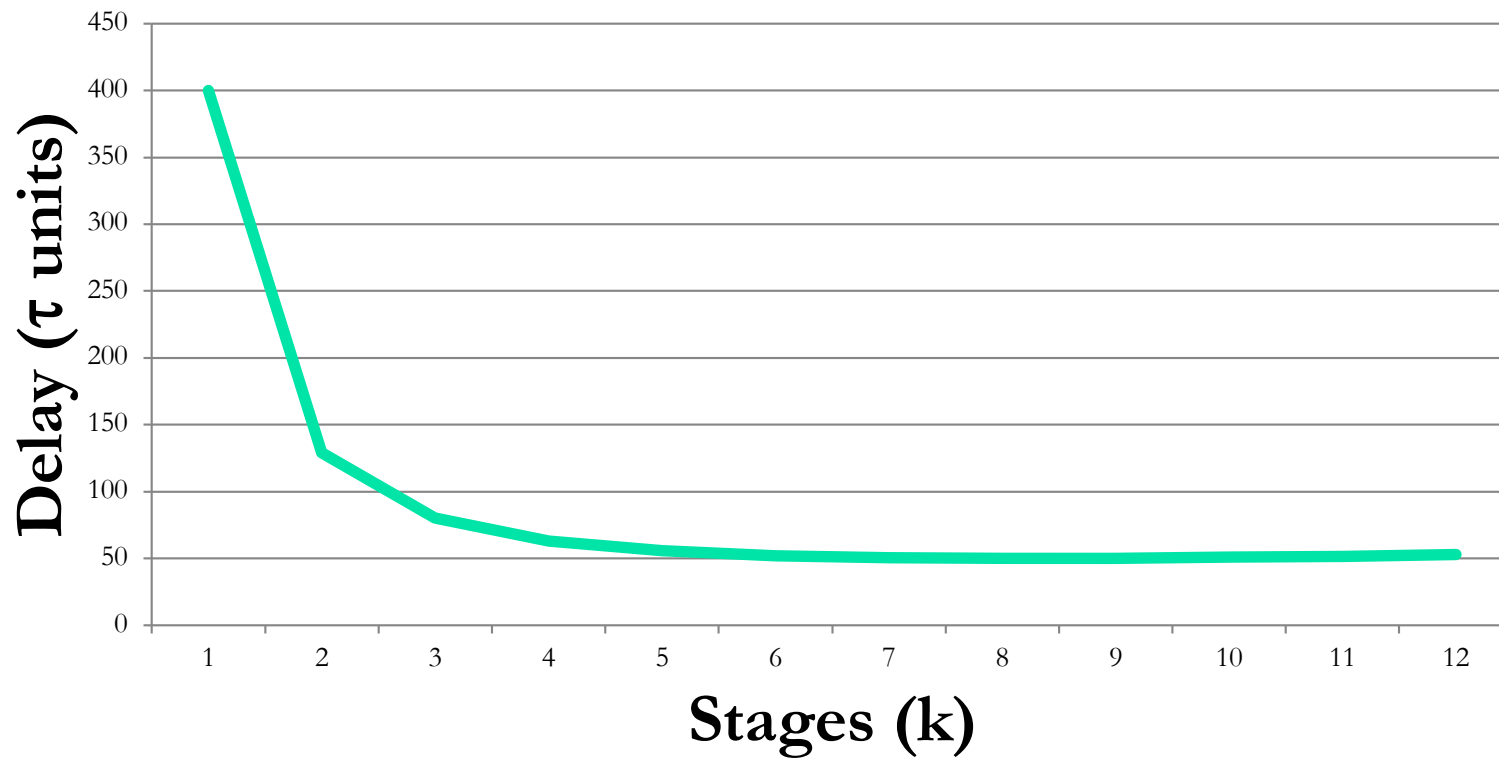
$$TotalDelay = 2\tau(k+1)\rho$$

$$TotalDelay = 2\tau(k+1) \left(\frac{C_{load}}{4C_0} \right)^{\left(\frac{1}{k+1} \right)}$$

Plot Delay vs. k ($C_{load}=4 \times 10^4 C_0$)

$$TotalDelay = 2\tau(k+1) \left(\frac{C_{load}}{4C_0} \right)^{\left(\frac{1}{k+1} \right)}$$

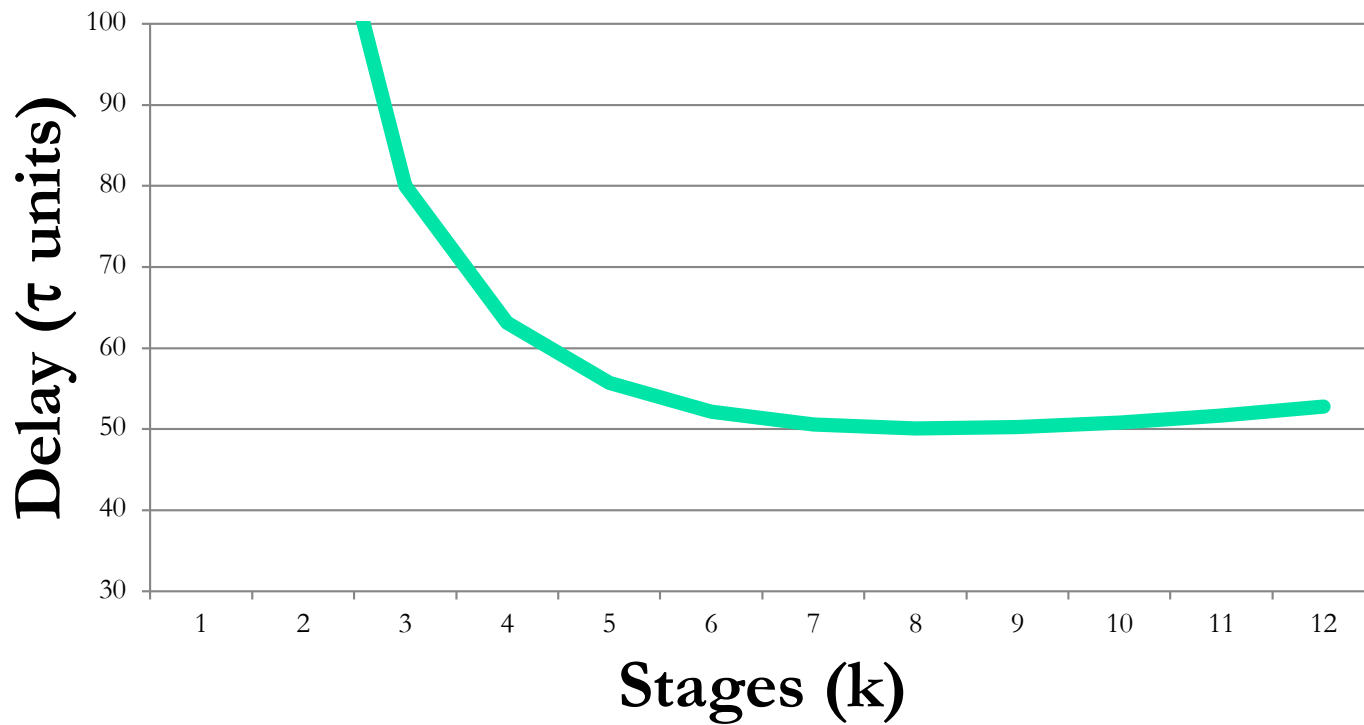
Delay vs. Number of Stages



Zoom: Plot Delay vs. k ($C_{load}=4 \times 10^4 C_0$)

$$TotalDelay = 2\tau(k+1) \left(\frac{C_{load}}{4C_0} \right)^{\left(\frac{1}{k+1} \right)}$$

Delay vs. Number of Stages



Minimize (Preclass 3a)

$$TotalDelay = 2\tau(k+1)\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}$$

$$0 = 2\tau \left[\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)} - (k+1) \cdot \ln\left(\frac{C_{load}}{4C_0}\right) \left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)} \left(\frac{1}{k+1}\right)^2 \right]$$

$$\frac{d(b^x)}{dx} = \ln(b) \cdot b^x$$
$$\frac{d(1/x)}{dx} = -\frac{1}{x^2}$$

Minimize (Preclass 3b)

$$TotalDelay = 2\tau(k+1)\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}$$

$$0 = 2\tau \left[\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)} - (k+1) \cdot \ln\left(\frac{C_{load}}{4C_0}\right) \left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)} \left(\frac{1}{k+1}\right)^2 \right]$$

$$0 = 1 - \left(\frac{1}{k+1}\right) \ln\left(\frac{C_{load}}{4C_0}\right)$$

$$\begin{aligned} \frac{d(b^x)}{dx} &= \ln(b) \cdot b^x \\ \frac{d(1/x)}{dx} &= -\frac{1}{x^2} \end{aligned}$$

$$k = \ln\left(\frac{C_{load}}{4C_0}\right) - 1$$



Concrete (Preclass 3d)

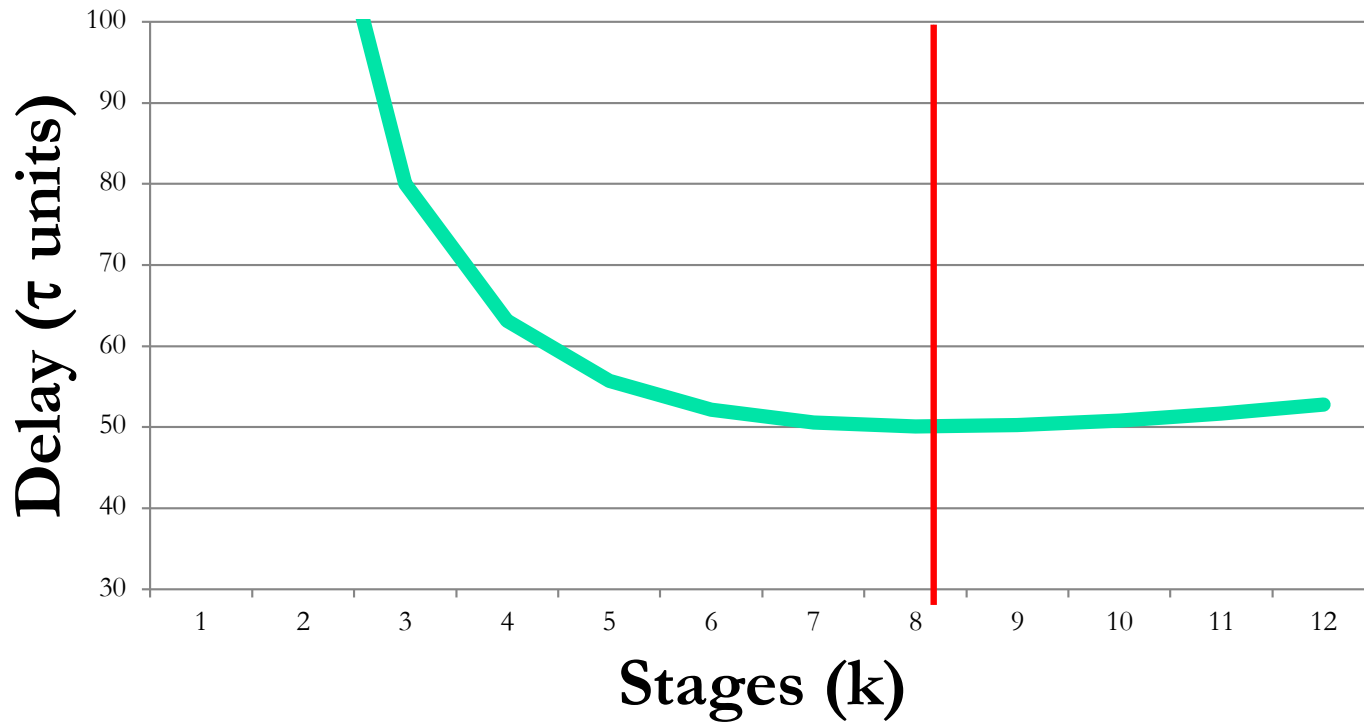
□ What is optimal k for $C_{load}=4 \times 10^4 C_0$?

$$k = \ln\left(\frac{C_{load}}{4C_0}\right) - 1$$

Zoom: Plot Delay vs. k ($C_{load}=4 \times 10^4 C_0$)

$$TotalDelay = 2\tau(k+1) \left(\frac{C_{load}}{4C_0} \right)^{\left(\frac{1}{k+1} \right)}$$

Delay vs. Number of Stages



Optimum Scale Up (Preclass 3c)

- For optimum delay

$$k = \ln\left(\frac{C_{load}}{4C_0}\right) - 1$$

- What is ρ ?

$$\rho = \left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}$$



Optimum Scale Up

$$\rho = \left(\frac{C_{load}}{4C_0} \right)^{\left(\frac{1}{\ln \left(\frac{C_{load}}{4C_0} \right)} \right)} = (Y)^{\left(\frac{1}{\ln(Y)} \right)}$$



Optimum Scale Up

$$\rho = \left(\frac{C_{load}}{4C_0} \right)^{\left(\frac{1}{\ln\left(\frac{C_{load}}{4C_0} \right)} \right)} = (Y)^{\left(\frac{1}{\ln(Y)} \right)}$$

$$\ln(\rho) = \frac{1}{\ln(Y)} \ln(Y) = 1$$

$$\rho = e$$

Call Back: Total Delay

$$\rho = \frac{W_{N1}}{2} = \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}} = \frac{C_{load}}{W_{Nk} (2C_0)}$$

$$2\tau \left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} \right) + \frac{R_0}{W_{Nk}} \times C_{load}$$

$$2\tau \left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} + \frac{R_0}{2\tau \cdot W_{Nk}} \times C_{load} \right)$$

$$\boxed{TotalDelay = 2\tau(k+1)\rho}$$



Delay at Optimum (Preclass 3e)

$$k = \ln\left(\frac{C_{load}}{4C_0}\right) - 1 \qquad \rho = e$$

$$TotalDelay = 2\tau(k + 1)\rho$$

Delay at Optimum (Preclass 3e)

$$k = \ln\left(\frac{C_{load}}{4C_0}\right) - 1 \qquad \rho = e$$

$$TotalDelay = 2\tau(k + 1)\rho$$

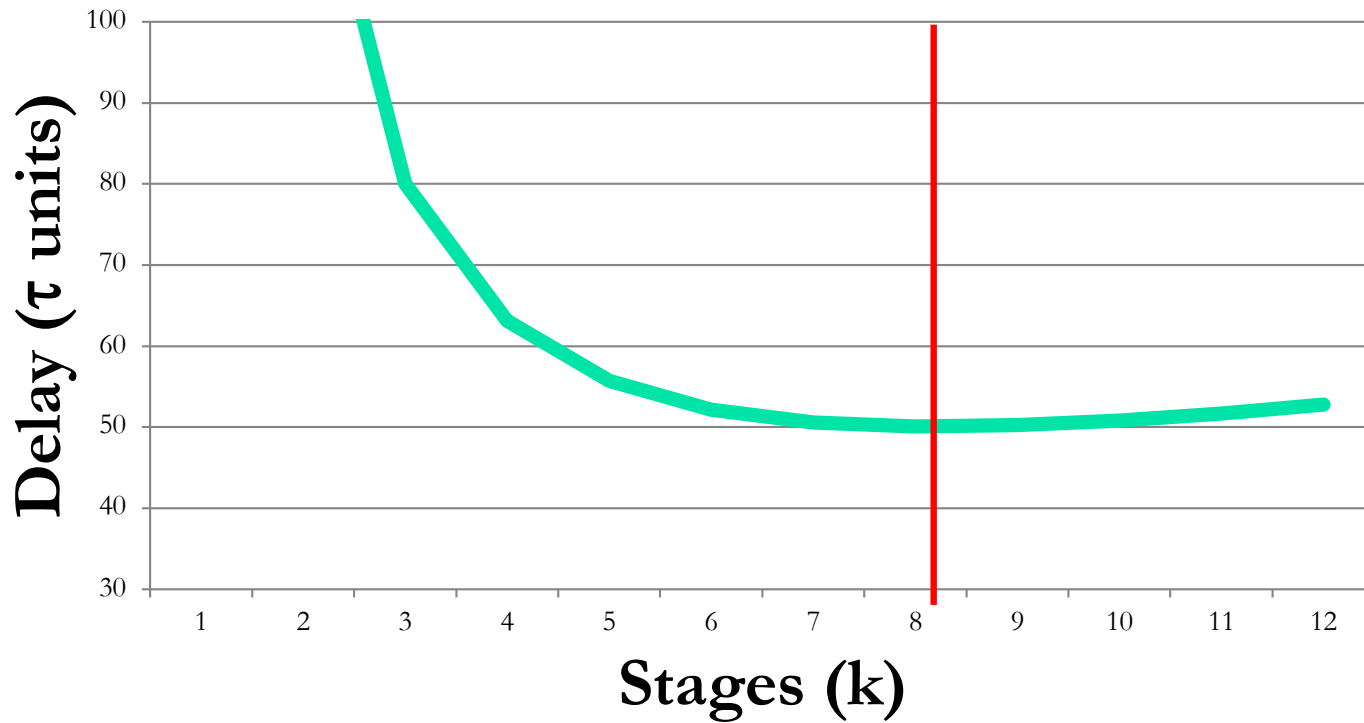
$$TotalDelay = 2\tau \cdot \ln\left(\frac{C_{load}}{4C_0}\right) \cdot e$$

□ What is optimal delay for $C_{load} = 4 \times 10^4 C_0$ in tau units?

Zoom: Plot Delay vs. k ($C_{load}=4 \times 10^4 C_0$)

$$TotalDelay = 2\tau(k+1) \left(\frac{C_{load}}{4C_0} \right)^{\left(\frac{1}{k+1} \right)}$$

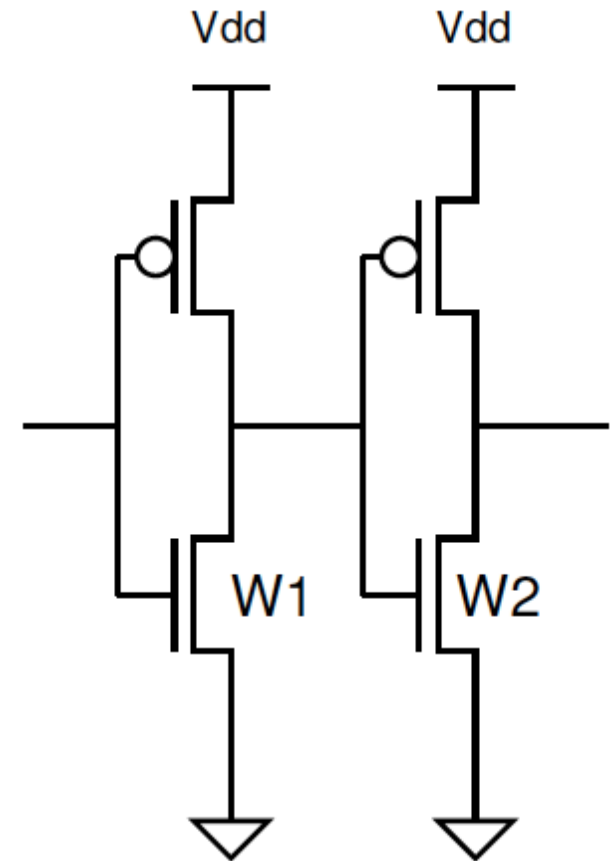
Delay vs. Number of Stages



$$C_{\text{diff}} = \gamma C_{\text{gate}}$$

Diffusion Capacitance (Preclass 4)

- What does this do to τ model?
 - Delay of middle stage cascade?



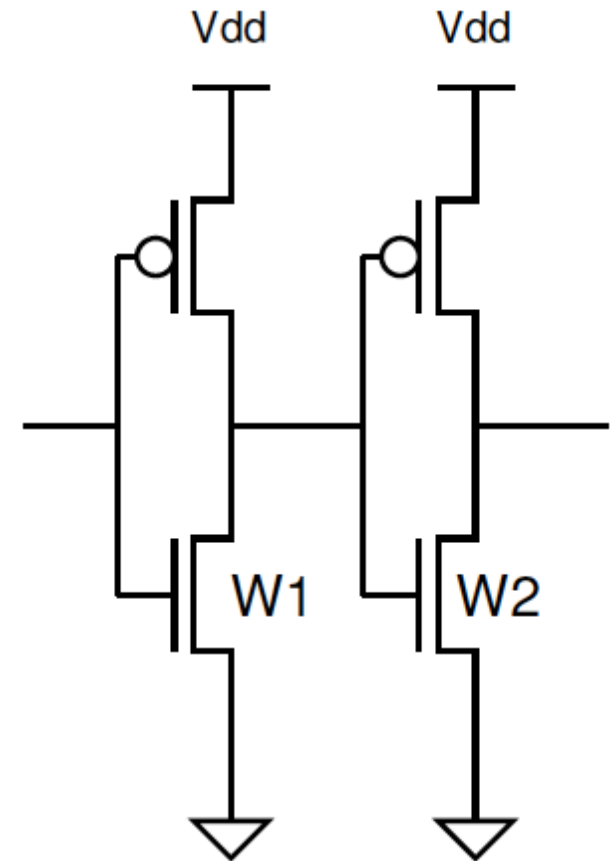
Diffusion Capacitance (Preclass 4)

- What does this do to τ model?
 - Delay of middle stage cascade?

$$\text{delay}_{W1 \rightarrow W2} = \left(\frac{R_0}{W_1} \right) (2W_1 \cdot C_{diff0} + 2W_2 \cdot C_0)$$

$$\text{delay}_{W1 \rightarrow W2} = 2 \left(\frac{R_0}{W_1} \right) (W_1 \cdot \gamma C_0 + W_2 \cdot C_0)$$

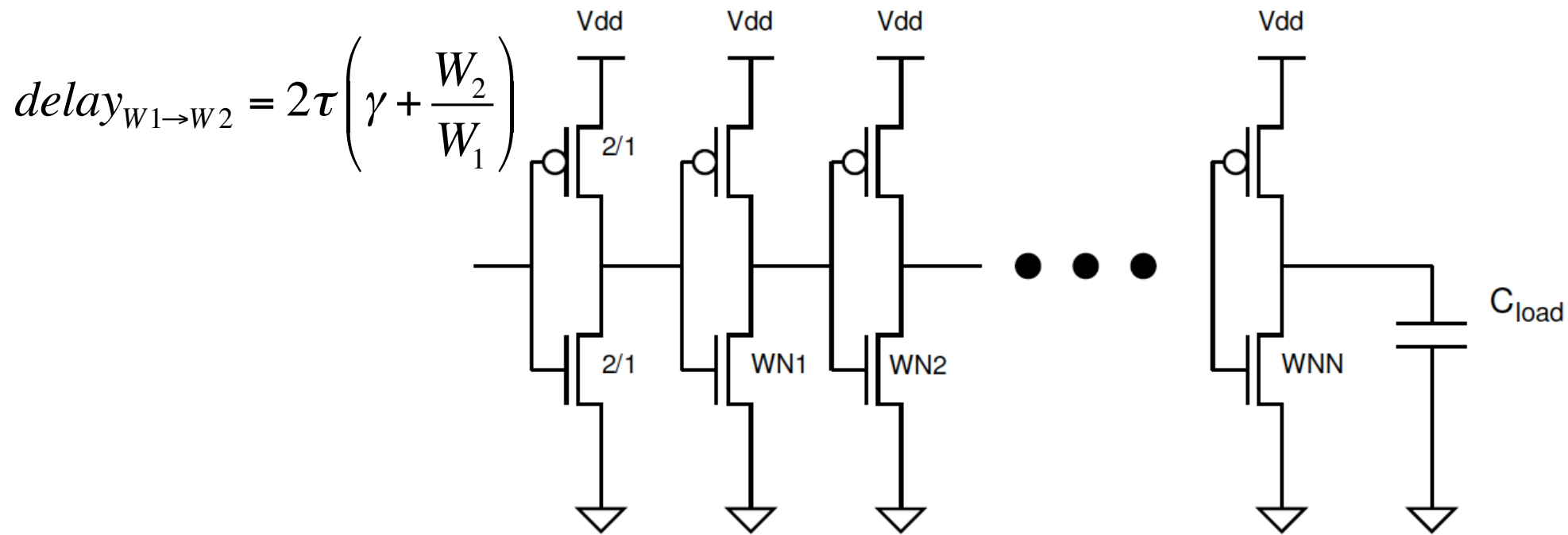
$$\text{delay}_{W1 \rightarrow W2} = 2\tau \left(\gamma + \frac{W_2}{W_1} \right)$$





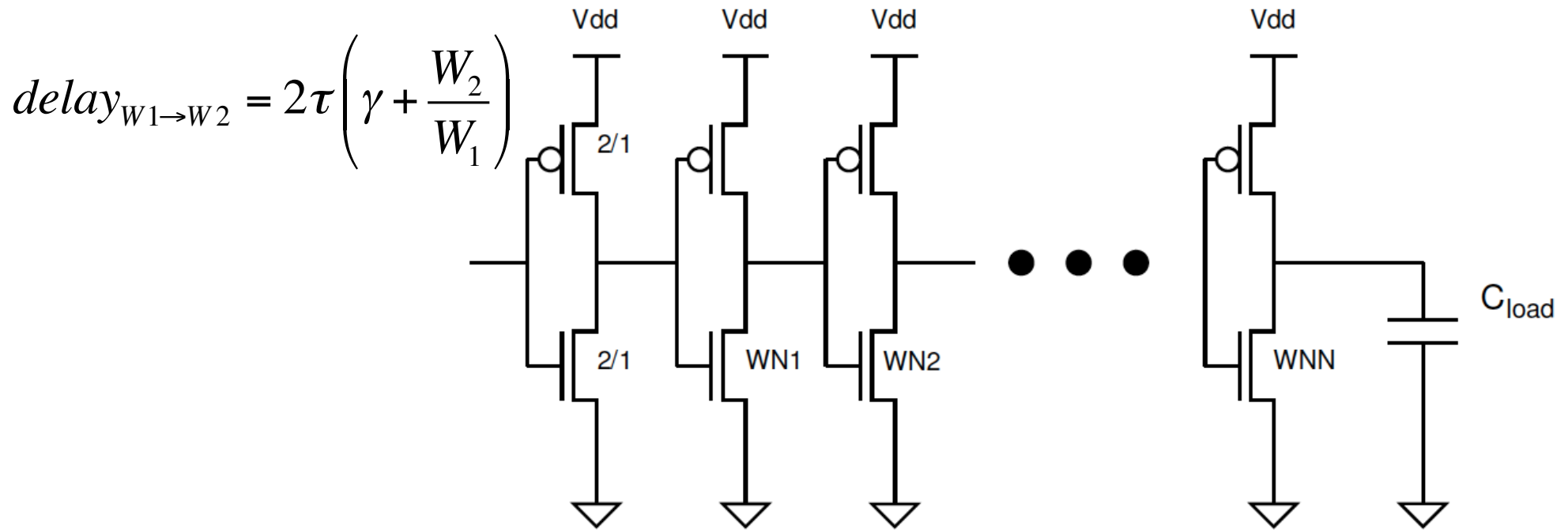
k-stage Delay

$$2\tau \left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} \right) + \frac{R_0}{W_{Nk}} \times C_{load}$$



k-stage Delay (Preclass 4a)

$$2\tau \left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} \right) + \frac{R_0}{W_{Nk}} \times C_{load}$$



$$2\tau \left(\gamma + \frac{W_{N1}}{2} + \gamma + \frac{W_{N2}}{W_{N1}} + \gamma + \frac{W_{N3}}{W_{N2}} + \dots + \gamma + \frac{W_{Ni}}{W_{N(i-1)}} + \gamma + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \gamma + \frac{W_{Nk}}{W_{N(k-1)}} \right)$$

$$+ \frac{R_0}{W_{Nk}} (C_{load} + 2\gamma W_{Nk} C_0)$$

k-stage Delay

$$2\tau \left(\gamma + \frac{W_{N1}}{2} + \gamma + \frac{W_{N2}}{W_{N1}} + \gamma + \frac{W_{N3}}{W_{N2}} + \dots + \gamma + \frac{W_{Ni}}{W_{N(i-1)}} + \gamma + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \gamma + \frac{W_{Nk}}{W_{N(k-1)}} \right) \\ + \frac{R_0}{W_{Nk}} (C_{load} + 2\gamma W_{Nk} C_0)$$

$$2\tau \left(\gamma k + \frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} \right) + \frac{R_0}{W_{Nk}} (C_{load} + 2\gamma W_{Nk} C_0)$$

k-stage Delay

$$2\tau \left(\gamma + \frac{W_{N1}}{2} + \gamma + \frac{W_{N2}}{W_{N1}} + \gamma + \frac{W_{N3}}{W_{N2}} + \dots + \gamma + \frac{W_{Ni}}{W_{N(i-1)}} + \gamma + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \gamma + \frac{W_{Nk}}{W_{N(k-1)}} \right) \\ + \frac{R_0}{W_{Nk}} (C_{load} + 2\gamma W_{Nk} C_0)$$

$$2\tau \left(\gamma k + \frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} \right) + \frac{R_0}{W_{Nk}} (C_{load} + 2\gamma W_{Nk} C_0)$$

$$2\tau \left(\gamma k + \frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} \right) + \frac{R_0}{W_{Nk}} C_{load} + 2\tau\gamma$$

k-stage Delay

$$2\tau \left(\gamma + \frac{W_{N1}}{2} + \gamma + \frac{W_{N2}}{W_{N1}} + \gamma + \frac{W_{N3}}{W_{N2}} + \dots + \gamma + \frac{W_{Ni}}{W_{N(i-1)}} + \gamma + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \gamma + \frac{W_{Nk}}{W_{N(k-1)}} \right) \\ + \frac{R_0}{W_{Nk}} (C_{load} + 2\gamma W_{Nk} C_0)$$

$$2\tau \left(\gamma k + \frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} \right) + \frac{R_0}{W_{Nk}} (C_{load} + 2\gamma W_{Nk} C_0)$$

$$2\tau \left(\gamma k + \frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} \right) + \frac{R_0}{W_{Nk}} C_{load} + 2\tau\gamma$$

$$2\tau \left(\gamma k + \frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} + \frac{R_0}{2\tau W_{Nk}} C_{load} + \gamma \right)$$

k-stage Delay

$$2\tau \left(\gamma + \frac{W_{N1}}{2} + \gamma + \frac{W_{N2}}{W_{N1}} + \gamma + \frac{W_{N3}}{W_{N2}} + \dots + \gamma + \frac{W_{Ni}}{W_{N(i-1)}} + \gamma + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \gamma + \frac{W_{Nk}}{W_{N(k-1)}} \right) \\ + \frac{R_0}{W_{Nk}} (C_{load} + 2\gamma W_{Nk} C_0)$$

$$2\tau \left(\gamma k + \frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} \right) + \frac{R_0}{W_{Nk}} (C_{load} + 2\gamma W_{Nk} C_0)$$

$$2\tau \left(\gamma k + \frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} \right) + \frac{R_0}{W_{Nk}} C_{load} + 2\tau\gamma$$

$$2\tau \left(\gamma k + \frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} + \frac{R_0}{2\tau W_{Nk}} C_{load} + \gamma \right)$$

$$2\tau \left(\gamma(k+1) + \frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} + \frac{R_0}{2\tau \cdot W_{Nk}} \cdot C_{load} \right)$$

Size W_{Ni} to minimize delay (Preclass 4b)

□ Take partial derivative with respect to $W_{Ni} = 0$

$$2\tau \left(\gamma(k+1) + \frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} + \frac{R_0}{2\tau \cdot W_{Nk}} \cdot C_{load} \right)$$

$$2\tau \left(0 + 0 + 0 + \dots + \frac{1}{W_{N(i-1)}} - \frac{W_{N(i+1)}}{(W_{Ni})^2} + \dots + 0 \right) + 0 = 0$$

$$\frac{1}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}^2} \rightarrow \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}}$$

Impact on Minimum W_{Ni} ? (Preclass 4c)

- Partial derivative unchanged

$$2\tau \left(\gamma(k+1) + \frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} + \frac{R_0}{2\tau \cdot W_{Nk}} \cdot C_{load} \right)$$

What does this say about ρ ?

Stage Delay: ρ unchanged (for fixed k)

$$\rho = \frac{W_{N1}}{2} = \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}} = \frac{C_{load}}{W_{Nk} (2C_0)}$$

$$\rho = \left(\frac{C_{load}}{4C_0} \right)^{\left(\frac{1}{k+1} \right)}$$

Stage Delay: (Preclass 4d)

$$\rho = \frac{W_{N1}}{2} = \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}} = \frac{C_{load}}{W_{Nk} (2C_0)}$$

$$\rho = \left(\frac{C_{load}}{4C_0} \right)^{\left(\frac{1}{k+1} \right)}$$

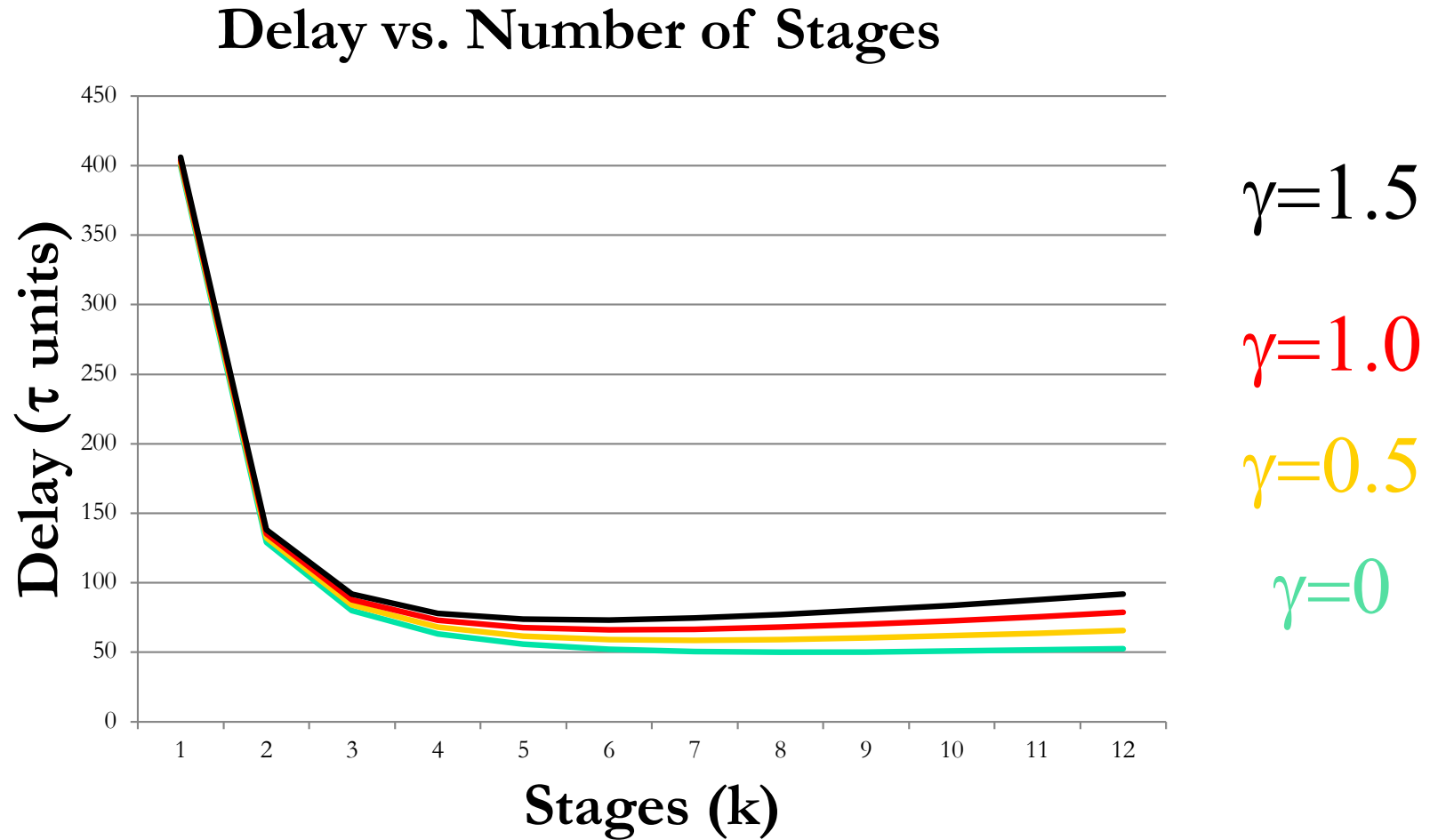
$$2\tau \left(\gamma(k+1) + \frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \dots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \dots + \frac{W_{Nk}}{W_{N(k-1)}} + \frac{R_0}{2\tau \cdot W_{Nk}} \cdot C_{load} \right)$$

$$TotalDelay = 2\tau(k+1)(\rho + \gamma)$$

$$TotalDelay = 2\tau(k+1) \left(\left(\frac{C_{load}}{4C_0} \right)^{\left(\frac{1}{k+1} \right)} + \gamma \right)$$

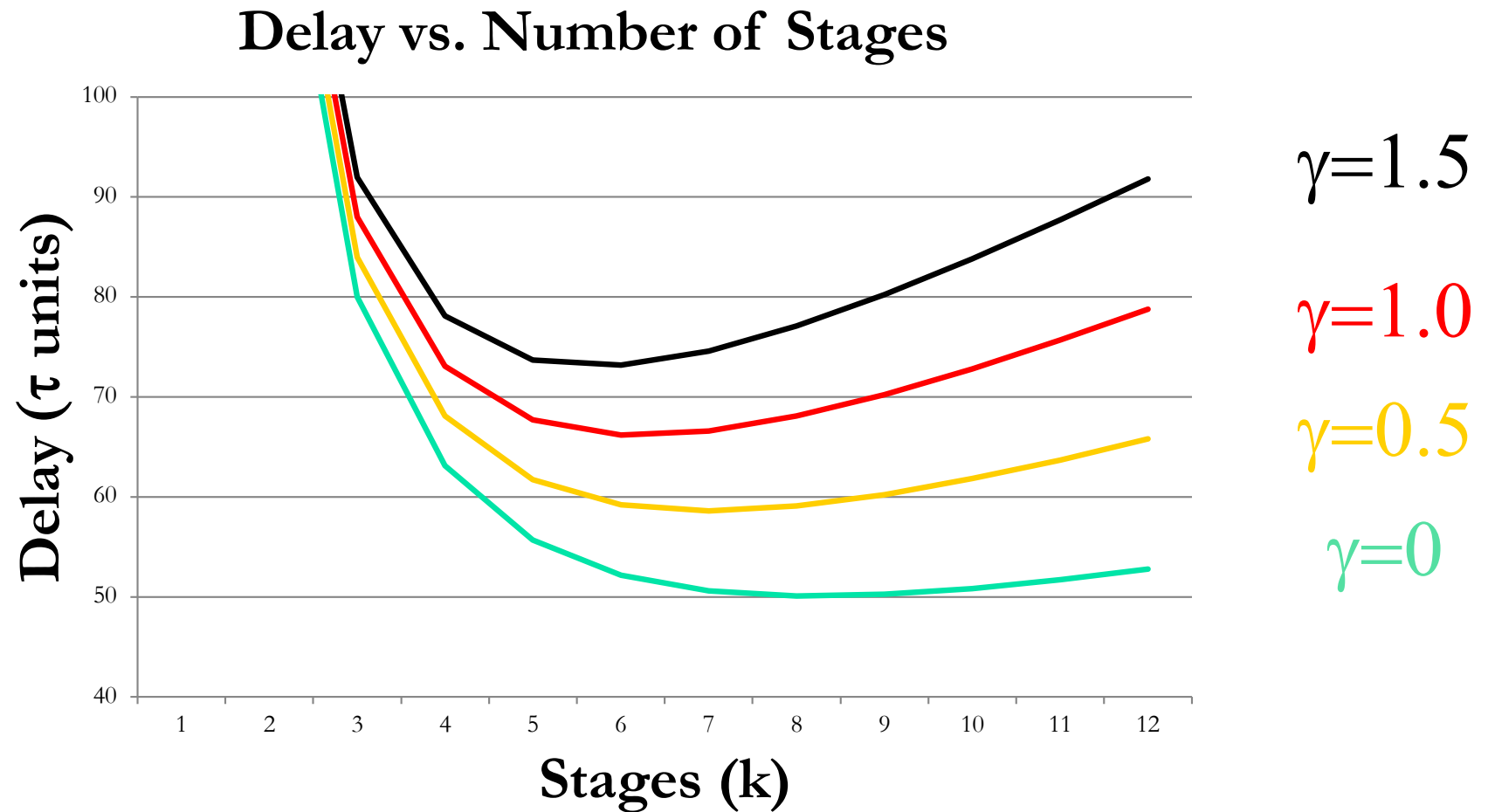


Impact of Gamma





Impact of Gamma





Minimize

$$TotalDelay = 2\tau(k+1)(\rho + \gamma)$$

$$TotalDelay = 2\tau(k+1)\left(\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)} + \gamma\right)$$

$$0 = 2\tau\left[\gamma + \left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)} - (k+1) \cdot \ln\left(\frac{C_{load}}{4C_0}\right)\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}\left(\frac{1}{k+1}\right)^2\right]$$

$$0 = \gamma + \rho - (k+1) \cdot \ln\left(\frac{C_{load}}{4C_0}\right)\rho\left(\frac{1}{k+1}\right)^2$$

$$\gamma + \rho = \ln\left(\frac{C_{load}}{4C_0}\right)\left(\frac{1}{k+1}\right)\rho$$



Solve

$$\gamma + \rho = \ln\left(\frac{C_{load}}{4C_0}\right)\left(\frac{1}{k+1}\right)\rho$$

$$\frac{\gamma}{\rho} + 1 = \ln\left(\frac{C_{load}}{4C_0}\right)\left(\frac{1}{k+1}\right)$$



Solve

$$\gamma + \rho = \ln\left(\frac{C_{load}}{4C_0}\right)\left(\frac{1}{k+1}\right)\rho$$

$$\frac{\gamma}{\rho} + 1 = \ln\left(\frac{C_{load}}{4C_0}\right)\left(\frac{1}{k+1}\right)$$

$$\frac{\gamma}{\rho} + 1 = \ln\left(\left(\frac{C_{load}}{4C_0}\right)^{\frac{1}{k+1}}\right)$$

$$\frac{\gamma}{\rho} + 1 = \ln(\rho)$$



Optimal Staging Any γ (Preclass 4e)

$$\rho = e^{\left(\frac{\gamma}{\rho} + 1\right)}$$



ρ and γ ? (Preclass 4f)

- $\rho=4$ is optimal for what γ ?
- $\rho=3$ is optimal for what γ ?

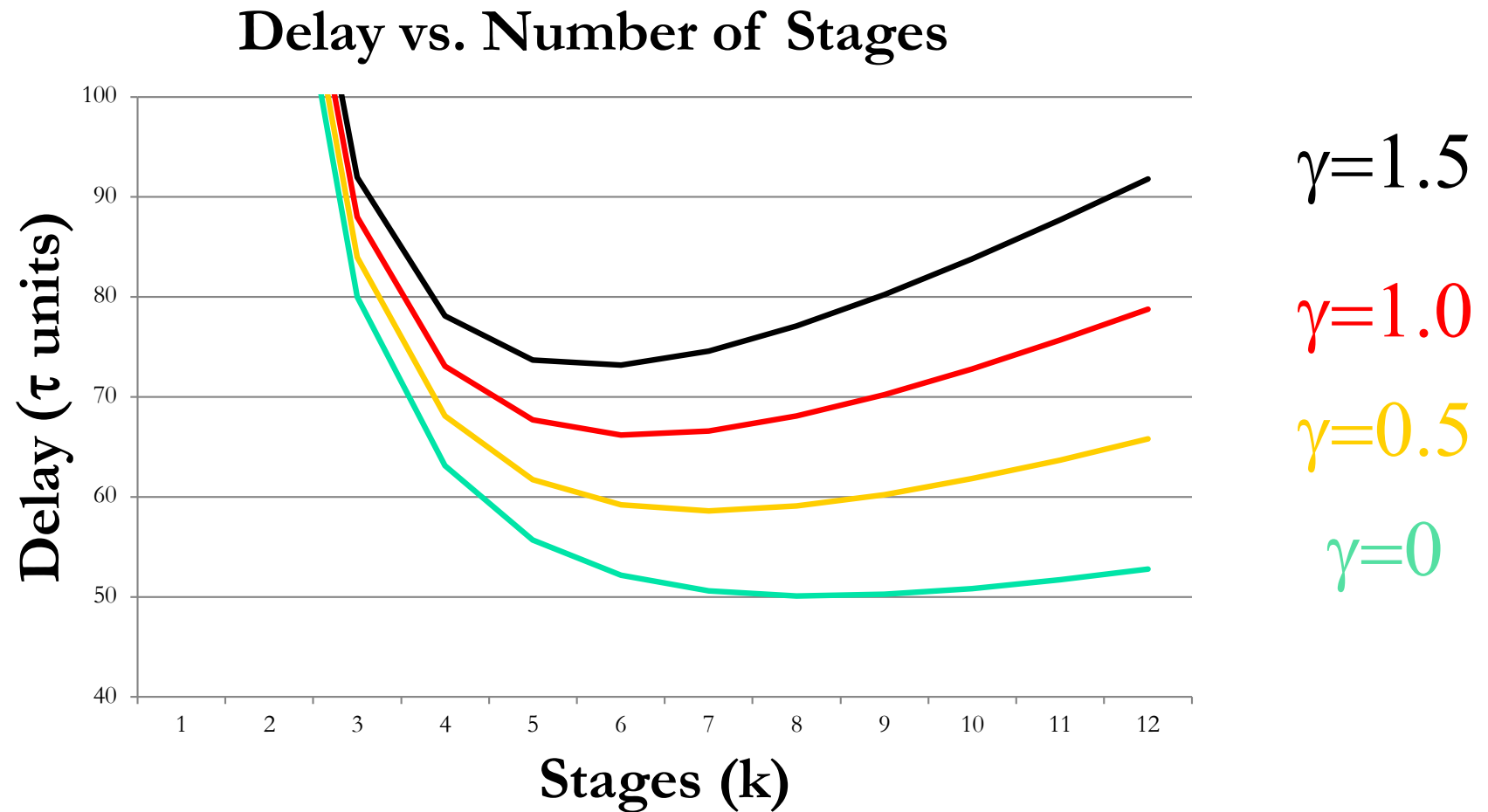
$$\rho = e^{\left(\frac{\gamma}{\rho} + 1\right)}$$

$$\ln \rho = \frac{\gamma}{\rho} + 1$$

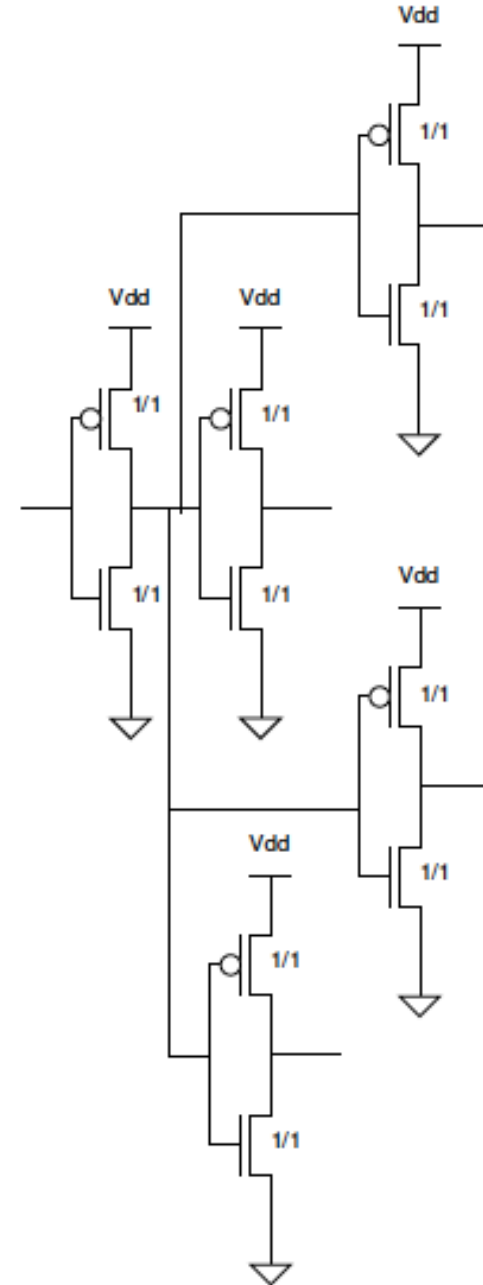
$$\rho \ln \rho - \rho = \gamma$$



Impact of Gamma



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Idea

- ❑ To drive large loads
 - Scale buffers geometrically
 - Exponential scale up in buffer size ($\rho = e$)
- ❑ Scale factor: 3—4 typically
 - One origin of fanout 4 target
- ❑ Drains contribute capacitance too (C_{diff})
- ❑ Can formulate sizing to optimize



Admin

- ❑ Project 1
 - Final report due **Friday 3/28** midnight
- ❑ HW 6 release Friday 3/28, due 4/11



Acknowledgement

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